GLOBAL ATTRACTOR AND ROBUST EXPONENTIAL ATTRACTORS FOR SOME CLASSES OF FOURTH-ORDER NONLINEAR EVOLUTION EQUATIONS

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Abstract. We study the long-time behaviour of solutions to some classes of fourth-order nonlinear PDEs with non-monotone nonlinearities, which include the Landau–Lifshitz–Baryakhtar (LLBar) equation (with all relevant fields and spin torques) and the convective Cahn–Hilliard/Allen–Cahn (CH-AC) equation with a proliferation term, in dimensions $d = 1, 2, 3$. Firstly, we show the global well-posedness, as well as the existence of global and exponential attractors with finite fractal dimensions for these problems. In the case of the exchange-dominated LLBar equation and the CH-AC equation without convection, an estimate for the rate of convergence of the solution to the corresponding stationary state is given. Finally, we show the existence of a robust family of exponential attractors when $d \leq 2$. As a corollary, exponential attractor of the LLBar equation is shown to converge to that of the Landau–Lifshitz–Bloch equation in the limit of vanishing exchange damping, while exponential attractor of the convective CH-AC equation is shown to converge to that of the convective Allen–Cahn equation in the limit of vanishing diffusion coefficient.

Contents

1. Introduction

This paper aims to show the existence of global attractor and a family of robust exponential attractors for some classes of fourth-order nonlinear PDEs, which include the vector-valued Landau–Lifshitz–Baryakhtar (LLBar) equation with spin-torques and the scalar-valued convective Cahn–Hilliard/Allen–Cahn (CH-AC) equation with a proliferation term, among others. The result also applies to their limiting cases, namely the Landau–Lifshitz–Bloch (LLB) equation with spin-torques and the convective Allen–Cahn equation. The existence of global and exponential attractors of finite fractal dimension allows a reduction, in some sense, of an infinite-dimensional dynamical system to a finite-dimensional one [43].

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We now describe the general form of the problem discussed in this paper. Let $\mathscr{O} \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, be an open bounded domain. Let $\boldsymbol{u}(t,\boldsymbol{x}) \in \mathbb{R}^m$, where $m=1$ or 3, be the unknown functions (which can be scalar- or vector-valued). Here, $\boldsymbol{x} \in \mathscr{O}$ is the spatial variable, and $t \in (0,T)$ is the temporal variable with $T>0$. The boundary of $\mathscr{O}$ is denoted by $\partial\mathscr{O}$, with exterior unit normal vector denoted by $\boldsymbol{n}$. The problem can be written as:

$$\begin{aligned} \partial_t \boldsymbol{u} &= \sigma\big(\boldsymbol{H} + \Phi_{\mathrm{d}}(\boldsymbol{u})\big) - \varepsilon\Delta\big(\boldsymbol{H} + \Phi_{\mathrm{d}}(\boldsymbol{u})\big) \\ &\qquad - \gamma \boldsymbol{u} \times \big(\boldsymbol{H} + \Phi_{\mathrm{d}}(\boldsymbol{u})\big) + \mathcal{R}(\boldsymbol{u}) + \mathcal{S}(\boldsymbol{u}) \qquad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\mathscr{O}, \end{aligned} \tag{1.1a}$$

$$\boldsymbol{H} = \Psi(\boldsymbol{u}) + \Phi_{\mathrm{a}}(\boldsymbol{u}) \qquad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\mathscr{O}, \tag{1.1b}$$

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \mathscr{O}, \tag{1.1c}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{0},\ \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{n}} = \boldsymbol{0} \qquad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\partial\mathscr{O}. \tag{1.1d}$$

The coefficients σ, ε, and γ are positive constants of physical significance, and in particular ε is called the exchange damping coefficient if $m=3$, or the diffusion coefficient if $m=1$. We set $\gamma=0$ if $\boldsymbol{u}$ is scalar-valued ($m=1$). The terms $\Psi(\boldsymbol{u}), \Phi_{\mathrm{d}}(\boldsymbol{u}), \Phi_{\mathrm{a}}(\boldsymbol{u}), \mathcal{R}(\boldsymbol{u})$, and $\mathcal{S}(\boldsymbol{u})$ are nonlinear functions of $\boldsymbol{u}$ and possibly its spatial gradient, whose exact forms are detailed in Section 2.2. While we consider the Neumann boundary condition in the above problem, similar arguments will also work for the Dirichlet or the periodic boundary conditions.

When $\boldsymbol{u}$ is vector-valued ($m=3$), problem (1.1) is the initial-boundary value problem associated with the Landau–Lifshitz–Baryakhtar (LLBar) equation [5, 6, 7, 16], which describes the evolution of the magnetisation vectors $\boldsymbol{u}(t,\boldsymbol{x}) \in \mathbb{R}^3$ on any point in a magnetic body $\mathscr{O}$ at elevated temperatures. The unknown field $\boldsymbol{H}$ is called the effective field. Spin-torque effects due to currents [52, 55] and anisotropy of the material [36] are also taken into account. Formally setting $\varepsilon=0$ in (1.1a) gives the Landau–Lifshitz–Bloch (LLB) equation [24, 25, 33] with spin-torques [3], which is a system of quasilinear second-order PDEs. In physical applications, often the limit $\varepsilon \to 0^+$ is taken in the LLBar equation when certain long-range interactions are negligible [15, 52].

When $\boldsymbol{u}$ is scalar-valued ($m=1$), we always set $\gamma=0$ and $\Phi_{\mathrm{d}}(\boldsymbol{u}) = \Phi_{\mathrm{a}}(\boldsymbol{u}) = 0$. In this case, problem (1.1) is the initial-boundary value problem associated with the convective Cahn–Hilliard/Allen–Cahn (CH-AC) equation which models multiple microscopic mechanisms involving diffusion, reaction, transport, and adsorption in cluster interface evolution [2, 28, 29, 30]. The unknown $\boldsymbol{u}$ is often called the order parameter and $\boldsymbol{H}$ is the potential. We further remark that the case $\sigma=0$ and $\mathcal{S}(\boldsymbol{u})=0$ gives the convective Cahn–Hilliard (CH) equation [18, 27], while the case $\mathcal{R}(\boldsymbol{u})=0$ gives the Cahn–Hilliard equation with a mass source term [23, 32, 37, 39]. The term $\mathcal{S}(u)$ represents a proliferation term [13, 23, 38], which is relevant in various biological applications. The problem (1.1) also describes a generalised diffusion model for growth and dispersal in a population [12]. Formally setting $\varepsilon=0$ gives a second-order PDE known as the convective Allen–Cahn (AC) equation [45, 46] or a reaction-diffusion-convection model with Allee effect in mathematical biology [53]. Thus, it is also of interest to examine the behaviour of (1.1) as $\varepsilon \to 0^+$ if $\boldsymbol{u}$ is scalar-valued.

Some mathematical results which are relevant to the present paper will be reviewed here and in the following paragraph. For problem (1.1) with $m=3$ and $\Phi_{\mathrm{d}}(\boldsymbol{u}) = \Phi_{\mathrm{a}}(\boldsymbol{u}) = \mathcal{R}(\boldsymbol{u}) = \mathcal{S}(\boldsymbol{u}) = \boldsymbol{0}$, i.e. the LLBar equation, the global existence and uniqueness of strong solution for any finite $T>0$ are shown in [48] (also in [26] for the stochastic case). Some numerical schemes to approximate the solution are proposed in [47, 49]. In the case of the exchange-dominated LLB equation ($\varepsilon=0$), the existence of weak solution is obtained in [33], while the existence and uniqueness of strong solution are shown in [34]. The LLB equation with spin torques is considered in [3], where the existence and uniqueness of weak solution for $d \le 2$ were shown under certain assumptions. However, to the best of our knowledge, the analysis of the LLBar or the LLB equations with full effective fields and spin-torques are not available yet in the literature. Asymptotic behaviour of the solutions to these equations in terms of finite-dimensional attractors has not been discussed before either.

For the problem (1.1) with $m = 1$ and $\mathcal{R}(\boldsymbol{u}) = \mathcal{S}(\boldsymbol{u}) = 0$, i.e. the CH-AC equation, the existence and uniqueness of weak and strong solutions are shown in [29, 35], while the existence of global attractor in 2D can be established by similar argument as in [30]. The existence and uniqueness of solution to the convective CH equation with periodic boundary conditions are shown in [18] (also in [58] for the case of unbounded domains), while the existence of global attractor is obtained in [18, 59]. We also mention several other papers [23, 32, 38, 39], which study the Cahn–Hilliard equation with a polynomial source term. While many results are available in the literature for the scalar-valued CH or CH-AC equations, none of them are sufficiently general to cover the nonlinearities present in problem (1.1), especially for $d = 3$. Moreover, the limiting case $\varepsilon \to 0^+$ (vanishing diffusion coefficient) has also not been studied.

This paper aims to unify and further develop the analysis of (1.1) by deriving the following results:

(i) global existence and uniqueness of weak and strong solution to (1.1) on $(0, T) \times \mathscr{O}$, for any $T > 0$ (Theorem 3.11),

(ii) existence of a (compact) global attractor for (1.1) with finite fractal dimension (Theorem 4.10, Theorem 5.2),

(iii) convergence of the solution of the LLBar equation to the corresponding stationary state, with an estimate on the rate of convergence, in the case of exchange-dominated field (Theorem 4.13),

(iv) existence of an exponential attractor for (1.1) and its characterisation (Theorem 4.16),

(v) existence of a robust (in ε) family of exponential attractors for (1.1) when $d \le 2$ (Theorem 5.10).

Existence of a solution to the problem (1.1) is obtained by means of the Faedo–Galerkin method. Owing to the nature of nonlinearities present in the problem (which are non-monotone), a detailed analysis is done to derive uniform a priori estimates on the approximate solutions in suitable function spaces, which extend the solution globally in time. The existence of global and exponential attractors is deduced by showing various dissipative and smoothing estimates. To obtain a robust family of exponential attractors, more careful analysis is needed to ensure these estimates are uniform in the parameter ε. We reiterate that while the existence of global solution to the LLBar equation has been shown in [48], the model considered there only include the exchange field in $\boldsymbol{H}$ and does not consider any convective terms. Most a priori estimates, especially the smoothing estimates and the uniform estimates independent of ε developed in this paper, are new.

As a corollary of our analysis, we deduce the existence of the global attractor and an exponential attractor with finite fractal dimensions for the LLB equations (taking into account all relevant effective fields and spin torques) when $d \le 2$. In this case, we show that exponential attractor of the LLBar equation converges (in the sense of the symmetric Hausdorff distance) at a given rate to that of the LLB equation as $\varepsilon \to 0^+$. Similar results are also obtained for the convective Cahn–Hilliard/Allen–Cahn equation, for which the convergence to the convective Allen–Cahn equation is shown in the limit of vanishing diffusion coefficient.

2. Preliminaries

2.1. **Notations.** We begin by defining some notations used in this paper. For $m = 1$ or $m = 3$, the function space $\mathbb{L}^p := \mathbb{L}^p(\mathscr{O}; \mathbb{R}^m)$ denotes the usual space of p-th integrable functions taking values in $\mathbb{R}^m$ and $\mathbb{W}^{k,p} := \mathbb{W}^{k,p}(\mathscr{O}; \mathbb{R}^m)$ denotes the usual Sobolev space of functions on $\mathscr{O} \subset \mathbb{R}^d$ taking values in $\mathbb{R}^m$. We write $\mathbb{H}^k := \mathbb{W}^{k,2}$. Here, $\mathscr{O} \subset \mathbb{R}^d$ for $d = 1, 2, 3$ is an open domain with C^∞-smooth boundary. The Laplacian operator acting on $\mathbb{R}^m$-valued functions is denoted by Δ.

If X is a Banach space, the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$ denote respectively the usual Lebesgue and Sobolev spaces of functions on $(0, T)$ taking values in X. The space $C([0, T]; X)$ denotes the space of continuous functions on $[0, T]$ taking values in X. Throughout this paper, we denote the scalar product in a Hilbert space H by $\langle \cdot, \cdot \rangle_H$ and its corresponding norm by $\| \cdot \|_H$. We will not distinguish between the scalar product of $\mathbb{L}^2$ vector-valued functions taking values in $\mathbb{R}^m$ and the scalar product of $\mathbb{L}^2$ matrix-valued functions taking values in $\mathbb{R}^{m \times m}$, and still denote them by $\langle \cdot, \cdot \rangle_{\mathbb{L}^2}$.

Throughout this paper, the constant C in the estimate always denotes a generic constant which depends only on the coefficients of (1.1), $\mathscr{O}$, and ν_∞ (to be defined in (2.6)), but is *independent* of t. If the dependence of C on some variable, e.g. T, is highlighted, we will write $C(T)$. The constants denoted by C might take different values at different occurrences, unless otherwise specified.

2.2. **Formulation of the problem and assumptions.** In this section, we provide further details on the formulation of problem (1.1). Recall that $m = 1$ or 3. The meaning of each term in (1.1) is as follows:

(i) $\boldsymbol{u}(t) : \mathscr{O} \to \mathbb{R}^m$ is the magnetic spin field if $m = 3$, or the order parameter if $m = 1$.
(ii) $\boldsymbol{H}(t) : \mathscr{O} \to \mathbb{R}^m$ is the effective magnetic field if $m = 3$, or the potential if $m = 1$.
(iii) $\mathcal{R}(\boldsymbol{u})$ is the convective term defined by

$$\mathcal{R}(\boldsymbol{u}) := \beta_1(\boldsymbol{\nu} \cdot \nabla)\boldsymbol{u} + \beta_2 \boldsymbol{u} \times (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{u} + \chi \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}), \tag{2.1}$$

where $\boldsymbol{\nu} : \mathscr{O} \to \mathbb{R}^d$ is the given current density [18, 27, 51] independent of t.
(iv) $\mathcal{S}(\boldsymbol{u})$ consist of other lower-order source term (spin-orbit torque or proliferation term) which grow at most quadratically in $\boldsymbol{u}$, whose properties are detailed in Section 2.2.
(v) $\Psi(\boldsymbol{u})$ is the sum of the exchange field and the Ginzburg–Landau (phase transition) field defined by

$$\Psi(\boldsymbol{u}) := \Delta \boldsymbol{u} + \kappa_1 \boldsymbol{u} - \kappa_2 |\boldsymbol{u}|^2 \boldsymbol{u}, \tag{2.2}$$

where $\kappa_2 > 0$.

Two of the terms in (1.1) are only relevant when $m = 3$, namely:

(i) $\Phi_{\mathrm{a}}(\boldsymbol{u})$ is the anisotropy field with cubic nonlinearities given by

$$\Phi_{\mathrm{a}}(\boldsymbol{u}) = \lambda_1(\boldsymbol{e} \cdot \boldsymbol{u})\boldsymbol{e} - \lambda_2(\boldsymbol{e} \cdot \boldsymbol{u})^3 \boldsymbol{e}, \tag{2.3}$$

where $\lambda_2 \geq 0$ and $\boldsymbol{e} \in \mathbb{R}^3$ is a given unit vector.
(ii) The continuous operator $\Phi_{\mathrm{d}} : \mathbb{L}^2(\mathscr{O}) \to \mathbb{L}^2(\mathbb{R}^3)$ defining the demagnetising field satisfies the static Maxwell-Ampère equations on $\mathbb{R}^3$:

$$\begin{cases} \operatorname{curl} \Phi_{\mathrm{d}}(\boldsymbol{u}) = \boldsymbol{0} & \text{in } \mathbb{R}^3, \\ \operatorname{div}(\Phi_{\mathrm{d}}(\boldsymbol{u}) + \boldsymbol{u}\mathbb{1}_{\mathscr{O}}) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{2.4}$$

where $\boldsymbol{u}\mathbb{1}_{\mathscr{O}} : \mathbb{R}^d \to \mathbb{R}^3$ is an extension of $\boldsymbol{u}$ by zero outside of $\mathscr{O}$, namely

$$\boldsymbol{u}\mathbb{1}_{\mathscr{O}}(\boldsymbol{x}) := \begin{cases} \boldsymbol{u}(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \mathscr{O}, \\ \boldsymbol{0}, & \text{if } \boldsymbol{x} \in \mathbb{R}^3 \setminus \mathscr{O}. \end{cases}$$

We remark that a more general anisotropy field such as the cubic anisotropy field or the uniaxial anisotropy field with spatially-dependent parameter, as well as other zero order contributions to the field (such as a spatially-varying applied field) can be considered without difficulties, but are omitted for simplicity.

The constants β_1, β_2, χ, κ_1, and λ_1 may be positive or negative, but without loss of generality they will be taken as positive throughout the paper. More precisely, according to the Ginzburg–Landau theory, the constant κ_1 is positive for temperatures above the Curie temperature and negative below it.

Assumptions made in this paper are stated in the following:

(1) The constant ε is generally taken to be small (say $\varepsilon \leq 1$), such that $\sigma - (\kappa_1 + \lambda_1)\varepsilon > 0$ for physical reasons and for simplicity of argument. If this is not the case, then the interpolation inequality can be used as in (3.12).
(2) The constant χ in (2.1) is assumed to be sufficiently small, say of order ε, or such that

$$2\chi^2 < \kappa_2 \sigma^2. \tag{2.5}$$

The current density $\boldsymbol{\nu} \in \mathbb{H}^2(\mathscr{O}; \mathbb{R}^d)$ satisfies

$$\|\boldsymbol{\nu}\|^2_{\mathbb{H}^2(\mathscr{O};\mathbb{R}^d))} \leq \nu_\infty, \tag{2.6}$$

for some positive constant ν_∞.
(3) The constant λ_1 in (2.3) can be positive or negative, while $\lambda_2 \geq 0$. More generally, the argument presented here will still hold when $\lambda_2 < 0$ such that $\lambda_2 + \kappa_2 > 0$. Without loss of generality, we assume $\lambda_1, \lambda_2 > 0$. In Section 5.2, we assume $\lambda_2 = 0$.
(4) The source term $\mathcal{S}(\boldsymbol{u})$ satisfies the following assumptions:

$$|\mathcal{S}(\boldsymbol{u})| \leq C|\boldsymbol{u}|(1 + |\boldsymbol{u}|), \tag{2.7}$$

$$\|\nabla \mathcal{S}(\boldsymbol{u})\|_{\mathbb{L}^2} \leq C\left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}\right) \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}, \tag{2.8}$$

$$\|\Delta\mathcal{S}(\boldsymbol{u})\|_{\mathbb{L}^2} \le C\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}\right)\|\Delta\boldsymbol{u}\|_{\mathbb{L}^2}. \tag{2.9}$$

Furthermore, the map $\boldsymbol{u}\mapsto\mathcal{S}(\boldsymbol{u})$ satisfies certain local Lipschitz conditions:

$$\|\mathcal{S}(\boldsymbol{v})-\mathcal{S}(\boldsymbol{w})\|_{\mathbb{L}^2} \le C\left(1+\|\boldsymbol{v}\|_{\mathbb{L}^\infty}+\|\boldsymbol{w}\|_{\mathbb{L}^\infty}\right)\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{L}^2}, \quad \forall \boldsymbol{v},\boldsymbol{w}\in\mathbb{L}^\infty, \tag{2.10}$$

where C is a constant depending only on $|\mathscr{O}|$. An example of such map is $\mathcal{S}(\boldsymbol{u})=\boldsymbol{u}+(\boldsymbol{a}\cdot\boldsymbol{u})\boldsymbol{u}$, where $\boldsymbol{a}$ is a given vector in $\mathbb{R}^3$.

The weak formulation of (1.1) used in this paper is stated below.

Definition 2.1. Given $T>0$ and initial data $\boldsymbol{u}_0\in\mathbb{H}^1$, a function $\boldsymbol{u}\in C([0,T];\mathbb{H}^1)\cap L^2(0,T;\mathbb{H}^3)$ is a weak solution to problem (1.1) if $\boldsymbol{u}(0)=\boldsymbol{u}_0$, and for any $\boldsymbol{\chi}\in\mathbb{H}^1$ and $t\in(0,T)$,

$$\begin{aligned}\langle\partial_t\boldsymbol{u}(t),\boldsymbol{\chi}\rangle_{\mathbb{L}^2} &= \sigma\left\langle\boldsymbol{H}(t)+\Phi_{\mathrm{d}}(\boldsymbol{u}(t)),\boldsymbol{\chi}\right\rangle_{\mathbb{L}^2}+\varepsilon\left\langle\nabla\boldsymbol{H}(t),\nabla\boldsymbol{\chi}\right\rangle_{\mathbb{L}^2}-\varepsilon\left\langle\Delta\Phi_{\mathrm{d}}(\boldsymbol{u}(t)),\boldsymbol{\chi}\right\rangle_{\mathbb{L}^2}\\ &\quad-\gamma\left\langle\boldsymbol{u}(t)\times\big(\boldsymbol{H}(t)+\Phi_{\mathrm{d}}(\boldsymbol{u}(t))\big),\boldsymbol{\chi}\right\rangle_{\mathbb{L}^2}+\left\langle\mathcal{R}(\boldsymbol{u}(t)),\boldsymbol{\chi}\right\rangle_{\mathbb{L}^2}+\left\langle\mathcal{S}(\boldsymbol{u}(t)),\boldsymbol{\chi}\right\rangle_{\mathbb{L}^2},\end{aligned}$$

where

$$\boldsymbol{H}(t)=\Psi(\boldsymbol{u}(t))+\Phi_{\mathrm{a}}(\boldsymbol{u}(t)) \quad \text{in} \quad \mathbb{L}^2. \tag{2.11}$$

A weak solution $\boldsymbol{u}$ is called a strong solution if

$$\boldsymbol{u}\in C([0,T];\mathbb{H}^2)\cap L^2(0,T;\mathbb{H}^4).$$

In this case, $\boldsymbol{u}$ satisfies (1.1) almost everywhere in $(0,T)\times\mathscr{O}$.

2.3. **Auxiliary results.** In this section, we collect some estimates and identities that will be needed for our analysis. For any vector-valued functions $\boldsymbol{v},\boldsymbol{w}:\mathscr{O}\to\mathbb{R}^3$, we have

$$\nabla(|\boldsymbol{v}|^2\boldsymbol{w})=2\boldsymbol{w}\ (\boldsymbol{v}\cdot\nabla\boldsymbol{v})+|\boldsymbol{v}|^2\nabla\boldsymbol{w}, \tag{2.12}$$

$$\frac{\partial\big(|\boldsymbol{v}|^2\boldsymbol{v}\big)}{\partial\boldsymbol{n}}=2\boldsymbol{v}\Big(\boldsymbol{v}\cdot\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}\Big)+|\boldsymbol{v}|^2\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}}, \tag{2.13}$$

$$\Delta(|\boldsymbol{v}|^2\boldsymbol{w})=2|\nabla\boldsymbol{v}|^2\boldsymbol{w}+2(\boldsymbol{v}\cdot\Delta\boldsymbol{v})\boldsymbol{w}+4\nabla\boldsymbol{w}\ (\boldsymbol{v}\cdot\nabla\boldsymbol{v})^\top+|\boldsymbol{v}|^2\Delta\boldsymbol{w}, \tag{2.14}$$

provided that the partial derivatives are well defined. As a consequence of (2.13), (2.11), (2.2), (2.3), and (1.1d), for a sufficiently regular solution $\boldsymbol{u}$ of equation (1.1), $\partial(\Delta\boldsymbol{u})/\partial\boldsymbol{n}=\boldsymbol{0}$ on $\partial\mathscr{O}$.

Lemma 2.2. Let $\epsilon>0$. There exists a positive constant C (depending only on $\mathscr{O}$) such that the following inequalities hold:

(i) for any $\boldsymbol{v}\in\mathbb{L}^2(\mathscr{O})$ such that $\Delta\boldsymbol{v}\in\mathbb{L}^2(\mathscr{O})$ and $\partial\boldsymbol{v}/\partial\boldsymbol{n}=0$ on $\partial\mathscr{O}$,

$$\|\boldsymbol{v}\|_{\mathbb{H}^2}^2\le C\left(\|\boldsymbol{v}\|_{\mathbb{L}^2}^2+\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2\right), \tag{2.15}$$

$$\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}^2\le\frac{1}{4\epsilon}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2+\epsilon\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \tag{2.16}$$

(ii) for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{H}^s$, where $s>d/2$,

$$\|\boldsymbol{v}\odot\boldsymbol{w}\|_{\mathbb{H}^s}\le C\|\boldsymbol{v}\|_{\mathbb{H}^s}\|\boldsymbol{w}\|_{\mathbb{H}^s}, \tag{2.17}$$

$$\|(\boldsymbol{u}\times\boldsymbol{v})\odot\boldsymbol{w}\|_{\mathbb{H}^s}\le C\|\boldsymbol{u}\|_{\mathbb{H}^s}\|\boldsymbol{v}\|_{\mathbb{H}^s}\|\boldsymbol{w}\|_{\mathbb{H}^s}. \tag{2.18}$$

Here $\odot$ denotes either the dot product or cross product in $\mathbb{R}^m$.

Proof. (2.15) and (2.16) are shown in [48, Lemma 3.3], while (2.17) and (2.18) follow from [8]. ☐

We show some estimates for the map Φ_{a}, given in (2.3), which defines the anisotropy field in the following lemma.

Lemma 2.3. Let Φ_{a} be as defined in (2.3), and let $p,q\in[1,\infty]$ be such that $2/p+1/q=1/2$.

(1) For any $\boldsymbol{v},\boldsymbol{w}\in\mathbb{L}^2$,

$$\langle\Phi_{\mathrm{a}}(\boldsymbol{v})-\Phi_{\mathrm{a}}(\boldsymbol{w}),\boldsymbol{v}-\boldsymbol{w}\rangle_{\mathbb{L}^2}\le\lambda_1\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{L}^2}^2. \tag{2.19}$$

(2) For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{L}^{\max\{p,q\}}$,

$$\|\Phi_{\mathrm{a}}(\boldsymbol{v}) - \Phi_{\mathrm{a}}(\boldsymbol{w})\|_{\mathbb{L}^2}^2 \le C\left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^p}^4 + \|\boldsymbol{w}\|_{\mathbb{L}^p}^4\right) \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^q}^2 . \tag{2.20}$$

(3) For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{W}^{1,p} \cap \mathbb{L}^\infty$,

$$\begin{aligned}\|\Phi_{\mathrm{a}}(\boldsymbol{v}) - \Phi_{\mathrm{a}}(\boldsymbol{w})\|_{\mathbb{H}^1}^2 &\le C\left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{w}\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{H}^1}^2 \\ &\quad + C\left(\|\boldsymbol{v}\|_{\mathbb{L}^p}^2 + \|\boldsymbol{w}\|_{\mathbb{L}^p}^2\right)\left(\|\boldsymbol{v}\|_{\mathbb{W}^{1,p}}^2 + \|\boldsymbol{w}\|_{\mathbb{W}^{1,p}}^2\right) \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^q}^2 .\end{aligned} \tag{2.21}$$

(4) For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{H}^3$,

$$\begin{aligned}\|\Delta\Phi_{\mathrm{a}}(\boldsymbol{v}) - \Delta\Phi_{\mathrm{a}}(\boldsymbol{w})\|_{\mathbb{L}^2}^2 &\le C\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^2}^2 + C\left(\|\boldsymbol{v}\|_{\mathbb{H}^1}^4 + \|\boldsymbol{w}\|_{\mathbb{H}^1}^4\right) \|\Delta\boldsymbol{v} - \Delta\boldsymbol{w}\|_{\mathbb{H}^1}^2 \\ &\quad + C\left(\|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \|\boldsymbol{w}\|_{\mathbb{H}^1}^2\right)\left(\|\boldsymbol{v}\|_{\mathbb{H}^3}^2 + \|\boldsymbol{w}\|_{\mathbb{H}^3}^2\right) \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{H}^1}^2 .\end{aligned} \tag{2.22}$$

(5) For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{H}^s$, where $s \ge 2$,

$$\|\Phi_{\mathrm{a}}(\boldsymbol{v})\|_{\mathbb{H}^s}^2 \le C\left(1 + \|\boldsymbol{v}\|_{\mathbb{H}^s}^6\right). \tag{2.23}$$

Proof. It follows from the definition of Φ_{a} that

$$\langle \Phi_{\mathrm{a}}(\boldsymbol{v}) - \Phi_{\mathrm{a}}(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w}\rangle_{\mathbb{L}^2} = \lambda_1 \|\boldsymbol{e}\cdot(\boldsymbol{v} - \boldsymbol{w})\|_{\mathbb{L}^2}^2 - \lambda_2 \left\langle (\boldsymbol{e}\cdot\boldsymbol{v})^3\boldsymbol{e} - (\boldsymbol{e}\cdot\boldsymbol{w})^3\boldsymbol{e}, \boldsymbol{v} - \boldsymbol{w}\right\rangle_{\mathbb{L}^2} . \tag{2.24}$$

Note that

$$\left\langle (\boldsymbol{e}\cdot\boldsymbol{v})^3\boldsymbol{e} - (\boldsymbol{e}\cdot\boldsymbol{w})^3\boldsymbol{e}, \boldsymbol{v} - \boldsymbol{w}\right\rangle_{\mathbb{L}^2} = (\boldsymbol{e}\cdot\boldsymbol{v})^4 + (\boldsymbol{e}\cdot\boldsymbol{w})^4 - (\boldsymbol{e}\cdot\boldsymbol{v})^3(\boldsymbol{e}\cdot\boldsymbol{w}) - (\boldsymbol{e}\cdot\boldsymbol{w})^3(\boldsymbol{e}\cdot\boldsymbol{v}). \tag{2.25}$$

By Young's inequality,

$$|(\boldsymbol{e}\cdot\boldsymbol{v})^3(\boldsymbol{e}\cdot\boldsymbol{w})| \le \frac{(\boldsymbol{e}\cdot\boldsymbol{v})^4}{4/3} + \frac{(\boldsymbol{e}\cdot\boldsymbol{w})^4}{4} \quad \text{and} \quad |(\boldsymbol{e}\cdot\boldsymbol{w})^3(\boldsymbol{e}\cdot\boldsymbol{v})| \le \frac{(\boldsymbol{e}\cdot\boldsymbol{w})^4}{4/3} + \frac{(\boldsymbol{e}\cdot\boldsymbol{v})^4}{4},$$

and thus from (2.25),

$$\left\langle (\boldsymbol{e}\cdot\boldsymbol{v})^3\boldsymbol{e} - (\boldsymbol{e}\cdot\boldsymbol{w})^3\boldsymbol{e}, \boldsymbol{v} - \boldsymbol{w}\right\rangle_{\mathbb{L}^2} \ge 0.$$

This, together with (2.24), implies (2.19).

Next, using the elementary identity

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2), \quad \forall a, b \in \mathbb{R}, \tag{2.26}$$

we have

$$\begin{aligned}\|\Phi_{\mathrm{a}}(\boldsymbol{v}) - \Phi_{\mathrm{a}}(\boldsymbol{w})\|_{\mathbb{L}^2}^2 &\le 2\lambda_1^2 \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^2}^2 + 2\lambda_2^2 \left\|\big(\boldsymbol{e}\cdot(\boldsymbol{v} - \boldsymbol{w})\big)\big((\boldsymbol{e}\cdot\boldsymbol{v})^2 + (\boldsymbol{e}\cdot\boldsymbol{v})(\boldsymbol{e}\cdot\boldsymbol{w}) + (\boldsymbol{e}\cdot\boldsymbol{w})^2\big)\right\|_{\mathbb{L}^2}^2 \\ &\le 2\lambda_1^2 \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^2}^2 + 4\lambda_2^2\left(\|\boldsymbol{v}\|_{\mathbb{L}^p}^4 + \|\boldsymbol{w}\|_{\mathbb{L}^p}^4\right)\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^q}^2 ,\end{aligned}$$

which implies (2.20).

Similarly, by the product rule for derivatives, Hölder's and Young's inequalities,

$$\begin{aligned}\|\Phi_{\mathrm{a}}(\boldsymbol{v}) - \Phi_{\mathrm{a}}(\boldsymbol{w})\|_{\mathbb{H}^1}^2 &\le 2\lambda_1^2 \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{H}^1}^2 + 2\lambda_2^2 \left\|\big(\boldsymbol{e}\cdot(\boldsymbol{v} - \boldsymbol{w})\big)\big((\boldsymbol{e}\cdot\boldsymbol{v})^2 + (\boldsymbol{e}\cdot\boldsymbol{v})(\boldsymbol{e}\cdot\boldsymbol{w}) + (\boldsymbol{e}\cdot\boldsymbol{w})^2\big)\right\|_{\mathbb{H}^1}^2 \\ &\le 2\lambda_1^2 \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{H}^1}^2 + 4\lambda_2^2\left(\|\boldsymbol{v}\|_{\mathbb{L}^p}^4 + \|\boldsymbol{w}\|_{\mathbb{L}^p}^4\right)\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^q}^2 \\ &\quad + 4\lambda_2^2\left(\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{w}\|_{\mathbb{L}^\infty}^4\right)\|\nabla\boldsymbol{v} - \nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2 \\ &\quad + 4\lambda_2^2\left(\|\boldsymbol{v}\|_{\mathbb{L}^p}^2 + \|\boldsymbol{w}\|_{\mathbb{L}^p}^2\right)\left(\|\nabla\boldsymbol{v}\|_{\mathbb{L}^p}^2 + \|\nabla\boldsymbol{w}\|_{\mathbb{L}^p}^2\right)\|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^q}^2 ,\end{aligned}$$

which yields (2.21).

Next, we aim to show (2.22). Note that by the identity (2.26), writing $a := \boldsymbol{e}\cdot\boldsymbol{u}$ and $b := \boldsymbol{e}\cdot\boldsymbol{v}$ and $\rho := a - b$, we have

$$\begin{aligned}\Delta(a^3) - \Delta(b^3) &= (\Delta\rho)(a^2 + ab + b^2) + \rho(2a\Delta a + |\nabla a|^2 + a\Delta b + b\Delta a + 2\nabla a\cdot\nabla b + 2b\Delta b + |\nabla b|^2) \\ &\quad + 2\nabla\rho\cdot(2a\nabla a + a\nabla b + b\nabla a + 2b\nabla b).\end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
&\left\|\Delta(a^3)-\Delta(b^3)\right\|_{L^2}^2 \\
&\leq 2\left\|\Delta\rho\right\|_{L^6}^2\left(\|a\|_{L^6}^4+\|b\|_{L^6}^4\right)+2\|\rho\|_{L^6}^2\left(\|a\|_{L^6}^2+\|b\|_{L^6}^2\right)\left(\|\Delta a\|_{L^6}^2+\|\Delta b\|_{L^6}^2\right) \\
&\quad+2\|\rho\|_{L^6}^2\left(\|\nabla a\|_{L^6}^4+\|\nabla b\|_{L^6}^4\right)+8\|\nabla\rho\|_{L^2}^2\left(\|a\|_{L^\infty}^2+\|b\|_{L^\infty}^2\right)\left(\|\nabla a\|_{L^\infty}^2+\|\nabla b\|_{L^\infty}^2\right). \qquad (2.27)
\end{aligned}$$

Note that by the Gagliardo–Nirenberg and interpolation inequalities, we have for any function $f\in H^3(\mathscr{O})$,

$$\begin{aligned}
\|f\|_{L^\infty}^2 &\leq C\|f\|_{H^1}\|f\|_{H^2}\leq C\|f\|_{H^1}^{3/2}\|f\|_{H^3}^{1/2}, \\
\|\nabla f\|_{L^\infty}^2 &\leq C\|\nabla f\|_{H^1}\|\nabla f\|_{H^2}\leq C\|f\|_{H^1}^{1/2}\|f\|_{H^3}^{3/2}, \\
\|\nabla f\|_{L^6}^4 &\leq C\|f\|_{H^2}^4\leq C\|f\|_{H^1}^2\|f\|_{H^3}^2.
\end{aligned}$$

Using these inequalities in (2.27) and applying the Sobolev embedding $H^1\hookrightarrow L^6$, we obtain

$$\begin{aligned}
&\left\|\Delta(a^3)-\Delta(b^3)\right\|_{L^2}^2 \\
&\leq C\left(\|a\|_{H^1}^4+\|b\|_{H^1}^4\right)\|\Delta\rho\|_{H^1}^2+C\left(\|a\|_{H^1}^2+\|b\|_{H^1}^2\right)\left(\|a\|_{H^3}^2+\|b\|_{H^3}^2\right)\|\rho\|_{H^1}^2 \\
&\quad+C\left(\|a\|_{H^1}^{3/2}\|a\|_{H^3}^{1/2}+\|b\|_{H^1}^{3/2}\|b\|_{H^3}^{1/2}\right)\left(\|a\|_{H^1}^{1/2}\|a\|_{H^3}^{3/2}+\|b\|_{H^1}^{1/2}\|b\|_{H^3}^{3/2}\right)\|\nabla\rho\|_{L^2}^2 \\
&\leq C\left(\|a\|_{H^1}^4+\|b\|_{H^1}^4\right)\|\Delta\rho\|_{H^1}^2+C\left(\|a\|_{H^1}^2+\|b\|_{H^1}^2\right)\left(\|a\|_{H^3}^2+\|b\|_{H^3}^2\right)\|\rho\|_{H^1}^2 \\
&\leq C\left(\|\boldsymbol{v}\|_{\mathbb{H}^1}^4+\|\boldsymbol{w}\|_{\mathbb{H}^1}^4\right)\|\Delta\boldsymbol{v}-\Delta\boldsymbol{w}\|_{\mathbb{H}^1}^2+C\left(\|\boldsymbol{v}\|_{\mathbb{H}^1}^2+\|\boldsymbol{w}\|_{\mathbb{H}^1}^2\right)\left(\|\boldsymbol{v}\|_{\mathbb{H}^3}^2+\|\boldsymbol{w}\|_{\mathbb{H}^3}^2\right)\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{H}^1}^2. \qquad (2.28)
\end{aligned}$$

where in the last step we used Young's inequality. Hence, by the triangle inequality,

$$\begin{aligned}
\|\Delta\Phi_{\mathrm{a}}(\boldsymbol{v})-\Delta\Phi_{\mathrm{a}}(\boldsymbol{w})\|_{\mathbb{L}^2}^2 &\leq \lambda_1^2\|\rho\boldsymbol{e}\|_{\mathbb{L}^2}^2+\lambda_2^2\left\|\left(\Delta(a^3)-\Delta(b^3)\right)\boldsymbol{e}\right\|_{\mathbb{L}^2}^2 \\
&\leq \lambda_1^2\|\boldsymbol{u}-\boldsymbol{v}\|_{\mathbb{L}^2}^2+\lambda_2^2\left\|\Delta(a^3)-\Delta(b^3)\right\|_{L^2}^2,
\end{aligned}$$

and thus the required inequality (2.22) follows from (2.28).

Finally, inequality (2.23) follows by applying (2.18) to the definition of $\Phi_{\mathrm{a}}(\boldsymbol{v})$. ☐

Next, we prove several estimates related to the map $\mathcal{R}$ (which defines the spin torque term, see (2.1)) in the following lemmas.

Lemma 2.4. Let $\boldsymbol{\nu}\in\mathbb{H}^2(\mathscr{O};\mathbb{R}^d)$ be given, satisfying assumption (2.6). Let $\mathcal{R}$ be the map defined by (2.1). For any $\epsilon,\sigma>0$, there exists a positive constant C_ϵ such that for any $\boldsymbol{v},\boldsymbol{w}\in\mathbb{H}^1\cap\mathbb{L}^\infty$,

$$\left|\langle\mathcal{R}(\boldsymbol{v}),\boldsymbol{v}\rangle_{\mathbb{L}^2}\right|\leq C_\epsilon\nu_\infty\|\boldsymbol{v}\|_{\mathbb{L}^2}^2+\chi^2\sigma^{-1}\|\boldsymbol{v}\|_{\mathbb{L}^4}^4+\epsilon\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}^2+\frac{\sigma}{4}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}^2, \qquad (2.29)$$

$$\left|\langle\mathcal{R}(\boldsymbol{v}),\boldsymbol{w}\rangle_{\mathbb{L}^2}\right|\leq C_\epsilon\nu_\infty\left(\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}^2+\||\boldsymbol{v}|\,|\nabla\boldsymbol{v}|\|_{\mathbb{L}^2}^2\right)+\epsilon\|\boldsymbol{w}\|_{\mathbb{L}^2}^2, \qquad (2.30)$$

$$\left|\langle\mathcal{R}(\boldsymbol{v})-\mathcal{R}(\boldsymbol{w}),\boldsymbol{v}-\boldsymbol{w}\rangle_{\mathbb{L}^2}\right|\leq C_\epsilon\nu_\infty\left(1+\|\boldsymbol{w}\|_{\mathbb{L}^\infty}^2\right)\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{L}^2}^2+\epsilon\|\nabla\boldsymbol{v}-\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2, \qquad (2.31)$$

where ν_∞ was defined in (2.6). Furthermore, for any $\boldsymbol{v},\boldsymbol{w}\in\mathbb{H}^{2k}$, where $k\in\mathbb{N}$,

$$\left|\left\langle\mathcal{R}(\boldsymbol{v})-\mathcal{R}(\boldsymbol{w}),\Delta^k\boldsymbol{v}-\Delta^k\boldsymbol{w}\right\rangle_{\mathbb{L}^2}\right|\leq C_\epsilon\nu_\infty\left(1+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2+\|\boldsymbol{w}\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{H}^1}^2+\epsilon\left\|\Delta^k\boldsymbol{v}-\Delta^k\boldsymbol{w}\right\|_{\mathbb{L}^2}^2. \qquad (2.32)$$

Proof. These can be shown in the same manner as in [26, Lemma 2.3]. ☐

Lemma 2.5. Let $\boldsymbol{\nu}\in\mathbb{H}^2(\mathscr{O};\mathbb{R}^d)$ be given, satisfying assumption (2.6). There exists a positive constant C such that for sufficiently regular $\boldsymbol{v}$,

$$\|\mathcal{R}(\boldsymbol{v})\|_{\mathbb{H}^1}^2\leq C\nu_\infty\left(1+\|\boldsymbol{v}\|_{\mathbb{H}^1}^4+\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^4\right), \qquad (2.33)$$

$$\|\mathcal{R}(\boldsymbol{v})\|_{\mathbb{H}^2}^2\leq C\nu_\infty\left(1+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{v}\|_{\mathbb{H}^3}^2. \qquad (2.34)$$

Proof. We first show (2.33). By Hölder's inequality,

$$\begin{aligned}\|\mathcal{R}(\boldsymbol{v})\|_{\mathbb{L}^2}^2 &\le C\,\|\boldsymbol{\nu}\|_{\mathbb{L}^4(\mathscr{O};\mathbb{R}^d)}^2 \|\nabla \boldsymbol{v}\|_{\mathbb{L}^4}^2 + C\,\|\boldsymbol{v}\|_{\mathbb{L}^6}^2 \|\boldsymbol{\nu}\|_{\mathbb{L}^6(\mathscr{O};\mathbb{R}^d)}^2 \|\nabla \boldsymbol{v}\|_{\mathbb{L}^6}^2 \\ &\le C\nu_\infty \left(1+\|\boldsymbol{v}\|_{\mathbb{H}^1}^2\right) \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2 . \end{aligned} \tag{2.35}$$

Next, by Hölder's and Young's inequalities, and the Sobolev embedding, we have

$$\begin{aligned}\|\nabla\mathcal{R}(\boldsymbol{v})\|_{\mathbb{L}^2}^2 &\le \beta_1^2\big(\,\|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{O};\mathbb{R}^d)}^2 \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 + \|\nabla\boldsymbol{\nu}\|_{\mathbb{L}^3(\mathscr{O};\mathbb{R}^d)}^2 \|\nabla \boldsymbol{v}\|_{\mathbb{L}^6}^2\,\big) \\ &\quad + (\beta_2+\chi)^2 \left(\|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{O};\mathbb{R}^d)}^2 \|\nabla \boldsymbol{v}\|_{\mathbb{L}^4}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^6}^2 \|\nabla\boldsymbol{\nu}\|_{\mathbb{L}^6}^2 \|\nabla \boldsymbol{v}\|_{\mathbb{L}^6}^2 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{O};\mathbb{R}^d)}^2 \|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right) \\ &\le C\nu_\infty\left(1+\|\boldsymbol{v}\|_{\mathbb{H}^1}^4 + \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^4\right), \end{aligned} \tag{2.36}$$

Adding (2.35) and (2.36) gives (2.33).

Finally, inequality (2.34) follows by applying (2.18) to the definition of $\mathcal{R}(\boldsymbol{v})$. □

Further estimates for the map $\mathcal{R}$ in the case $d \le 2$ are stated below. These will be needed in Section 5.2.

Lemma 2.6. Let $d\le 2$ and $\boldsymbol{\nu} \in \mathbb{H}^2(\mathscr{O};\mathbb{R}^d)$ be given, satisfying assumption (2.6). Let $\mathcal{R}$ be the map defined by (2.1). For any $\epsilon>0$, there exists a positive constant C_ϵ such that for sufficiently regular $\boldsymbol{v}$,

$$\big|\,\langle \mathcal{R}(\boldsymbol{v}), \Delta \boldsymbol{v}\rangle_{\mathbb{L}^2}\big| \le C_\epsilon \nu_\infty \left(1+\|\boldsymbol{v}\|_{\mathbb{L}^4}^4\right)\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon\,\|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2, \tag{2.37}$$

$$\big|\,\langle \nabla\mathcal{R}(\boldsymbol{v}), \nabla\Delta \boldsymbol{v}\rangle_{\mathbb{L}^2}\big| \le C_\epsilon \nu_\infty \left(1+\|\boldsymbol{v}\|_{\mathbb{H}^1}^4 + \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^4\right) + \epsilon\,\|\nabla\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2, \tag{2.38}$$

$$\big|\,\langle \Delta\mathcal{R}(\boldsymbol{v}), \Delta^2 \boldsymbol{v}\rangle_{\mathbb{L}^2}\big| \le C_\epsilon \nu_\infty \left(1+\|\boldsymbol{v}\|_{\mathbb{H}^2}^4\right) + C_\epsilon\nu_\infty\left(1+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right)\|\nabla\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon\,\big\|\Delta^2 \boldsymbol{v}\big\|_{\mathbb{L}^2}^2, \tag{2.39}$$

where ν_∞ was defined in (2.6). Furthermore, for sufficiently regular $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$\begin{aligned}\big|\,\langle \nabla\mathcal{R}(\boldsymbol{v}) - \nabla\mathcal{R}(\boldsymbol{w}), \nabla\Delta \boldsymbol{v} - \nabla\Delta\boldsymbol{w}\rangle_{\mathbb{L}^2}\big| &\le C_\epsilon\nu_\infty\left(1+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2 + \|\boldsymbol{w}\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{H}^2}^2 \\ &\quad + \epsilon\,\|\nabla\Delta\boldsymbol{v} - \nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2 . \end{aligned} \tag{2.40}$$

Proof. Firstly, by Hölder's and Young's inequalities, and the Gagliardo–Nirenberg inequalities (for $d\le 2$) we obtain

$$\begin{aligned}\big|\,\langle \mathcal{R}(\boldsymbol{v}), \Delta \boldsymbol{v}\rangle_{\mathbb{L}^2}\big| &\le \beta_1\,\|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{O};\mathbb{R}^d)} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2} \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2} + \left(\beta_2\,\|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{O};\mathbb{R}^d)} + \chi\right)\left(\|\boldsymbol{v}\|_{\mathbb{L}^4}\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^{1/2}\|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^{3/2}\right) \\ &\le C_\epsilon\nu_\infty\left(1+\|\boldsymbol{v}\|_{\mathbb{L}^4}^4\right)\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon\,\|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2, \end{aligned}$$

showing (2.37). Next, (2.38) follows from (2.33) and Young's inequality. Similarly, the estimate (2.39) can be deduced from (2.34) and Young's inequality. The inequality (2.40) can be shown in a similar manner as (2.32) and (2.38). This completes the proof of the lemma. □

Next, estimates on $\mathcal{S}(\boldsymbol{u})$ are stated in the following lemma.

Lemma 2.7. Let $\mathcal{S}$ be the map satisfying (2.7) and $k\ge 0$. For any $\epsilon>0$, there exists a positive constant C_ϵ such that for sufficiently regular $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$\big|\,\langle \mathcal{S}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\mathbb{L}^2}\big| \le C_\epsilon\left(\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \|\boldsymbol{v}\|_{\mathbb{L}^4}^4\right) + \epsilon\,\|\boldsymbol{w}\|_{\mathbb{L}^2}^2, \tag{2.41}$$

$$\big|\,\langle \nabla\mathcal{S}(\boldsymbol{v}), \nabla\boldsymbol{w}\rangle_{\mathbb{L}^2}\big| \le C_\epsilon\left(\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \||\boldsymbol{v}||\nabla\boldsymbol{v}|\|_{\mathbb{L}^2}^2\right) + \epsilon\,\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2, \tag{2.42}$$

$$\big|\,\langle \Delta\mathcal{S}(\boldsymbol{v}), \Delta^2\boldsymbol{w}\rangle_{\mathbb{L}^2}\big| \le C_\epsilon\,\|\nabla\boldsymbol{v}\|_{\mathbb{L}^4}^4 + C_\epsilon\left(1+\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2\right)\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon\,\big\|\Delta^2\boldsymbol{w}\big\|_{\mathbb{L}^2}^2, \tag{2.43}$$

$$\left|\,\left\langle \mathcal{S}(\boldsymbol{v}) - \mathcal{S}(\boldsymbol{w}), \Delta^k\boldsymbol{v} - \Delta^k\boldsymbol{w}\right\rangle_{\mathbb{L}^2}\right| \le C_\epsilon\left(1+\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{w}\|_{\mathbb{L}^\infty}^2\right)\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \epsilon\left\|\Delta^k\boldsymbol{v} - \Delta^k\boldsymbol{w}\right\|_{\mathbb{L}^2}^2, \tag{2.44}$$

$$\left|\,\langle \nabla\mathcal{S}(\boldsymbol{v}) - \nabla\mathcal{S}(\boldsymbol{w}), \nabla\Delta\boldsymbol{v} - \nabla\Delta\boldsymbol{w}\rangle_{\mathbb{L}^2}\right| \le C_\epsilon\left(1+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2 + \|\boldsymbol{w}\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{v}-\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \epsilon\,\|\nabla\Delta\boldsymbol{v} - \nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2 . \tag{2.45}$$

Proof. These inequalities follow from (2.7), (2.8), (2.9), (2.10), and Young's inequality. Inequality (2.45) can be shown in a similar manner as (2.40). □

Some properties of the operator Φ_{d} defining the demagnetising field are recalled below (see also [42]).

Theorem 2.8. The solution to (2.4) can be written as

$$\Phi_{\mathrm{d}}(\boldsymbol{v}) = -\nabla\big(G * \operatorname{div}(\boldsymbol{v})\big), \tag{2.46}$$

where G is the fundamental solution of the Laplace operator and $*$ denotes the convolution operator. Furthermore, the following statements hold

(1) For any $\boldsymbol{v} \in \mathbb{H}^s(\mathscr{O})$, where $s \geq 0$, we have

$$\|\Phi_{\mathrm{d}}(\boldsymbol{v})\|_{\mathbb{H}^s(\mathbb{R}^3)} \leq \|\boldsymbol{v}\|_{\mathbb{H}^s(\mathscr{O})}\,. \tag{2.47}$$

(2) If $\boldsymbol{v} \in \mathbb{W}^{k,p}(\mathscr{O})$ for some $p \in (1,\infty)$ and $k \in \mathbb{N}$, then the restriction of $\Phi_{\mathrm{d}}(\boldsymbol{v})$ to $\mathscr{O}$ belongs to $\mathbb{W}^{k,p}(\mathscr{O})$ and satisfies

$$\|\Phi_{\mathrm{d}}(\boldsymbol{v})\|_{\mathbb{W}^{k,p}(\mathscr{O})} \leq C\,\|\boldsymbol{v}\|_{\mathbb{W}^{k,p}(\mathscr{O})}\,, \tag{2.48}$$

where the positive constant C is independent of $\boldsymbol{v}$.

Proof. Refer to [44, Section 2.5], [9, Lemma 2.3] and [14, Lemma 3.1]. □

2.4. **Faedo–Galerkin method.** The Faedo–Galerkin approximation will be used to establish the existence of solution to (1.1). Let $\{\boldsymbol{e}_i\}_{i=1}^\infty$ denote an orthonormal basis of $\mathbb{L}^2$ consisting of eigenfunctions of $-\Delta$ such that

$$-\Delta \boldsymbol{e}_i = \mu_i \boldsymbol{e}_i \;\text{ in } \mathscr{O} \quad\text{and}\quad \frac{\partial \boldsymbol{e}_i}{\partial \boldsymbol{n}} = \boldsymbol{0} \;\text{ on } \partial\mathscr{O}, \quad \forall i \in \mathbb{N},$$

where μ_i are the eigenvalues of $-\Delta$ associated with $\boldsymbol{e}_i$.

Let $\mathbb{V}_n := \operatorname{span}\{\boldsymbol{e_1},\dots,\boldsymbol{e_n}\}$ and $\Pi_n : \mathbb{L}^2 \to \mathbb{V}_n$ be the orthogonal projection defined by

$$\langle \Pi_n \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2} = \langle \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2}\,, \quad \forall \boldsymbol{\phi} \in \mathbb{V}_n,\; \boldsymbol{v} \in \mathbb{L}^2.$$

Note that Π_n is self-adjoint and satisfies

$$\begin{aligned} \|\Pi_n \boldsymbol{v}\|_{\mathbb{L}^2} &\leq \|\boldsymbol{v}\|_{\mathbb{L}^2}\,, \quad \forall \boldsymbol{v} \in \mathbb{L}^2, \\ \|\nabla \Pi_n \boldsymbol{v}\|_{\mathbb{L}^2} &\leq \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}\,, \quad \forall \boldsymbol{v} \in \mathbb{H}^1. \end{aligned}$$

Also,

$$\langle \Pi_n \Delta \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2} = \langle \Delta \Pi_n \boldsymbol{v}, \boldsymbol{\phi}\rangle_{\mathbb{L}^2}\,, \quad \forall \boldsymbol{\phi} \in \mathbb{V}_n,\; \boldsymbol{v} \in \mathrm{D}(\Delta).$$

The Faedo–Galerkin method seeks to approximate the solution to (1.1) by $\boldsymbol{u}_n(t) \in \mathbb{V}_n$ satisfying the equation

$$\begin{cases} \partial_t \boldsymbol{u}_n = \sigma\big(\boldsymbol{H}_n + \Pi_n \Phi_{\mathrm{d}}(\boldsymbol{u}_n)\big) - \varepsilon \Delta \boldsymbol{H}_n - \varepsilon \Pi_n \Delta \Phi_{\mathrm{d}}(\boldsymbol{u}_n) & \\ \qquad\qquad - \gamma \Pi_n\big(\boldsymbol{u}_n \times (\boldsymbol{H}_n + \Phi_{\mathrm{d}}(\boldsymbol{u}_n))\big) + \Pi_n \mathcal{R}(\boldsymbol{u}_n) + \Pi_n \mathcal{S}(\boldsymbol{u}_n) & \text{in } (0,T)\times\mathscr{O}, \\ \boldsymbol{H}_n = \Pi_n\big(\Psi(\boldsymbol{u}_n)\big) + \Phi_{\mathrm{a}}(\boldsymbol{u}_n)\big) & \text{in } (0,T)\times\mathscr{O}, \\ \boldsymbol{u}_n(0) = \boldsymbol{u}_{0n} & \text{in } \mathscr{O}, \end{cases} \tag{2.49}$$

where the maps $\mathcal{R}$, $\mathcal{S}$, Ψ, Φ_{a}, and Φ_{d} are specified in Section 2.2, and $\boldsymbol{u}_{0n} := \Pi_n \boldsymbol{u}_0 \in \mathbb{V}_n$.

The existence of solutions to the above ordinary differential equation in $\mathbb{V}_n$ defined on the interval $(0,t_n) \subseteq (0,T)$ is guaranteed by the Cauchy–Lipschitz theorem. In the next section, we will prove several a priori estimates, which are used to ensure the Faedo–Galerkin solutions $(\boldsymbol{u}_n, \boldsymbol{H}_n)$ can be continued globally to $(0,+\infty)$ for any initial data $\boldsymbol{u}_{0n} \in \mathbb{V}_n$.

3. Uniform estimates

In the following, we will derive various estimates on $\boldsymbol{u}_n$ and $\boldsymbol{H}_n$ which are uniform in n to show global existence and uniqueness of solution to (1.1), as well as the existence of an absorbing set. Several types of bounds are proved in this section, namely for $k=0,1,2,3$, we derive:

(1) bounds for $\|\boldsymbol{u}_n\|_{L^\infty(0,T;\mathbb{H}^k)}$ which depend on $\|\boldsymbol{u}_0\|_{\mathbb{H}^k}$, and bounds for $\|\boldsymbol{u}_n\|_{L^2(0,T;\mathbb{H}^{k+2})}$ which depend on $\|\boldsymbol{u}_0\|_{\mathbb{H}^k}$ and T,
(2) for sufficiently large $t_k := t_k(\boldsymbol{u}_0)$, bounds for $\|\boldsymbol{u}_n\|_{L^\infty(t_k,\infty;\mathbb{H}^k)}$ which are independent of $\boldsymbol{u}_0$ and t,
(3) bounds for $\|\boldsymbol{u}_n\|_{L^\infty(0,\infty;\mathbb{H}^k)}$ which depend on $\|\boldsymbol{u}_0\|_{\mathbb{H}^{k-1}}$, but are independent of $\|\boldsymbol{u}_0\|_{\mathbb{H}^k}$.

Corresponding estimates for $\boldsymbol{H}_n$ will also be shown. These will be essential in the proof of existence of attractors in the next section. For ease of presentation, we often omit the dependence of the functions on t in the proof of these estimates. For some of the estimates, we highlight the dependence of the constant on ε as this will be used subsequently.

Proposition 3.1. For any $n\in\mathbb{N}$ and $t\geq 0$, the following bounds hold:

$$\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s + \int_0^t \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \varepsilon \int_0^t \|\Delta \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \varepsilon \int_0^t \||\boldsymbol{u}_n(s)||\nabla \boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \leq C(1+t)\|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2, \tag{3.1}$$

where C is independent of ε and $\boldsymbol{u}_0$, and

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 \leq M, \tag{3.2}$$

where M depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of ε and t.

Furthermore, there exists t_0 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that for all $t\geq t_0$,

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 + \int_t^{t+1} \Big(\|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4 + \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon \||\boldsymbol{u}_n(s)||\nabla \boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2 \Big) \mathrm{d}s \leq \rho_0, \tag{3.3}$$

where the constant ρ_0 is independent of $\boldsymbol{u}_0$, ε, and t.

Finally, for any $t\geq 0$ and $\delta>0$,

$$\int_t^{t+\delta} \Big(\|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4 + \|\nabla \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon \||\boldsymbol{u}_n(s)||\nabla \boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2 \Big) \mathrm{d}s \leq \mu_0, \tag{3.4}$$

where the constant μ_0 depends only on M and δ.

Proof. Taking the inner product of the first equation in (2.49) with $\boldsymbol{u}_n$ and integrating by parts give

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &= \sigma\langle \boldsymbol{H}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \sigma\langle \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \varepsilon\langle \nabla \boldsymbol{H}_n, \nabla \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \varepsilon\langle \Delta\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &\quad + \langle \mathcal{R}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{S}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2}\end{aligned} \tag{3.5}$$

Taking the inner product of the second equation in (2.49) with $\boldsymbol{u}_n$ and $\Delta\boldsymbol{u}_n$, successively, give

$$\sigma\langle \boldsymbol{H}_n, \boldsymbol{u}_n\rangle_{\mathbb{L}^2} = -\sigma\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \kappa_1\sigma\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \kappa_2\sigma\|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 + \lambda_1\|\boldsymbol{e}\cdot\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \lambda_2\|\boldsymbol{e}\cdot\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \tag{3.6}$$

$$\begin{aligned}\varepsilon\langle \nabla\boldsymbol{H}_n, \nabla\boldsymbol{u}_n\rangle_{\mathbb{L}^2} &= -\varepsilon\|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_1\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \varepsilon\kappa_2\||\boldsymbol{u}_n||\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 - 2\varepsilon\kappa_2\|\boldsymbol{u}_n\cdot\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \\ &\quad + \varepsilon\lambda_1\|\boldsymbol{e}\cdot\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - 3\varepsilon\lambda_2\|(\boldsymbol{e}\cdot\boldsymbol{u}_n)(\boldsymbol{e}\cdot\nabla\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2 .\end{aligned} \tag{3.7}$$

Adding equations (3.5), (3.6), and (3.7), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon\|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \sigma\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_2\||\boldsymbol{u}_n||\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 + 2\varepsilon\kappa_2\|\boldsymbol{u}_n\cdot\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \kappa_2\sigma\|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \\ &= \sigma\langle \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \varepsilon\langle \Delta\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{R}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \kappa_1\sigma\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_1\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \\ &\quad + \lambda_1\|\boldsymbol{e}\cdot\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \lambda_2\|\boldsymbol{e}\cdot\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 + \varepsilon\lambda_1\|\boldsymbol{e}\cdot\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - 3\varepsilon\lambda_2\|(\boldsymbol{e}\cdot\boldsymbol{u}_n)(\boldsymbol{e}\cdot\nabla\boldsymbol{u}_n)\|_{\mathbb{L}^2}^2 + \langle \mathcal{S}(\boldsymbol{u}_n), \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &= I_1 + I_2 + \cdots + I_{10}.\end{aligned} \tag{3.8}$$

For the first and the second term, we apply Hölder's inequality and (2.47) to obtain

$$|I_1| \le \sigma \|\Phi_{\rm d}(\boldsymbol{u}_n)\|_{\mathbb{L}^2} \|\boldsymbol{u}_n\|_{\mathbb{L}^2} \le C \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2, \tag{3.9}$$

$$|I_2| \le \varepsilon \|\Delta\Phi_{\rm d}(\boldsymbol{u}_n)\|_{\mathbb{L}^2} \|\boldsymbol{u}_n\|_{\mathbb{L}^2} \le C\varepsilon \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{4} \|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2. \tag{3.10}$$

For the term I_3, by (2.29),

$$|I_3| \le C\nu_\infty \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \chi^2\sigma^{-1} \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 + \frac{\sigma}{4} \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2. \tag{3.11}$$

For the terms I_5 and I_8, by Young's inequality and interpolation inequality (2.16), we have

$$|I_5| + |I_8| \le C\varepsilon \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{4} \|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2. \tag{3.12}$$

For the last term, by (2.7) we have

$$|I_{10}| \le C \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C \|\boldsymbol{u}_n\|_{\mathbb{L}^3}^3 + \frac{\sigma}{4} \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2. \tag{3.13}$$

Substituting (3.9), (3.10), (3.11), and (3.12) into (3.8) (noting assumption (2.5) and that $\boldsymbol{e}$ is a unit vector), then rearranging the terms give

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \sigma \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_2 \||\boldsymbol{u}_n||\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 + \kappa_1\sigma \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \kappa_2\sigma \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \le C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{L}^3}^3\right)$$

which can be rearranged into

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \sigma \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_2 \||\boldsymbol{u}_n||\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 + \kappa_1\sigma \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{1}{2}\kappa_2\sigma \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \\ \le C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{L}^3}^3\right) - \frac{1}{2}\kappa_2\sigma \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4, \end{aligned} \tag{3.14}$$

where C is independent of ε. Now, note that for any $\alpha, \beta > 0$, we have

$$\alpha \|\boldsymbol{u}_n\|_{\mathbb{L}^3}^3 - \beta \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 = \int_{\mathscr{O}} \alpha|\boldsymbol{u}_n|^3 - \beta|\boldsymbol{u}_n|^4 \,\mathrm{d}x \le \frac{\alpha^4|\mathscr{O}|}{\beta^3}.$$

Using this inequality with $\alpha = C$ and $\beta = \kappa_2\sigma/2$, we obtain from (3.14),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + 2\sigma \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_2 \||\boldsymbol{u}_n||\nabla\boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 + \kappa_1\sigma \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \kappa_2\sigma \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \le C. \tag{3.15}$$

The Gronwall inequality then yields

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 \le e^{-\kappa_1\sigma t} \|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2 + C(\kappa_1\sigma)^{-1},$$

which implies the existence of positive constants M and ρ_0 such that

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 \le M \quad \text{and} \quad \limsup_{t\to\infty} \|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 \le \frac{1}{4}\rho_0. \tag{3.16}$$

Here, M depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ but is independent of ε, while ρ_0 is independent of $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$. Thus, the inequality (3.2) is shown. Integrating (3.15) over $(0,t)$ and rearranging the terms then yield (3.1).

To prove (3.3), we note that the second inequality in (3.16) also implies

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 \le \frac{1}{2}\rho_0, \quad \forall t \ge t_0, \tag{3.17}$$

for some sufficiently large t_0 (depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$). Integrating (3.15) over $(t, t+1)$, rearranging the terms, and using (3.17). we obtain the second inequality in (3.3). By the same argument, but using (3.2) and integrating over $(t, t+\delta)$ instead, we obtain (3.4). This completes the proof of the proposition. □

The following proposition establishes a parabolic smoothing estimate, which will be used later to deduce the existence of an exponential attractor and obtain an estimate for the dimension of the attractor.

Proposition 3.2. For all $t > 0$,

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}^2 \le M_0\varepsilon^{-1}(1 + t + t^{-1}),$$

where M_0 depends only on the coefficients of (1.1), $|\mathscr{O}|$, ν_∞, and $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ (but is independent of ε).

Proof. Taking the inner product of the first equation in (2.49) with $\boldsymbol{H}_n$ and the second equation with $\partial_t \boldsymbol{u}_n$, we obtain

$$\begin{aligned}\langle \partial_t \boldsymbol{u}_n, \boldsymbol{H}_n\rangle_{\mathbb{L}^2} &= \sigma \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \sigma \langle \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \varepsilon \langle \Delta\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} \\ &\quad - \gamma \langle \boldsymbol{u}_n \times \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{R}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{S}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2}, \end{aligned} \tag{3.18}$$

$$\begin{aligned}\langle \boldsymbol{H}_n, \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{\kappa_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \frac{\kappa_2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \\ &\quad + \frac{\lambda_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}\cdot \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \frac{\lambda_2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{e}\cdot \boldsymbol{u}_n\|_{\mathbb{L}^4}^4. \end{aligned} \tag{3.19}$$

For the third and the fourth term on the right-hand side of (3.18), by Young's inequality and (2.47),

$$\sigma \left| \langle \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} \right| + \varepsilon \left| \langle \Delta\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} \right| \le C\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C\varepsilon \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{\sigma}{8}\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2.$$

For the fifth term on the right-hand side of (3.18), we use (2.48) and Young's inequality to obtain

$$\gamma \left| \langle \boldsymbol{u}_n \times \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \boldsymbol{H}_n\rangle_{\mathbb{L}^2} \right| \le C\|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 + \frac{\sigma}{8}\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2.$$

For the last two terms in (3.18), we apply (2.30) and (2.41) respectively. Altogether, from (3.18) and (3.19) we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{\kappa_2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left\||\boldsymbol{u}_n|^2 - \kappa_1/\kappa_2\right\|_{\mathbb{L}^2}^2 + \frac{\lambda_2}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left\|(\boldsymbol{e}\cdot \boldsymbol{u}_n)^2 - \lambda_1/\lambda_2\right\|_{\mathbb{L}^2}^2 + \sigma\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \\ &\quad \le C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4\right) + C\varepsilon\|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C\nu_\infty \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right) + \frac{\sigma}{2}\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2, \end{aligned} \tag{3.20}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \le C\nu_\infty \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 + \varepsilon\|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \nu_\infty \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right).$$

On the other hand, due to (3.4), we have

$$\int_t^{t+1} \left(\|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4 + \varepsilon\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^2}^2 + \varepsilon \||\boldsymbol{u}_n(s)||\nabla \boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2\right) \mathrm{d}s \le \mu_0.$$

Therefore, by using Corollary A.2 and noting (3.1), we obtain, for any $t \ge 0$,

$$\|\nabla \boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-1} t + C\varepsilon^{-1}(1 + t^{-1}),$$

which implies the required result. ☐

Proposition 3.3. For any $n \in \mathbb{N}$ and $t \ge 0$,

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}^2 \le C\varepsilon^{-1}\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2, \tag{3.21}$$

$$\int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \varepsilon \int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s + \varepsilon^2 \int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2\,\mathrm{d}s \le C\varepsilon^{-1}(1+t^3)\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2, \tag{3.22}$$

where C is a constant which is independent of ε, t, and $\boldsymbol{u}_0$.

Moreover, there exists t_1 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that for all $t \ge t_1$,

$$\varepsilon\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}^2 + \int_t^{t+1} \left(\varepsilon\|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon^2\|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon^3\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2\right)\mathrm{d}s \le \rho_1, \tag{3.23}$$

where ρ_1 is independent of $\boldsymbol{u}_0$, ε, and t.

Finally, let $\delta > 0$ be arbitrary. Then for all $t \ge \delta$,

$$\varepsilon\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}^2 + \int_t^{t+\delta} \left(\varepsilon\|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon^2\|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 + \varepsilon^3\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2\right)\mathrm{d}s \le \mu_1, \tag{3.24}$$

where μ_1 depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of ε and t.

Proof. Integrating (3.20) with respect to t and rearranging (noting (3.1)), we obtain

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}^2 + \sigma \int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \varepsilon \int_0^t \|\nabla \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-1}(1+t)\,\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2. \tag{3.25}$$

Note that the bound for $\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}$ still depends on t. Taking the inner product of the second equation in (2.49) with $\Delta^2 \boldsymbol{u}_n$ and integrating by parts as necessary give

$$\begin{aligned}
\|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &= \kappa_1 \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \langle \nabla \boldsymbol{H}_n, \nabla\Delta\boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \kappa_2 \left\langle \nabla(|\boldsymbol{u}_n|^2\boldsymbol{u}_n), \nabla\Delta\boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\
&\quad - \lambda_1 \langle \boldsymbol{e}(\boldsymbol{e}\cdot\nabla\boldsymbol{u}_n), \nabla\Delta\boldsymbol{u}_n\rangle_{\mathbb{L}^2} + 3\lambda_2 \left\langle \boldsymbol{e}(\boldsymbol{e}\cdot\boldsymbol{u}_n)^2\boldsymbol{e}(\boldsymbol{e}\cdot\nabla\boldsymbol{u}_n), \nabla\Delta\boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\
&\le \kappa_1 \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\nabla\boldsymbol{H}_n\|_{\mathbb{L}^2} \|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2} + \kappa_2 \left\|\nabla(|\boldsymbol{u}_n|^2\boldsymbol{u}_n)\right\|_{\mathbb{L}^2} \|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2} \\
&\quad + \lambda_1 \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2} \|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2} + 3\lambda_2 \|\boldsymbol{u}_n\|_{\mathbb{L}^6}^2 \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^6} \|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2} \\
&\le C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 + C\|\nabla\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + C\|\boldsymbol{u}_n\|_{\mathbb{L}^6}^4 \|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^6}^2 + \frac{1}{2}\|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \\
&\le C\left(1+\|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\right)\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 + C\|\nabla\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{1}{2}\|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2,
\end{aligned} \tag{3.26}$$

where in the penultimate step we used Young's inequality and the Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$. Rearranging the terms, integrating both sides over $(0,t)$, then applying (3.25) and (3.1), we obtain

$$\int_0^t \|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-3}(1+t^3)\,\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2.$$

This, together with (3.25) shows (3.22).

Next, from (3.20) and noting (3.4), we have by the uniform Gronwall inequality that,

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}^2 + \left\||\boldsymbol{u}_n(t)|^2 - \kappa_1/\kappa_2\right\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-1}, \quad \forall t \ge \delta, \tag{3.27}$$

where C is independent of ε. We now take $\delta = 1$. For $t \in (0,1)$, (3.25) gives

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^1}^2 \le C\varepsilon^{-1}\,\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2,$$

where C is independent of t and ε. This bound, together with (3.27), yields (3.21).

Noting (3.3), we apply the uniform Gronwall inequality to (3.20) to obtain

$$\|\nabla\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{\kappa_2}{2}\left\||\boldsymbol{u}_n|^2 - \kappa_1/\kappa_2\right\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-1}, \quad \forall t \ge t_0 + 1, \tag{3.28}$$

where C is independent of $\boldsymbol{u}_0$, ε, and t. Inserting this into (3.20) and integrating over $(t, t+1)$ give

$$\int_t^{t+1} \sigma\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \varepsilon\|\nabla\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-1}, \quad \forall t \ge t_0 + 1. \tag{3.29}$$

Now, integrating (3.26) over $(t,t+1)$ and rearranging the terms (and noting (3.28)), we obtain

$$\int_t^{t+1} \|\nabla\Delta\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-3}, \quad \forall t \ge t_0 + 1, \tag{3.30}$$

where C is independent of $\boldsymbol{u}_0$, ε, and t. Altogether, (3.28), (3.29), and (3.30) yield (3.23).

By similar argument, but integrating (3.20) over $(t, t+\delta)$ instead and applying (3.27), we obtain (3.24). This completes the proof of the proposition. ☐

Proposition 3.4. For any $n \in \mathbb{N}$ and $t \ge 0$,

$$\|\boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-4}\,\|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2, \tag{3.31}$$

$$\int_0^t \|\partial_t \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-3}(1+t^3)\,\|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2, \tag{3.32}$$

where C is a constant which is independent of ε, t, and $\boldsymbol{u}_0$.

Moreover, there exists t_2 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that for all $t \ge t_2$,

$$\varepsilon^4\,\|\boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 + \int_t^{t+1} \varepsilon^3\,\|\partial_t\boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le \rho_2. \tag{3.33}$$

Here, ρ_2 is independent of $\boldsymbol{u}_0$, ε, and t.

Finally, let $\delta > 0$ be arbitrary. Then for all $t \geq \delta$,

$$\varepsilon^4 \|\boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 + \int_t^{t+\delta} \varepsilon^3 \|\partial_t \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \leq \mu_2, \tag{3.34}$$

where μ_2 depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of ε and t.

Proof. Taking the inner product of the first equation in (2.49) with $\partial_t \boldsymbol{u}_n$ yields

$$\begin{aligned}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &= \sigma \langle \boldsymbol{H}_n, \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \varepsilon \langle \nabla \boldsymbol{H}_n, \nabla \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \sigma \langle \Pi_n \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &\quad - \varepsilon \langle \Delta \Pi_n \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \gamma \langle \boldsymbol{u}_n \times \boldsymbol{H}_n, \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \gamma \langle \boldsymbol{u}_n \times \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &\quad + \langle \mathcal{R}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{S}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2}. \end{aligned} \tag{3.35}$$

Differentiating the second equation in (2.49) with respect to t, then taking inner product with $\varepsilon \boldsymbol{H}_n$ yields

$$\begin{aligned}\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 &= -\varepsilon \langle \nabla \partial_t \boldsymbol{u}_n, \nabla \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \kappa_1 \varepsilon \langle \partial_t \boldsymbol{u}_n, \boldsymbol{H}_n\rangle_{\mathbb{L}^2} - \kappa_2 \varepsilon \left\langle \partial_t(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n), \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} \\ &\quad + \lambda_1 \varepsilon \langle (\boldsymbol{e}\cdot \partial_t \boldsymbol{u}_n)\boldsymbol{e}, \boldsymbol{H}_n\rangle_{\mathbb{L}^2} - 3\lambda_2 \varepsilon \left\langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)^2 \boldsymbol{e}(\boldsymbol{e}\cdot \partial_t \boldsymbol{u}_n), \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2}\end{aligned}$$

Adding this to (3.35), then applying Hölder's and Young's inequalities give

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \\ &= (\sigma + \kappa_1 \varepsilon) \langle \partial_t \boldsymbol{u}_n, \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \sigma \langle \Pi_n \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \varepsilon \langle \Delta \Pi_n \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &\quad - \gamma \langle \boldsymbol{u}_n \times \boldsymbol{H}_n, \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \gamma \langle \boldsymbol{u}_n \times \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} - \kappa_2 \varepsilon \left\langle \partial_t(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n), \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} \\ &\quad + \lambda_1 \varepsilon \langle (\boldsymbol{e}\cdot \partial_t \boldsymbol{u}_n)\boldsymbol{e}, \boldsymbol{H}_n\rangle_{\mathbb{L}^2} - 3\lambda_2 \varepsilon \left\langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)^2 \boldsymbol{e}(\boldsymbol{e}\cdot \partial_t \boldsymbol{u}_n), \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} + \langle \mathcal{R}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{S}(\boldsymbol{u}_n), \partial_t \boldsymbol{u}_n\rangle_{\mathbb{L}^2} \\ &= J_1 + J_2 + \cdots + J_{10}.\end{aligned} \tag{3.36}$$

We will estimate each term on the last line. For the first three terms, by Young's inequality (noting (2.47)), we have

$$|J_1| + |J_2| + |J_3| \leq C\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C\varepsilon \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2.$$

For the term J_5, similarly we have

$$|J_5| \leq \gamma \|\boldsymbol{u}_n\|_{\mathbb{L}^\infty} \|\boldsymbol{u}_n\|_{\mathbb{L}^2} \|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2} \leq C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2.$$

For the terms J_4 and J_6, by Hölder's and Young's inequalities,

$$|J_4| \leq \gamma \|\boldsymbol{u}_n\|_{\mathbb{L}^4} \|\boldsymbol{H}_n\|_{\mathbb{L}^4} \|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2} \leq C\|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2 \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2,$$

$$|J_6| \leq \kappa_2 \varepsilon \|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2} \|\boldsymbol{u}_n\|_{\mathbb{L}^6}^2 \|\boldsymbol{H}_n\|_{\mathbb{L}^6} \leq C\varepsilon^2 \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4 \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2.$$

For the terms J_7 and J_8, by similar argument we have

$$|J_7| \leq C\varepsilon^2 \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2,$$

$$|J_8| \leq C\varepsilon^2 \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4 \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2.$$

For the terms J_9 and J_{10}, we apply (2.30) and (2.41) respectively to obtain

$$|J_9| \leq C\nu_\infty \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right) + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2,$$

$$|J_{10}| \leq C\left(\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4\right) + \frac{1}{9}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2.$$

Altogether, from (3.36) we infer

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \leq C\|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + C\left(\|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2 + \varepsilon^2 \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\right)\|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + C\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2$$

$$+ C\nu_\infty \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \right) + C\left(\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \right). \tag{3.37}$$

Integrating this with respect to t and using (3.1), (3.21), and (3.22), we obtain

$$\varepsilon \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \int_0^t \|\partial_t \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \, \mathrm{d}s \le C\varepsilon^{-3}(1+t^3) \|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2, \tag{3.38}$$

which gives (3.32), but not exactly (3.31) since this bound still depends on t. Inequality (3.31) will be derived after the rest of the proposition is shown.

Next, note that for $t \ge t_1$ (as defined in Proposition 3.3), the estimates (3.3), (3.23), and (3.37) imply

$$\begin{aligned}\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 &\le C \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + C\varepsilon^{-1}(1+\rho_1^2) \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + C\rho_0 \left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\right) \\ &\quad + C\nu_\infty \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \right) + C \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 .\end{aligned}$$

The uniform Gronwall inequality (noting (3.3) and (3.23)) yields

$$\|\boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-4}, \quad \forall t \ge t_1 + 1, \tag{3.39}$$

where C is a constant independent of t, ε, and $\boldsymbol{u}_0$. Noting (3.39), integrating (3.37) over $(t, t+1)$ and rearranging the terms, we obtain (3.33). Similarly, integrating over $(t, t+\delta)$ instead, by the uniform Gronwall inequality and (3.24), we have (3.34).

Finally, applying (3.34) with $\delta = 1$, we have

$$\varepsilon^4 \|\boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 \le \mu_2, \quad \forall t \ge 1.$$

where μ_2 is independent of ε and t. This, together with the bound (3.38) for $t \le 1$, implies (3.31). Thus, the proof is completed. ☐

Proposition 3.5. For any $n \in \mathbb{N}$ and $t \ge 0$,

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^2}^2 \le C\varepsilon^{-4} \|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2, \tag{3.40}$$

$$\int_0^t \varepsilon \|\Delta \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \, \mathrm{d}s + \int_0^t \varepsilon \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^4}^2 \, \mathrm{d}s \le C\varepsilon^{-4}(1+t^3) \|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2, \tag{3.41}$$

where C is a constant which is independent of ε, t, and $\boldsymbol{u}_0$.

Moreover, there exists t_3 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that for all $t \ge t_3$,

$$\varepsilon^4 \|\boldsymbol{u}_n(t)\|_{\mathbb{H}^2}^2 + \int_t^{t+1} \varepsilon^5 \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^2}^2 \, \mathrm{d}s + \int_t^{t+1} \varepsilon^5 \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^4}^2 \, \mathrm{d}s \le \rho_3, \tag{3.42}$$

where ρ_3 is independent of $\boldsymbol{u}_0$, ε, and t.

Finally, let $\delta > 0$ be arbitrary. Then for all $t \ge \delta$,

$$\varepsilon^4 \|\boldsymbol{u}_n(t)\|_{\mathbb{H}^2}^2 + \int_t^{t+\delta} \varepsilon^5 \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^2}^2 \, \mathrm{d}s + \int_t^{t+\delta} \varepsilon^5 \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^4}^2 \, \mathrm{d}s \le \mu_3, \tag{3.43}$$

where μ_3 depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of ε and t.

Proof. Taking the inner product of the second equation in (2.49) with $\Delta \boldsymbol{u}_n$, we obtain

$$\begin{aligned}\langle \boldsymbol{H}_n, \Delta \boldsymbol{u}_n \rangle_{\mathbb{L}^2} &= \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \kappa_1 \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 - \kappa_2 \left\langle |\boldsymbol{u}_n|^2 \boldsymbol{u}_n, \Delta \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \\ &\quad + \lambda_1 \langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)\boldsymbol{e}, \Delta \boldsymbol{u}_n \rangle_{\mathbb{L}^2} - \lambda_2 \left\langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)^3 \boldsymbol{e}, \Delta \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2}\end{aligned}$$

Therefore, after rearranging the terms, we have

$$\begin{aligned}\|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &= \kappa_1 \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \langle \boldsymbol{H}_n, \Delta \boldsymbol{u}_n \rangle_{\mathbb{L}^2} + \kappa_2 \left\langle |\boldsymbol{u}_n|^2 \boldsymbol{u}_n, \Delta \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \\ &\quad - \lambda_1 \langle \boldsymbol{e}(\boldsymbol{e}\cdot \boldsymbol{u}_n), \Delta \boldsymbol{u}_n \rangle_{\mathbb{L}^2} + \lambda_2 \left\langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)^3 \boldsymbol{e}, \Delta \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \\ &\le \kappa_1 \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{1}{2} \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + C \|\boldsymbol{u}_n\|_{\mathbb{L}^6}^6 + C \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \\ &\le C \left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\right) \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2 + \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{1}{2} \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 ,\end{aligned} \tag{3.44}$$

where we used Young's inequality and the Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$. Rearranging the terms in this inequality, noting (3.21) and (3.31), we then have

$$\|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-4} \|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2 , \tag{3.45}$$

which, together with (3.2), implies (3.40).

Similarly, taking the inner product of the first equation in (2.49) with $\Delta \boldsymbol{H}_n$, rearranging the terms, and applying Hölder's inequality, we have

$$\begin{aligned}
&\varepsilon \|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \sigma \|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \\
&= -\langle \partial_t \boldsymbol{u}_n, \Delta \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \sigma \langle \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \Delta \boldsymbol{H}_n\rangle_{\mathbb{L}^2} - \varepsilon \langle \Delta\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \Delta \boldsymbol{H}_n\rangle_{\mathbb{L}^2} \\
&\quad - \gamma \langle \boldsymbol{u}_n \times \boldsymbol{H}_n, \Delta \boldsymbol{H}_n\rangle_{\mathbb{L}^2} - \gamma \langle \boldsymbol{u}_n \times \Phi_{\mathrm{d}}(\boldsymbol{u}_n), \Delta \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{R}(\boldsymbol{u}_n), \Delta \boldsymbol{H}_n\rangle_{\mathbb{L}^2} + \langle \mathcal{S}(\boldsymbol{u}_n), \Delta \boldsymbol{H}_n\rangle_{\mathbb{L}^2} \\
&= I_1 + I_2 + \cdots + I_7.
\end{aligned} \tag{3.46}$$

For the terms I_1, I_2, and I_3, we apply Young's inequality and (2.47) to infer

$$|I_1| + |I_2| + |I_3| \le C\varepsilon^{-1} \|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + C\|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2 + C\varepsilon \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \frac{\sigma}{8}\|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{8}\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 .$$

For the terms I_4 and I_5, by Hölder's and Young's inequalities (noting (2.47)), we have

$$\begin{aligned}
|I_4| &\le \gamma \|\boldsymbol{u}_n\|_{\mathbb{L}^4} \|\boldsymbol{H}_n\|_{\mathbb{L}^4} \|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2} \le C\varepsilon^{-1} \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2 \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + \frac{\varepsilon}{8}\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 , \\
|I_5| &\le \gamma \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^2 \|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2} \le C\varepsilon^{-1} \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 + \frac{\varepsilon}{8}\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 .
\end{aligned}$$

For the last two terms, we use (2.30) and (2.41) respectively, to obtain

$$|I_6| + |I_7| \le C\nu_\infty \varepsilon^{-1} \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \right) + C\varepsilon^{-1} \left(\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \right) + \frac{\varepsilon}{8}\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 .$$

Altogether, substituting these estimates into (3.46) and integrating both sides with respect to t (noting (3.32), (3.45), and (3.21)), we infer

$$\int_0^t \|\Delta \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-5}(1+t^3) \|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2 . \tag{3.47}$$

Next, applying the operator Δ to the second equation in (2.49) and taking the inner product with $\Delta^2 \boldsymbol{u}_n$, we obtain by similar argument as in (3.44),

$$\int_0^t \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-5}(1+t^3) \|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2 . \tag{3.48}$$

This, together with (3.45) and (3.47), implies (3.41).

Moreover, rearranging the terms in (3.44), then applying (3.23) and (3.33) give

$$\varepsilon^4 \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \le C(1+\rho_1^3+\rho_2), \quad \forall t \ge t_1 + 1. \tag{3.49}$$

Integrating both sides of (3.46) over $(t, t+1)$, noting (3.23), (3.33), and (3.33), we obtain

$$\int_t^{t+1} \varepsilon^5 \|\Delta \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C(\rho_0 + \rho_1^2 + \rho_2), \quad \forall t \ge t_1 + 1. \tag{3.50}$$

Similarly, corresponding to (3.48), we have

$$\int_t^{t+1} \varepsilon^5 \left\|\Delta^2 \boldsymbol{u}_n(s)\right\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C, \quad \forall t \ge t_1 + 1, \tag{3.51}$$

where C depends only on ρ_0, ρ_1, ρ_2, and $|\mathscr{O}|$. Altogether, we infer the inequality (3.42) for all $t \ge t_1 + 1$ from (3.49), (3.50), and (3.51).

Finally, noting (3.24) and (3.34), we repeat the argument leading to (3.49), (3.50), and (3.51), but integrating over $(t, t+\delta)$ instead. This yields (3.43), thus completing the proof of the proposition. $\square$

Proposition 3.6. For all $t > 0$,

$$\|\boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n(t)\|_{\mathbb{H}^2}^2 \le M_1 \varepsilon^{-4}(1 + t^3 + t^{-1}) \tag{3.52}$$

where M_1 depends on $|\mathscr{O}|$, ν_∞, and $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$ (but is independent of ε).

Proof. From inequalities (3.37), (3.2), and (3.21), we have

$$\begin{aligned}
\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &\le C \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + C \left(\|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2 + \varepsilon^2 \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4 \right) \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + C \|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 \\
&\quad + C\nu_\infty \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \right) + C \left(\|\boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \right) \\
&\le C \|\boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + C\varepsilon^{-1} \left(1 + \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^4 \right) \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + CM \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 \\
&\quad + C\nu_\infty \left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n|\,|\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2 \right) + C \left(M + \|\boldsymbol{u}_n\|_{\mathbb{L}^4}^4 \right).
\end{aligned} \tag{3.53}$$

Now, by using (3.22) for $t \le 1$ and (3.24) for $t \ge 1$, we have

$$\int_t^{t+1} \varepsilon^2 \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^1}^2 \,\mathrm{d}s \le C_1, \quad \forall t \ge 0, \tag{3.54}$$

where C_1 depends only on the coefficients of (1.1), $|\mathscr{O}|$, ν_∞, and $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$. Moreover, by (3.4) for $\delta = 1$, we have

$$\int_t^{t+1} \left(\|\boldsymbol{u}_n(s)\|_{\mathbb{L}^4}^4 + \varepsilon \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^2}^2 + \varepsilon \||\boldsymbol{u}_n(s)||\nabla \boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2 \right) \mathrm{d}s \le C_0, \quad \forall t \ge 0, \tag{3.55}$$

where C_0 depends only on the coefficients of (1.1), $|\mathscr{O}|$, ν_∞, and $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$. Furthermore, by (3.1) and (3.22),

$$\int_0^t \left(\varepsilon^2 \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^1}^2 + \varepsilon \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^2}^2 + \varepsilon \||\boldsymbol{u}_n(s)||\nabla \boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2 \right) \mathrm{d}s \le C(1+t^3) \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2, \quad \forall t \ge 0. \tag{3.56}$$

Altogether, inequalities (3.54), (3.55), and (3.56) imply the required inequality for the first term by Corollary A.2.

Finally, using the second equation in (2.49), we have

$$\begin{aligned}
\|\Delta \boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 &\le \|\boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 + (\kappa_1 + \lambda_1) \|\boldsymbol{u}_n(t)\|_{\mathbb{L}^2}^2 + (\kappa_2 + \lambda_2) \|\boldsymbol{u}_n(t)\|_{\mathbb{L}^6}^6 \\
&\le M_1 \varepsilon^{-4}(1 + t^3 + t^{-1}) + CM + C\varepsilon^{-3} \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^6,
\end{aligned}$$

where in the last step we used (3.2), the Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$, and (3.21). Thus, the proof is now complete. ∎

Proposition 3.7. For any $n \in \mathbb{N}$ and $t \ge 0$,

$$\|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-8} \|\boldsymbol{u}_0\|_{\mathbb{H}^3}^2, \tag{3.57}$$

$$\int_0^t \|\nabla \partial_t \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^{-7}(1+t^3) \|\boldsymbol{u}_0\|_{\mathbb{H}^3}^2, \tag{3.58}$$

where C is a constant which is independent of ε, t, and $\boldsymbol{u}_0$.

Moreover, there exists t_3 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that for all $t \ge t_3$,

$$\varepsilon^8 \|\nabla \boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 + \int_t^{t+1} \varepsilon^7 \|\nabla \partial_t \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le \rho_3, \tag{3.59}$$

where ρ_3 is independent of $\boldsymbol{u}_0$, ε, and t.

Finally, let $\delta > 0$ be arbitrary. Then for all $t \ge \delta$,

$$\varepsilon^8 \|\nabla \boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 + \int_t^{t+\delta} \varepsilon^7 \|\nabla \partial_t \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le \mu_3, \tag{3.60}$$

where μ_3 depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of ε and t.

Proof. Taking the inner product of the first equation in (2.49) with $-\Delta\partial_t \boldsymbol{u}_n$ gives

$$\begin{aligned}\|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &= \sigma\left\langle \nabla \boldsymbol{H}_n, \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \sigma\left\langle \nabla\Pi_n\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \varepsilon\left\langle \Delta \boldsymbol{H}_n, \Delta\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\ &\quad + \varepsilon\left\langle \nabla\Delta\Pi_n\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} - \gamma\left\langle \nabla(\boldsymbol{u}_n\times \boldsymbol{H}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\ &\quad - \gamma\left\langle \nabla(\boldsymbol{u}_n\times \Phi_{\mathrm{d}}(\boldsymbol{u}_n)), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \left\langle \nabla\mathcal{R}(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \left\langle \nabla\mathcal{S}(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2}. \end{aligned} \tag{3.61}$$

Differentiating the second equation in (2.49) with respect to t, then taking the inner product of the resulting equation with $-\varepsilon\Delta\partial_t \boldsymbol{u}_n$ yields

$$\begin{aligned}\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 &= -\varepsilon\left\langle \Delta\partial_t \boldsymbol{u}_n, \Delta \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} + \kappa_1\varepsilon\left\langle \nabla\partial_t \boldsymbol{u}_n, \nabla \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} + \kappa_2\varepsilon\left\langle \partial_t(|\boldsymbol{u}_n|^2\boldsymbol{u}_n), \Delta \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} \\ &\quad - \lambda_1\varepsilon\left\langle (\boldsymbol{e}\cdot\partial_t \boldsymbol{u}_n)\boldsymbol{e}, \Delta \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} - 3\lambda_2\varepsilon\left\langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)^2\boldsymbol{e}(\boldsymbol{e}\cdot\partial_t \boldsymbol{u}_n), \Delta \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2}. \end{aligned} \tag{3.62}$$

Adding (3.61) and (3.62) gives

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \\ &= (\sigma+\kappa_1\varepsilon)\left\langle \nabla\partial_t \boldsymbol{u}_n, \nabla \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} + \sigma\left\langle \nabla\Pi_n\Phi_d(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \varepsilon\left\langle \nabla\Delta\Pi_n\Phi_{\mathrm{d}}(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\ &\quad - \gamma\left\langle \nabla(\boldsymbol{u}_n\times \boldsymbol{H}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} - \gamma\left\langle \nabla(\boldsymbol{u}_n\times\Phi_{\mathrm{d}}(\boldsymbol{u}_n)), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} + \kappa_2\varepsilon\left\langle \partial_t(|\boldsymbol{u}_n|^2\boldsymbol{u}_n), \Delta \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} \\ &\quad - \lambda_1\varepsilon\left\langle (\boldsymbol{e}\cdot\partial_t \boldsymbol{u}_n)\boldsymbol{e}, \Delta \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} - 3\lambda_2\varepsilon\left\langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)^2\boldsymbol{e}(\boldsymbol{e}\cdot\partial_t \boldsymbol{u}_n), \Delta \boldsymbol{H}_n\right\rangle_{\mathbb{L}^2} + \left\langle \nabla\mathcal{R}(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\ &\quad + \left\langle \nabla\mathcal{S}(\boldsymbol{u}_n), \nabla\partial_t \boldsymbol{u}_n\right\rangle_{\mathbb{L}^2} \\ &= I_1 + I_2 + \cdots + I_{10}. \end{aligned} \tag{3.63}$$

Each term on the last line can be estimated analogously to (3.36). For the first three terms, by Young's inequality,

$$|I_1| + |I_2| + |I_3| \le C\|\boldsymbol{u}_n\|_{\mathbb{H}^3}^2 + C\|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + \frac{1}{8}\|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2.$$

For the term I_4 and I_5, by Hölder's and Young's inequalities, and the Sobolev embedding we have

$$\begin{aligned}|I_4| &\le \gamma\|\nabla(\boldsymbol{u}_n\times \boldsymbol{H}_n)\|_{\mathbb{L}^2}\|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2} \le C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + C\|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2\|\boldsymbol{H}_n\|_{\mathbb{H}^2}^2 + \frac{1}{8}\|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2, \\ |I_5| &\le \gamma\|\nabla(\boldsymbol{u}_n\times\Phi_{\mathrm{d}}(\boldsymbol{u}_n))\|_{\mathbb{L}^2}\|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2} \le C\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\|\boldsymbol{u}_n\|_{\mathbb{H}^1}^2 + \frac{1}{8}\|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2. \end{aligned}$$

For the terms I_6, I_7, and I_8, similarly we have

$$\begin{aligned}|I_6| &\le 3\kappa_2\varepsilon\|\boldsymbol{u}_n\|_{\mathbb{L}^6}^2\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^6}\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2} \le \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{1}{8}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{H}^1}^2, \\ |I_7| &\le \lambda_1\varepsilon\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2} \le C\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + C\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2, \\ |I_8| &\le 3\lambda_2\varepsilon\|\boldsymbol{u}_n\|_{\mathbb{L}^6}^2\|\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^6}\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2} \le \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\|\Delta \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{1}{8}\|\partial_t \boldsymbol{u}_n\|_{\mathbb{H}^1}^2. \end{aligned}$$

For the last two terms in (3.63), we apply Young's inequality, (2.33), and (2.42) to obtain

$$|I_9| + |I_{10}| \le C\nu_\infty\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4 + \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^4\right) + C\left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n||\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right) + \frac{1}{8}\|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2.$$

Altogether, the above estimates for I_j, where $j = 1, 2, \ldots, 9$, imply

$$\begin{aligned}\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \|\nabla\partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &\le C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\right)\|\boldsymbol{H}_n\|_{\mathbb{H}^2}^2 \\ &\quad + C\nu_\infty\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4 + \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^4\right) + C\left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n||\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right). \end{aligned} \tag{3.64}$$

Integrating this over $(0, t)$, noting (3.1), (3.40), and (3.41), we obtain

$$\varepsilon\|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \int_0^t \|\nabla\partial_t \boldsymbol{u}_n(s)\|_{\mathbb{L}^2}^2\,\mathrm{d}s \le C\varepsilon^{-7}(1+t^3)\|\boldsymbol{u}_0\|_{\mathbb{H}^3}^2,$$

which implies (3.58), but not (3.57) due to the dependence on t. The proof of (3.57) will be given after we show (3.59) and (3.60).

Next, by using (3.42) and applying the uniform Gronwall inequality, we obtain

$$\|\nabla \boldsymbol{H}_n(t)\|_{\mathbb{L}^2}^2 \le C\varepsilon^{-8}, \quad \forall t \ge t_3,$$

where C is independent of $\boldsymbol{u}_0$ and t. Integrating (3.64) over $(t,t+1)$ and using the above inequality then yields (3.59). By similar argument using the uniform Gronwall inequality, but applying (3.43) and integrating over $(t,t+\delta)$ instead, we obtain (3.60).

Finally, it remains to show (3.57). By (3.60) for $\delta = 1$, we obtain a bound on $\|\nabla \boldsymbol{H}_n(t)\|_{\mathbb{L}^2}$ for $t \ge 1$ which is independent of t. This, together with (3.41) for $t \le 1$, yields (3.57). This completes the proof of the proposition. □

Proposition 3.8. For any $n \in \mathbb{N}$ and $t \ge 0$,

$$\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^3}^2 \le C\varepsilon^{-8} \|\boldsymbol{u}_0\|_{\mathbb{H}^3}^2 , \tag{3.65}$$

$$\int_0^t \|\nabla\Delta \boldsymbol{H}_n(s)\|_{\mathbb{L}^2}^2 \, \mathrm{d}s + \int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^5}^2 \, \mathrm{d}s \le C\varepsilon^{-9}(1+t^3) \|\boldsymbol{u}_0\|_{\mathbb{H}^3}^2 , \tag{3.66}$$

where C is a constant which is independent of ε, t, and $\boldsymbol{u}_0$.

Moreover, there exists t_4 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that for all $t \ge t_4$,

$$\varepsilon^8 \|\boldsymbol{u}_n(t)\|_{\mathbb{H}^3}^2 + \int_t^{t+1} \varepsilon^9 \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^3}^2 \, \mathrm{d}s + \int_t^{t+1} \varepsilon^9 \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^5}^2 \, \mathrm{d}s \le \rho_4, \tag{3.67}$$

where ρ_4 is independent of $\boldsymbol{u}_0$, ε, and t.

Finally, let $\delta > 0$ be arbitrary. Then for all $t \ge \delta$,

$$\varepsilon^8 \|\boldsymbol{u}_n(t)\|_{\mathbb{H}^3}^2 + \int_t^{t+\delta} \varepsilon^9 \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^3}^2 \, \mathrm{d}s + \int_t^{t+\delta} \varepsilon^9 \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^5}^2 \, \mathrm{d}s \le \mu_4, \tag{3.68}$$

where μ_4 depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of ε and t.

Proof. The proof of this proposition follows by similar argument as in Proposition 3.5. Firstly, taking the inner product of the second equation in (2.49) with $\Delta^2 \boldsymbol{u}_n$ and integrating by parts, we have

$$\begin{aligned}
\|\nabla\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 &\le \kappa_1 \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \left| \langle \nabla \boldsymbol{H}_n, \nabla\Delta \boldsymbol{u}_n \rangle_{\mathbb{L}^2} \right| + \kappa_2 \left| \left\langle \nabla(|\boldsymbol{u}_n|^2 \boldsymbol{u}_n), \nabla\Delta \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \right| \\
&\quad + \lambda_1 \left| \langle \boldsymbol{e}(\boldsymbol{e}\cdot\nabla \boldsymbol{u}_n), \nabla\Delta \boldsymbol{u}_n \rangle_{\mathbb{L}^2} \right| + 3\lambda_2 \left| \left\langle (\boldsymbol{e}\cdot \boldsymbol{u}_n)^2 (\boldsymbol{e}\cdot\nabla \boldsymbol{u}_n)\boldsymbol{e}, \nabla\Delta \boldsymbol{u}_n \right\rangle_{\mathbb{L}^2} \right| \\
&\le \kappa_1 \|\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2} \|\nabla\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2} + \lambda_1 \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2} \|\nabla\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2} \\
&\quad + (\kappa_2 + 3\lambda_2) \|\boldsymbol{u}_n\|_{\mathbb{L}^6}^2 \|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^6} \|\nabla\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2} \\
&\le C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\right) \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2 + C \|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \frac{1}{2} \|\nabla\Delta \boldsymbol{u}_n\|_{\mathbb{L}^2}^2
\end{aligned}$$

by the same argument as in (3.44). By (3.40) and (3.57), we then have (3.65).

Next, taking the inner product of the first equation in (2.49) with $\Delta^2 \boldsymbol{H}_n$ and continuing along the same line as in (3.46), (3.47), and (3.50), we obtain (3.66).

Finally, the proof of (3.67) and (3.68) follows the same argument as that of (3.42) and (3.43), respectively. Further details are omitted for brevity. □

Proposition 3.9. For all $t > 0$,

$$\|\boldsymbol{H}_n(t)\|_{\mathbb{H}^1}^2 + \|\boldsymbol{u}_n(t)\|_{\mathbb{H}^3}^2 \le M_2 \varepsilon^{-8}(1 + t^3 + t^{-1}),$$

where M_2 depends on $|\mathscr{O}|$, ν_∞, and $\|\boldsymbol{u}_0\|_{\mathbb{H}^2}$ (but is independent of ε).

Proof. From (3.64), noting (3.40), we have

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{H}_n\|_{\mathbb{L}^2}^2 + \|\nabla \partial_t \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 \le C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\right) \|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + C\left(1 + \|\boldsymbol{u}_n\|_{\mathbb{H}^1}^4\right) \|\boldsymbol{H}_n\|_{\mathbb{H}^2}^2$$

$$
\begin{aligned}
&\quad + C\nu_\infty \left(1+\|\boldsymbol{u}_n\|_{\mathbb{H}^2}^2\right) \|\boldsymbol{u}_n\|_{\mathbb{H}^3}^2 + C\left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n||\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right) \\
&\le C\left(1+\varepsilon^{-4}\|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{H}_n\|_{\mathbb{H}^1}^2 + C\left(1+\varepsilon^{-2}\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^4\right)\|\boldsymbol{H}_n\|_{\mathbb{H}^2}^2 \\
&\quad + C\nu_\infty\left(1+\varepsilon^{-4}\|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{u}_n\|_{\mathbb{H}^3}^2 + C\left(\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}_n||\nabla \boldsymbol{u}_n|\|_{\mathbb{L}^2}^2\right).
\end{aligned}
$$

Now, note that using (3.4) and (3.43) with $\delta = 1$, and (3.22) for $t \le 1$, we have

$$
\int_t^{t+1} \varepsilon^5 \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^2}^2 \,\mathrm{d}s + \int_t^{t+1} \varepsilon^3 \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2 \,\mathrm{d}s + \int_t^{t+1} \varepsilon \||\boldsymbol{u}_n(s)||\nabla \boldsymbol{u}_n(s)|\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C, \quad \forall t \ge 0.
$$

Furthermore, by (3.41), for all $t \ge 0$,

$$
\int_0^t \varepsilon^5 \left(1+\|\boldsymbol{u}_0\|_{\mathbb{H}^2}^4\right)\|\boldsymbol{H}_n(s)\|_{\mathbb{H}^2}^2 \,\mathrm{d}s + \int_0^t \varepsilon^3\left(1+\|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^3}^2\,\mathrm{d}s \le C(1+t^3),
$$

where C depends on $\|\boldsymbol{u}_0\|_{\mathbb{H}^2}$. The required result then follows from Corollary A.2. ☐

The bounds proved so far are summarised in the following proposition for ease of reference later. We do not track the dependence of the constants on ε here.

Proposition 3.10. Let $k = 0, 1, 2$, or 3.

(1) For all $t \ge 0$,

$$
\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^k}^2 \le C\|\boldsymbol{u}_0\|_{\mathbb{H}^k}^2, \tag{3.69}
$$

$$
\int_0^t \|\boldsymbol{u}_n(s)\|_{\mathbb{H}^{k+2}}^2 \,\mathrm{d}s + \int_0^t \|\boldsymbol{H}_n(s)\|_{\mathbb{H}^k}^2\,\mathrm{d}s \le C(1+t^3)\|\boldsymbol{u}_0\|_{\mathbb{H}^k}^2, \tag{3.70}
$$

where C is a constant depending only on the coefficients of (1.1).

(2) There exists t_k depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that, for all $t \ge t_k$,

$$
\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^k}^2 + \int_t^{t+1}\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^{k+2}}^2\,\mathrm{d}s + \int_t^{t+1}\|\boldsymbol{H}_n(s)\|_{\mathbb{H}^k}^2\,\mathrm{d}s \le \alpha_k, \tag{3.71}
$$

where α_k is a constant independent of t and $\boldsymbol{u}_0$.

(3) For all $t \ge \delta$, where $\delta > 0$ is arbitrary,

$$
\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^k}^2 + \int_t^{t+\delta}\|\boldsymbol{u}_n(s)\|_{\mathbb{H}^{k+2}}^2\,\mathrm{d}s + \int_t^{t+\delta}\|\boldsymbol{H}_n(s)\|_{\mathbb{H}^k}^2\,\mathrm{d}s \le \beta_k, \tag{3.72}
$$

where β_k is a constant independent of t (but may depend on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$).

(4) For all $t > 0$,

$$
\|\boldsymbol{u}_n(t)\|_{\mathbb{H}^k}^2 \le M_s(1+t^3+t^{-1}). \tag{3.73}
$$

Here, M_s is a constant depending on $\|\boldsymbol{u}_0\|_{\mathbb{H}^s}$, where $s := \max\{0, k-1\}$, but is independent of t.

Suppose now that $k = 2$ or 3.

(1) For all $t \ge 0$,

$$
\int_0^t \|\partial_t \boldsymbol{u}_n(s)\|_{\mathbb{H}^{k-2}}^2\,\mathrm{d}s \le C(1+t^3)\|\boldsymbol{u}_0\|_{\mathbb{H}^k}^2, \tag{3.74}
$$

where C is a constant depending only on the coefficients of (1.1).

(2) There exists t_k depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ such that, for all $t \ge t_k$,

$$
\|\boldsymbol{H}_n(t)\|_{\mathbb{H}^{k-2}}^2 + \int_t^{t+1}\|\partial_t \boldsymbol{u}_n(s)\|_{\mathbb{H}^{k-2}}^2\,\mathrm{d}s \le \alpha_k, \tag{3.75}
$$

where α_k is a constant independent of t and $\boldsymbol{u}_0$.

(3) For all $t \geq \delta$, where $\delta > 0$ is arbitrary,

$$\|\boldsymbol{H}_n(t)\|_{\mathbb{H}^{k-2}}^2 + \int_t^{t+\delta} \|\partial_t \boldsymbol{u}_n(s)\|_{\mathbb{H}^{k-2}}^2 \, \mathrm{d}s \leq \beta_k, \tag{3.76}$$

where β_k is a constant independent of t (but may depend on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$).

(4) For all $t \geq 0$,

$$\|\boldsymbol{H}_n(t)\|_{\mathbb{H}^{k-2}}^2 \leq M_k(1 + t^3 + t^{-1}), \tag{3.77}$$

where M_k is a constant depending on $\|\boldsymbol{u}_0\|_{\mathbb{H}^k}$, but is independent of t.

The following result on existence and uniqueness of solution to (1.1) is immediate.

Theorem 3.11. Let $\boldsymbol{u}_0 \in \mathbb{H}^1$ be a given initial data. There exists a unique global weak solution $\boldsymbol{u}$ to (1.1) in the sense of Definition 2.1. This solution satisfies

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2 + \|\boldsymbol{H}(t)\|_{\mathbb{L}^2}^2 \leq M_1(1 + t^3 + t^{-1}). \tag{3.78}$$

If $\boldsymbol{u}_0 \in \mathbb{H}^2$, then this solution is a strong solution satisfying

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^3}^2 + \|\boldsymbol{H}(t)\|_{\mathbb{H}^1}^2 \leq M_2(1 + t^3 + t^{-1}), \tag{3.79}$$

where the constants M_k, where $k = 1, 2$, depend only on the coefficients of (1.1), $|\mathscr{O}|$, ν_∞, and $\|\boldsymbol{u}_0\|_{\mathbb{H}^k}$.

Furthermore, if $\boldsymbol{u}_0 \in \mathbb{H}^3$, then the solution $\boldsymbol{u}$ belongs to $C([0,T];\mathbb{H}^3) \cap L^2(0,T;\mathbb{H}^5)$.

Proof. This follows from a standard compactness argument, making use of the uniform estimates summarised in Proposition 3.10 and the Aubin–Lions lemma (cf. [48, Theorem 2.2]). The inequalities (3.78) and (3.79) follow from (3.73) and (3.77), respectively. ☐

4. Long-time behaviour of the solution

4.1. **Auxiliary results on dynamical systems.** First, we recall some basic facts and terminologies in the theory of dynamical systems [40, 43]. A well-posed system of time-dependent PDEs on a Banach space $(X, \|\cdot\|_X)$ generates a strongly continuous (nonlinear) semigroup

$$S(t) : X \to X, \quad S(t)u_0 = u(t) \quad \text{for } t \geq 0.$$

Therefore, $\big(X, \{S(t)\}_{t\geq 0}\big)$ is a semi-dynamical system.

Definition 4.1 (Global attractor)**.** A subset $\mathcal{A} \subset X$ is a (compact) *global attractor* for $S(t)$ if

(1) it is compact in X,
(2) it is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \geq 0$,
(3) for any bounded set $B \subset X$,

$$\lim_{t\to+\infty} \operatorname{dist}\big(S(t)B, \mathcal{A}\big) = 0,$$

where dist denotes the Hausdorff semi-metric between sets defined by

$$\operatorname{dist}(A, B) := \sup_{a\in A} \inf_{b\in B} \|a - b\|_X .$$

Note that the global attractor, if it exists, is unique. Next, a bounded set $B_0 \subset X$ is a bounded absorbing set for $S(t)$ if, for any bounded set $B \subset X$, there exists $t_0 := t_0(B)$ such that $S(t)B \subset B_0$ for all $t \geq t_0$. The semigroup $S(t)$ is said to be dissipative in X if it possesses a bounded absorbing set $B_0 \subset X$. Moreover, the ω-limit set of a set B is defined as

$$\omega(B) := \{y \in X : \exists t_n \to +\infty \text{ and } x_n \in B \text{ such that } S(t_n)x_n \to y\} = \overline{\bigcap_{t\geq 0} \bigcup_{s\geq t} S(s)B}.$$

If $B = \{v\}$ is a singleton, then we write $\omega(v)$ in lieu of $\omega(\{v\})$. For any $u_0 \in X$, it can be seen that $\mathrm{d}\big(u(t), \omega(u_0)\big) \to 0$ as $t \to +\infty$, where $\mathrm{d}(u(t), B) := \inf_{\varphi\in B} \|u(t) - \varphi\|_X$. The following abstract theorem shows a relation between the existence of a compact absorbing set and the global attractor.

Theorem 4.2. If $S(t)$ is a dissipative semigroup on X which has a compact absorbing set K, then there exists a connected global attractor $\mathcal{A} = \omega(K)$.

If the semigroup admits a global Lyapunov function, then more regular structures on the global attractor can be deduced. We mention the following results from [40].

Definition 4.3 (Global Lyapunov function)**.** Let $E \subset X$ and $\mathcal{L} : E \to \mathbb{R}$ be a continuous function. The function $\mathcal{L}$ is a *global Lyapunov function* for $S(t)$ on E if

(1) for all $u_0 \in E$, the function $t \mapsto \mathcal{L}(S(t)u_0)$ is non-increasing,
(2) if $\mathcal{L}(S(t)u_0) = \mathcal{L}(u_0)$ for some $t > 0$, then u_0 is a fixed point of $S(t)$.

In particular, the second condition above implies that the system can have no periodic orbits. Next, denote the set of fixed points of $S(t)$ by $\mathcal{N}$, and define the unstable set of B to be the set

$$\mathcal{M}^{\mathrm{un}}(B) := \left\{ u_0 \in B : S(t)u_0 \text{ is defined for all } t \in \mathbb{R} \text{ and } \lim_{t\to-\infty} \mathrm{d}(u(t), B) = 0 \right\}. \tag{4.1}$$

Note that if the semigroup $S(t)$ is injective, then $S(t)u_0$ is defined for all $t \in \mathbb{R}$, i.e. $S(t)$ defines a dynamical system. In this case, the first condition in (4.1) is redundant. The following result shows that if the semigroup possesses a global Lyapunov function, then the only possible limit set of individual trajectories are the fixed points.

Proposition 4.4. Let $S(t)$ be a semigroup with global attractor $\mathcal{A}$, which admits a global Lyapunov function on E. Then

(1) $\omega(u_0) \subset \mathcal{N}$ for every $u_0 \in X$ (i.e. $\mathrm{d}(u(t), \mathcal{N}) \to 0$ as $t \to +\infty$),
(2) $\mathcal{A} = \mathcal{M}^{\mathrm{un}}(\mathcal{N})$.

We also need the following notion of the fractal dimension of a set.

Definition 4.5. Let X be a compact subset of E. For $\epsilon > 0$, let $N_\epsilon(X)$ be the minimal number of balls of radius ϵ which are necessary to cover X. The *fractal dimension* of X is the quantity

$$\dim_{\mathrm{F}} X := \limsup_{\epsilon\to 0^+} \frac{\log_2 N_\epsilon(X)}{\log_2(1/\epsilon)}.$$

Note that $\dim_{\mathrm{F}} X \in [0, \infty]$. The quantity $\mathcal{H}_\epsilon(X) := \log_2 N_\epsilon(X)$ is called the Kolmogorov ϵ-entropy of X.

To show that the global attractor has a finite fractal dimension, we follow a general method based on the smoothing (or squeezing) property of the semigroup proposed in [20, 21, 56, 57], whose ideas can be traced back to Ladyzhenskaya [31].

Theorem 4.6. Let X be a compact subset of a Banach space E_1. Suppose that $E_1 \hookrightarrow E$ is a compact embedding. Let $L : X \to X$ be a map such that $L(X) = X$ and

$$\|Lx_1 - Lx_2\|_{E_1} \le \alpha \|x_1 - x_2\|_E, \quad \forall x_1, x_2 \in X. \tag{4.2}$$

Then the fractal dimension of X is finite and satisfies

$$\dim_{\mathrm{F}} X \le \mathcal{H}_{1/4\alpha}(B_{E_1}),$$

where α is the constant in (4.2), $\mathcal{H}_{1/4\alpha}$ is the Kolmogorov $1/4\alpha$-entropy as defined in Definition 4.5, and B_{E_1} is the unit ball in E_1 (which is relatively compact in E).

The above theorem will be applied in our case with $X = \mathcal{A}$, where $\mathcal{A}$ is the global attractor, and $L := S(t_0)$ for some $t_0 > 0$.

4.2. **Uniform estimates on difference of two solutions.** To infer the existence of an exponential attractor and estimate the dimension of the global attractor, we need to establish continuous dependence and smoothing estimates for the difference of two solutions in various norms. Without loss of generality, we will assume that $\sigma - \varepsilon\kappa_1 > 0$ to simplify presentation (otherwise, the interpolation inequality (2.16) could be used to complete the proof in the general case).

Lemma 4.7. Let $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ be solutions of (1.1) corresponding to initial data $\boldsymbol{u}_0 \in \mathbb{H}^1$ and $\boldsymbol{v}_0 \in \mathbb{H}^1$, respectively. Then for any $t > 0$,

$$\|\boldsymbol{u}(t)-\boldsymbol{v}(t)\|_{\mathbb{L}^2}^2 + \varepsilon \int_0^t \|\Delta \boldsymbol{u}(s) - \Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \, \mathrm{d}s + \int_0^t \|\nabla \boldsymbol{u}(s) - \nabla \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \, \mathrm{d}s \le Ce^{Ct} \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{L}^2}^2 ,$$

where C depends only on $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$, $\|\boldsymbol{v}_0\|_{\mathbb{H}^1}$, and the coefficients of the equation (1.1).

Proof. Let $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ be the effective field corresponding to $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. Let $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$ and $\boldsymbol{B} := \boldsymbol{H}_1 - \boldsymbol{H}_2$. Then, noting (2.46), $\boldsymbol{w}$ and $\boldsymbol{B}$ satisfy

$$\begin{aligned} \partial_t \boldsymbol{w} &= \sigma \boldsymbol{B} + \sigma \Phi_{\mathrm{d}}(\boldsymbol{w}) - \varepsilon \Delta \boldsymbol{B} - \varepsilon \Delta \Phi_{\mathrm{d}}(\boldsymbol{w}) - \gamma(\boldsymbol{w} \times \boldsymbol{H}_1 + \boldsymbol{v} \times \boldsymbol{B}) \\ &\quad - \gamma\big(\boldsymbol{w} \times \Phi_{\mathrm{d}}(\boldsymbol{u}) + \boldsymbol{v} \times \Phi_{\mathrm{d}}(\boldsymbol{w})\big) + \mathcal{R}(\boldsymbol{u}) - \mathcal{R}(\boldsymbol{v}) + \mathcal{S}(\boldsymbol{u}) - \mathcal{S}(\boldsymbol{v}), \end{aligned} \tag{4.3}$$

$$\boldsymbol{B} = \Delta \boldsymbol{w} + \kappa_1 \boldsymbol{w} - \kappa_2 \left(|\boldsymbol{u}|^2 \boldsymbol{w} + ((\boldsymbol{u}+\boldsymbol{v})\cdot \boldsymbol{w})\boldsymbol{v}\right) + \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \tag{4.4}$$

with initial data $\boldsymbol{w}(0) = \boldsymbol{u}_0 - \boldsymbol{v}_0$.

Taking the inner product of (4.3) with $\boldsymbol{w}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_{\mathbb{L}^2}^2 &= \sigma \langle \boldsymbol{B}, \boldsymbol{w}\rangle_{\mathbb{L}^2} + \sigma \langle \Phi_{\mathrm{d}}(\boldsymbol{w}), \boldsymbol{w}\rangle_{\mathbb{L}^2} - \varepsilon \langle \Delta \boldsymbol{B}, \boldsymbol{w}\rangle_{\mathbb{L}^2} - \varepsilon \langle \Delta \Phi_{\mathrm{d}}(\boldsymbol{w}), \boldsymbol{w}\rangle_{\mathbb{L}^2} - \gamma \langle \boldsymbol{v}\times \boldsymbol{B}, \boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad - \gamma \langle \boldsymbol{v} \times \Phi_d(\boldsymbol{w}), \boldsymbol{w}\rangle_{\mathbb{L}^2} + \langle \mathcal{R}(\boldsymbol{u}) - \mathcal{R}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\mathbb{L}^2} + \langle \mathcal{S}(\boldsymbol{u}) - \mathcal{S}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\mathbb{L}^2}. \end{aligned} \tag{4.5}$$

Taking the inner product of (4.4) with $\sigma\boldsymbol{w}$, we have

$$\begin{aligned} \sigma \langle \boldsymbol{B}, \boldsymbol{w}\rangle_{\mathbb{L}^2} &= -\sigma \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \sigma\kappa_1 \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 - \sigma\kappa_2 \||\boldsymbol{u}||\boldsymbol{w}|\|_{\mathbb{L}^2}^2 - \sigma\kappa_2 \|\boldsymbol{v}\cdot \boldsymbol{w}\|_{\mathbb{L}^2}^2 - \sigma\kappa_2 \langle (\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v}, \boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad + \sigma \langle \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\mathbb{L}^2}. \end{aligned} \tag{4.6}$$

Furthermore, taking the inner product of (4.4) with $-\varepsilon\Delta\boldsymbol{w}$ we obtain

$$\begin{aligned} -\varepsilon \langle \Delta \boldsymbol{B}, \boldsymbol{w}\rangle_{\mathbb{L}^2} &= -\varepsilon \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_1 \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \varepsilon\kappa_2 \left\langle |\boldsymbol{u}|^2 \boldsymbol{w}, \Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \varepsilon\kappa_2 \langle ((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v}, \Delta\boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad - \varepsilon \langle \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2}. \end{aligned} \tag{4.7}$$

Similarly, using (4.4) again and noting that $\boldsymbol{v}\times\boldsymbol{w} = \boldsymbol{u}\times\boldsymbol{w}$,

$$-\gamma \langle \boldsymbol{v}\times \boldsymbol{B}, \boldsymbol{w}\rangle_{\mathbb{L}^2} = -\gamma \langle \boldsymbol{v}\times \Delta\boldsymbol{w}, \boldsymbol{w}\rangle_{\mathbb{L}^2} + \gamma \langle \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \boldsymbol{u}\times\boldsymbol{w}\rangle_{\mathbb{L}^2}. \tag{4.8}$$

Let $\eta := \sigma - \varepsilon\kappa_1 > 0$. Substituting (4.6), (4.7), and (4.8) into (4.5), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \eta \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \sigma\kappa_2 \||\boldsymbol{u}||\boldsymbol{w}|\|_{\mathbb{L}^2}^2 + \sigma\kappa_2 \|\boldsymbol{v}\cdot\boldsymbol{w}\|_{\mathbb{L}^2}^2 \\ &= \sigma\kappa_1 \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 - \sigma\kappa_2 \langle (\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v}, \boldsymbol{w}\rangle_{\mathbb{L}^2} + \sigma \langle \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\mathbb{L}^2} - \varepsilon \langle \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \Delta\boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad + \sigma \langle \Phi_{\mathrm{d}}(\boldsymbol{w}), \boldsymbol{w}\rangle_{\mathbb{L}^2} - \varepsilon \langle \Delta\Phi_{\mathrm{d}}(\boldsymbol{w}), \boldsymbol{w}\rangle_{\mathbb{L}^2} + \varepsilon\kappa_2 \left\langle |\boldsymbol{u}|^2\boldsymbol{w}, \Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \varepsilon\kappa_2 \langle ((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v}, \Delta\boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad - \gamma \langle \boldsymbol{v}\times\Delta\boldsymbol{w}, \boldsymbol{w}\rangle_{\mathbb{L}^2} + \gamma \langle \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \boldsymbol{u}\times\boldsymbol{w}\rangle_{\mathbb{L}^2} - \gamma \langle \boldsymbol{v}\times\Phi_{\mathrm{d}}(\boldsymbol{w}), \boldsymbol{w}\rangle_{\mathbb{L}^2} + \langle \mathcal{R}(\boldsymbol{u}) - \mathcal{R}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad + \langle \mathcal{S}(\boldsymbol{u}) - \mathcal{S}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &=: I_1 + I_2 + \cdots + I_{13}. \end{aligned} \tag{4.9}$$

We will estimate each term on the right-hand side above. The first term is kept as is, while the second term is estimated using Young's inequality to obtain

$$|I_2| \le \frac{\sigma\kappa_2}{2} \||\boldsymbol{u}||\boldsymbol{w}|\|_{\mathbb{L}^2}^2 + \frac{\sigma\kappa_2}{2} \|\boldsymbol{v}\cdot\boldsymbol{w}\|_{\mathbb{L}^2}^2.$$

For the terms I_3 and I_4, we apply (2.19) and (2.20) respectively to obtain

$$\begin{aligned} |I_3| &\le \sigma\lambda_1 \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 \\ |I_4| &\le C\varepsilon \left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{8}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2. \end{aligned}$$

For the next two terms, by (2.23) and Young's inequality, we have

$$|I_5| + |I_6| \le \frac{\varepsilon}{8}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2 + C\|\boldsymbol{w}\|_{\mathbb{L}^2}^2.$$

For the terms I_7 and I_8, by Young's inequality we have

$$|I_7| + |I_8| \le \frac{\varepsilon}{8} \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + C\varepsilon \left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 .$$

For the term I_9, similarly we have

$$|I_9| \le \frac{\gamma^2}{\varepsilon} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{4} \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 .$$

For the term I_{10}, we use Young's inequality and (2.20) to obtain

$$|I_{10}| \le C \left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \frac{\sigma\kappa_2}{4} \||\boldsymbol{u}||\boldsymbol{w}|\|_{\mathbb{L}^2}^2 .$$

For the term I_{11}, we apply (2.48) and Young's inequality to obtain

$$|I_{11}| \le C \|\boldsymbol{v}\|_{\mathbb{L}^4} \|\boldsymbol{w}\|_{\mathbb{L}^4} \|\boldsymbol{w}\|_{\mathbb{L}^2} \le C \|\boldsymbol{v}\|_{\mathbb{L}^4}^2 \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \frac{\eta}{4} \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 .$$

Finally, for the last two terms, applying (2.31) and (2.44) we have

$$|I_{12}| + |I_{13}| \le C \left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2\right) \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \frac{\eta}{4} \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 .$$

Collecting all the above estimates and combining them with (4.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 \le C \left(1 + \varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 . \tag{4.10}$$

We note that by Agmon's inequality, Proposition 3.1, and Proposition 3.3,

$$\int_0^t \left(1 + \varepsilon^{-2} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right) \mathrm{d}s \le C(1+t) \left(1 + \varepsilon^{-2} \|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2 + \varepsilon^{-2} \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^4 + \varepsilon^{-2} \|\boldsymbol{v}_0\|_{\mathbb{H}^1}^4\right).$$

Therefore, by the Gronwall inequality, we have the required inequality. □

Lemma 4.8. Let $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ be solutions of (1.1) corresponding to initial data $\boldsymbol{u}_0 \in \mathbb{H}^1$ and $\boldsymbol{v}_0 \in \mathbb{H}^1$, respectively. Then for any $t > 0$,

$$\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_{\mathbb{H}^1}^2 + \varepsilon \int_0^t \|\nabla \Delta \boldsymbol{u}(s) - \nabla \Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \int_0^t \|\Delta \boldsymbol{u}(s) - \Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le Ce^{Ct^3} \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{H}^1}^2 ,$$

where C depends only on $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$, $\|\boldsymbol{v}_0\|_{\mathbb{H}^1}$, and the coefficients of the equation (1.1).

Proof. Taking the inner product of (4.3) with $-\Delta \boldsymbol{w}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 &= -\sigma \langle \boldsymbol{B}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} - \sigma \langle \Phi_{\mathrm{d}}(\boldsymbol{w}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} + \varepsilon \langle \Delta \boldsymbol{B}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} + \varepsilon \langle \Delta \Phi_{\mathrm{d}}(\boldsymbol{w}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad + \gamma \langle \boldsymbol{w} \times \boldsymbol{H}_1, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} + \gamma \langle \boldsymbol{v} \times \boldsymbol{B}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} + \gamma \langle \boldsymbol{w} \times \Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad + \gamma \langle \boldsymbol{v} \times \Phi_{\mathrm{d}}(\boldsymbol{w}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} - \langle \mathcal{R}(\boldsymbol{u}) - \mathcal{R}(\boldsymbol{v}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} - \langle \mathcal{S}(\boldsymbol{u}) - \mathcal{S}(\boldsymbol{v}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} .\end{aligned} \tag{4.11}$$

Taking the inner product of (4.4) with $-\sigma\Delta \boldsymbol{w}$, we have

$$\begin{aligned}-\sigma \langle \boldsymbol{B}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} &= -\sigma \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \kappa_1\sigma \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \kappa_2\sigma \left\langle |\boldsymbol{u}|^2 \boldsymbol{w}, \Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \kappa_2\sigma \langle ((\boldsymbol{u}+\boldsymbol{v})\cdot \boldsymbol{w})\boldsymbol{v}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} \\ &\quad - \sigma \langle \Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} .\end{aligned} \tag{4.12}$$

Furthermore, applying the operator Δ to (4.4) then taking the inner product of the result with $\varepsilon\Delta \boldsymbol{w}$, we have

$$\begin{aligned}\varepsilon \langle \Delta \boldsymbol{B}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} &= -\varepsilon \|\nabla \Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \kappa_1\varepsilon \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 - \kappa_2\varepsilon \left\langle \Delta\big(|\boldsymbol{u}|^2 \boldsymbol{w}\big), \Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad - \kappa_2\varepsilon \left\langle \Delta\big(((\boldsymbol{u}+\boldsymbol{v})\cdot \boldsymbol{w})\boldsymbol{v}\big), \Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \varepsilon \langle \Delta\Phi_{\mathrm{a}}(\boldsymbol{u}) - \Delta\Phi_{\mathrm{a}}(\boldsymbol{v}), \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} .\end{aligned} \tag{4.13}$$

Similarly, using (4.4), we have

$$\begin{aligned}\gamma \langle \boldsymbol{v} \times \boldsymbol{B}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} &= \kappa_1\gamma \langle \boldsymbol{v} \times \boldsymbol{w}, \Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} - \kappa_2\gamma \left\langle \boldsymbol{v} \times |\boldsymbol{u}|^2 \boldsymbol{w}, \Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad + \gamma \left\langle \boldsymbol{v} \times \big(\Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v})\big), \Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} .\end{aligned} \tag{4.14}$$

Writing $\eta = \sigma - \kappa_1\varepsilon$, we add (4.11), (4.12), (4.13), and (4.14) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \varepsilon\|\nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \eta\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2 \\
&= \kappa_1\sigma\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \kappa_2\sigma\left\langle|\boldsymbol{u}|^2\boldsymbol{w},\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \kappa_2\sigma\left\langle((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v},\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - \sigma\left\langle\Phi_{\mathrm{a}}(\boldsymbol{u})-\Phi_{\mathrm{a}}(\boldsymbol{v}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\
&\quad - \sigma\left\langle\Phi_{\mathrm{d}}(\boldsymbol{w}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - \kappa_2\varepsilon\left\langle\Delta\big(|\boldsymbol{u}|^2\boldsymbol{w}\big),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - \kappa_2\varepsilon\left\langle\Delta\big(((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v}\big),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\
&\quad + \varepsilon\left\langle\Delta\Phi_{\mathrm{a}}(\boldsymbol{u})-\Delta\Phi_{\mathrm{a}}(\boldsymbol{v}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \varepsilon\left\langle\Delta\Phi_{\mathrm{d}}(\boldsymbol{w}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \gamma\left\langle\boldsymbol{w}\times\boldsymbol{H}_1,\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\
&\quad + \kappa_1\gamma\left\langle\boldsymbol{v}\times\boldsymbol{w},\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - \kappa_2\gamma\left\langle\boldsymbol{v}\times|\boldsymbol{u}|^2\boldsymbol{w},\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \gamma\left\langle\boldsymbol{v}\times\big(\Phi_{\mathrm{a}}(\boldsymbol{u})-\Phi_{\mathrm{a}}(\boldsymbol{v})\big),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\
&\quad + \gamma\left\langle\boldsymbol{w}\times\Phi_{\mathrm{d}}(\boldsymbol{u}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + \gamma\left\langle\boldsymbol{v}\times\Phi_{\mathrm{d}}(\boldsymbol{w}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - \left\langle\mathcal{R}(\boldsymbol{u})-\mathcal{R}(\boldsymbol{v}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - \left\langle\mathcal{S}(\boldsymbol{u})-\mathcal{S}(\boldsymbol{v}),\Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\
&= I_1 + I_2 + \cdots + I_{17}.
\end{aligned} \tag{4.15}$$

We will estimate each of the seventeen terms above in the following. The first term is kept as is. For the terms I_2 and I_3, we apply Young's inequality and the Sobolev embedding to obtain

$$\begin{aligned}
|I_2| &\le \kappa_2\sigma\|\boldsymbol{u}\|_{\mathbb{L}^6}^2\|\boldsymbol{w}\|_{\mathbb{L}^6}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\|\boldsymbol{u}\|_{\mathbb{H}^1}^4\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2, \\
|I_3| &\le \kappa_2\sigma\|\boldsymbol{u}+\boldsymbol{v}\|_{\mathbb{L}^6}\|\boldsymbol{v}\|_{\mathbb{L}^6}\|\boldsymbol{w}\|_{\mathbb{L}^6}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\left(\|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \|\boldsymbol{v}\|_{\mathbb{H}^1}^4\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2.
\end{aligned}$$

For the next term, using (2.20), Young's inequality, and Sobolev embedding, we have

$$|I_4| \le \sigma\|\Phi_{\mathrm{a}}(\boldsymbol{u})-\Phi_{\mathrm{a}}(\boldsymbol{v})\|_{\mathbb{L}^2}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^1}^4+\|\boldsymbol{v}\|_{\mathbb{H}^1}^4\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2.$$

For the terms I_6 and I_7, integrating by parts, then applying (2.12) and Young's inequality, we have

$$|I_6| \le \kappa_2\varepsilon\left\|\nabla\big(|\boldsymbol{u}|^2\boldsymbol{w}\big)\right\|_{\mathbb{L}^2}\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2} \le 2\kappa_2^2\varepsilon\left(\|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^2\|\Delta\boldsymbol{u}\|_{\mathbb{L}^2}^2\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\varepsilon}{8}\|\nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2,$$

and

$$\begin{aligned}
|I_7| &\le 4\kappa_2^2\varepsilon\left(\|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4+\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right)\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^2 \\
&\quad + 4\kappa_2^2\varepsilon\left(\|\boldsymbol{u}\|_{\mathbb{H}^1}^2+\|\boldsymbol{v}\|_{\mathbb{H}^1}^2\right)\left(\|\Delta\boldsymbol{u}\|_{\mathbb{L}^2}^2+\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\varepsilon}{8}\|\nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2.
\end{aligned}$$

For the term I_8, we integrate by parts, then apply Young's inequality and (2.21) with $p=q=6$ to obtain

$$\begin{aligned}
|I_8| &\le \varepsilon\|\nabla\Phi_{\mathrm{a}}(\boldsymbol{u})-\nabla\Phi_{\mathrm{a}}(\boldsymbol{v})\|_{\mathbb{L}^2}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \\
&\le C\varepsilon\left(1+\|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4+\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + C\varepsilon\left(\|\boldsymbol{u}\|_{\mathbb{H}^1}^2+\|\boldsymbol{v}\|_{\mathbb{H}^1}^2\right)\left(\|\boldsymbol{u}\|_{\mathbb{H}^2}^2+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\varepsilon}{8}\|\nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2,
\end{aligned}$$

where in the last step we also used the Sobolev embedding $\mathbb{H}^2 \hookrightarrow \mathbb{W}^{1,6}$. For the terms I_5 and I_9, by (2.47) and Young's inequality,

$$\begin{aligned}
|I_5| &\le \sigma\|\Phi_{\mathrm{d}}(\boldsymbol{w})\|_{\mathbb{L}^2}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2, \\
|I_9| &\le \varepsilon\|\Delta\Phi_{\mathrm{d}}(\boldsymbol{w})\|_{\mathbb{L}^2}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\varepsilon\|\boldsymbol{w}\|_{\mathbb{H}^2}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2.
\end{aligned}$$

For the term I_{10}, by Hölder's and Young's inequalities, Sobolev embedding, and the definition of $\boldsymbol{H}$, we obtain

$$\begin{aligned}
|I_{10}| &\le \gamma\|\boldsymbol{w}\|_{\mathbb{L}^4}\|\boldsymbol{H}_1\|_{\mathbb{L}^4}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\|\boldsymbol{H}_1\|_{\mathbb{H}^1}^2\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2 \\
&\le C\left(\|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4\|\boldsymbol{u}\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2.
\end{aligned}$$

Similarly, for the next two terms, we have

$$\begin{aligned}
|I_{11}| &\le \kappa_1\gamma\|\boldsymbol{v}\|_{\mathbb{L}^4}\|\boldsymbol{w}\|_{\mathbb{L}^4}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\|\boldsymbol{v}\|_{\mathbb{H}^1}^2\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2, \\
|I_{12}| &\le \kappa_2\gamma\|\boldsymbol{v}\|_{\mathbb{L}^\infty}\|\boldsymbol{u}\|_{\mathbb{L}^6}^2\|\boldsymbol{w}\|_{\mathbb{L}^6}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2} \le C\|\boldsymbol{u}\|_{\mathbb{H}^1}^4\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2.
\end{aligned}$$

For the next term, we used Young's inequality and (2.20) with $p=q=6$ to infer

$$\begin{aligned}|I_{13}| &\leq C\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \|\Phi_{\mathrm{a}}(\boldsymbol{u})-\Phi_{\mathrm{a}}(\boldsymbol{v})\|_{\mathbb{L}^2}^2 + \frac{\eta}{16}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 \\ &\leq C\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \left(1+\|\boldsymbol{u}\|_{\mathbb{H}^1}^2+\|\boldsymbol{v}\|_{\mathbb{H}^1}^2\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 .\end{aligned}$$

For the terms I_{14} and I_{15}, we have by Young's inequality and (2.48),

$$\begin{aligned}|I_{14}| &\leq C\|\Phi_{\mathrm{d}}(\boldsymbol{u})\|_{\mathbb{L}^4}^2 \|\boldsymbol{w}\|_{\mathbb{L}^4}^2 + \frac{\eta}{16}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 \leq C\|\boldsymbol{u}\|_{\mathbb{H}^1}^2\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 , \\ |I_{15}| &\leq C\|\Phi_{\mathrm{d}}(\boldsymbol{w})\|_{\mathbb{L}^4}^2 \|\boldsymbol{v}\|_{\mathbb{L}^4}^2 + \frac{\eta}{16}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 \leq C\|\boldsymbol{v}\|_{\mathbb{H}^1}^2\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 .\end{aligned}$$

Finally, for the last two terms we apply (2.32) and (2.44) to obtain

$$|I_{16}|+|I_{17}| \leq C\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}^2+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{\eta}{16}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 .$$

Altogether, substituting these estimates into (4.15), we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}(t)\|_{\mathbb{H}^1}^2 + \varepsilon\|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}(t)\|_{\mathbb{L}^2}^2 + C\mathcal{B}(\boldsymbol{u},\boldsymbol{v})\|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + C\varepsilon\|\boldsymbol{w}\|_{\mathbb{H}^2}^2 , \tag{4.16}$$

where

$$\mathcal{B}(\boldsymbol{u},\boldsymbol{v}) := 1+\|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4+\|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4+\|\boldsymbol{u}\|_{\mathbb{H}^3}^2+\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^1}^4+\|\boldsymbol{v}\|_{\mathbb{H}^1}^4\right)\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}^2+\|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right).$$

We note that by Agmon's inequality, Proposition 3.10, and Theorem 3.11,

$$\int_0^t \mathcal{B}(\boldsymbol{u},\boldsymbol{v})\,\mathrm{d}s \leq C(1+t^3)\left(1+\varepsilon^{-1}\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2+\|\boldsymbol{u}_0\|_{\mathbb{H}^1}^6+\|\boldsymbol{v}_0\|_{\mathbb{H}^1}^6\right).$$

Therefore, applying the Gronwall inequality on (4.16) and noting Lemma 4.7, we have the required result. ∎

The following lemma shows a smoothing estimate for the difference of two solutions originating from different initial data.

Lemma 4.9. Let $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ be solutions of (1.1) corresponding to initial data $\boldsymbol{u}_0\in\mathbb{H}^1$ and $\boldsymbol{v}_0\in\mathbb{H}^1$, respectively. Then for any $t>0$,

$$\|\boldsymbol{u}(t)-\boldsymbol{v}(t)\|_{\mathbb{H}^2}^2 \leq C(1+t^{-1})e^{Ct^3}\|\boldsymbol{u}_0-\boldsymbol{v}_0\|_{\mathbb{H}^1}^2 ,$$

where C depends only on $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$, $\|\boldsymbol{v}_0\|_{\mathbb{H}^1}$, and the coefficients of the equation (1.1).

Proof. Taking the inner product of (4.3) with $t\Delta^2\boldsymbol{w}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2\right) &= \frac{1}{2}\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + t\sigma\langle\Delta \boldsymbol{B},\Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} + t\sigma\left\langle\Phi_{\mathrm{d}}(\boldsymbol{w}),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\varepsilon\left\langle\Delta \boldsymbol{B},\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad - t\varepsilon\left\langle\Delta\Phi_{\mathrm{d}}(\boldsymbol{w}),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\gamma\left\langle\boldsymbol{w}\times\boldsymbol{H}_1,\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\gamma\left\langle\boldsymbol{v}\times\boldsymbol{B},\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad - t\gamma\left\langle\boldsymbol{w}\times\Phi_{\mathrm{d}}(\boldsymbol{u}),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\gamma\left\langle\boldsymbol{v}\times\Phi_{\mathrm{d}}(\boldsymbol{w}),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad + t\left\langle\mathcal{R}(\boldsymbol{u})-\mathcal{R}(\boldsymbol{v}),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\left\langle\mathcal{S}(\boldsymbol{u})-\mathcal{S}(\boldsymbol{v}),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} .\end{aligned} \tag{4.17}$$

Applying the operator Δ to (4.4), then taking the inner product of the result with $t\sigma\Delta\boldsymbol{w}$, we have

$$\begin{aligned}t\sigma\langle\Delta \boldsymbol{B},\Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} &= -t\sigma\|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + t\kappa_1\sigma\|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 - t\kappa_2\sigma\left\langle\Delta(|\boldsymbol{u}|^2\boldsymbol{w}),\Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad + t\kappa_2\sigma\left\langle\Delta\big(((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v}\big),\Delta \boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\sigma\langle\Delta\Phi_{\mathrm{a}}(\boldsymbol{u})-\Delta\Phi_{\mathrm{a}}(\boldsymbol{v}),\Delta \boldsymbol{w}\rangle_{\mathbb{L}^2} .\end{aligned} \tag{4.18}$$

Similarly, applying the operator Δ to (4.4) then taking the inner product of the result with $-t\varepsilon\Delta^2\boldsymbol{w}$, we obtain

$$\begin{aligned}-t\varepsilon\left\langle\Delta \boldsymbol{B},\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} &= -t\varepsilon\left\|\Delta^2\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 + t\kappa_1\varepsilon\|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + t\kappa_2\varepsilon\left\langle\Delta\big(|\boldsymbol{u}|^2\boldsymbol{w}\big),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad + t\kappa_2\varepsilon\left\langle\Delta\big(((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v}\big),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\varepsilon\left\langle\Delta\Phi_{\mathrm{a}}(\boldsymbol{u})-\Delta\Phi_{\mathrm{a}}(\boldsymbol{v}),\Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} .\end{aligned} \tag{4.19}$$

Similarly, using (4.4), we have

$$\begin{aligned}-t\gamma \left\langle \boldsymbol{v}\times \boldsymbol{B}, \Delta^2 \boldsymbol{w}\right\rangle_{\mathbb{L}^2} &= -t\kappa_1\gamma \left\langle \boldsymbol{v}\times\boldsymbol{w}, \Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\kappa_2\gamma\left\langle \boldsymbol{v}\times |\boldsymbol{u}|^2\boldsymbol{w}, \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad - t\gamma\left\langle \boldsymbol{v}\times \big(\Phi_{\mathrm{a}}(\boldsymbol{u})-\Phi_{\mathrm{a}}(\boldsymbol{v})\big), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2}. \end{aligned}\tag{4.20}$$

Writing $\eta = \sigma - \kappa_1\varepsilon$, we add (4.17), (4.18), (4.19), and (4.20) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t\left\|\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2\right) + t\varepsilon\left\|\Delta^2\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 + t\eta\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 \\ &= \left(\frac{1}{2}+t\kappa_1\sigma\right)\left\|\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 - t\kappa_2\sigma\left\langle \Delta(|\boldsymbol{u}|^2\boldsymbol{w}), \Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\kappa_2\sigma\left\langle \Delta\big(((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v}\big), \Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad + t\sigma\left\langle \Delta\Phi_{\mathrm{a}}(\boldsymbol{u})-\Delta\Phi_{\mathrm{a}}(\boldsymbol{v}), \Delta\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\sigma\left\langle \Phi_{\mathrm{d}}(\boldsymbol{w}), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\kappa_2\varepsilon\left\langle \Delta\big(|\boldsymbol{u}|^2\boldsymbol{w}\big), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad + t\kappa_2\varepsilon\left\langle \Delta\big(((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w})\boldsymbol{v}\big), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\varepsilon\left\langle \Delta\Phi_{\mathrm{a}}(\boldsymbol{u})-\Delta\Phi_{\mathrm{a}}(\boldsymbol{v}), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\varepsilon\left\langle \Delta\Phi_{\mathrm{d}}(\boldsymbol{w}), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad - t\gamma\left\langle \boldsymbol{w}\times\boldsymbol{H}_1, \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\kappa_1\gamma\left\langle \boldsymbol{v}\times\boldsymbol{w}, \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\kappa_2\gamma\left\langle \boldsymbol{v}\times|\boldsymbol{u}|^2\boldsymbol{w}, \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad - t\gamma\left\langle \boldsymbol{v}\times\big(\Phi_{\mathrm{a}}(\boldsymbol{u})-\Phi_{\mathrm{a}}(\boldsymbol{v})\big), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\gamma\left\langle \boldsymbol{w}\times\Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} - t\gamma\left\langle \boldsymbol{v}\times\Phi_{\mathrm{d}}(\boldsymbol{w}), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &\quad + t\left\langle \mathcal{R}(\boldsymbol{u})-\mathcal{R}(\boldsymbol{v}), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} + t\left\langle \mathcal{S}(\boldsymbol{u})-\mathcal{S}(\boldsymbol{v}), \Delta^2\boldsymbol{w}\right\rangle_{\mathbb{L}^2} \\ &= J_1 + J_2 + \cdots + J_{17}. \end{aligned}\tag{4.21}$$

It remains to estimate each of the terms in the last line. The first term is left as is. For the second term, integrating by parts, then applying Hölder's and Young's inequalities, we obtain

$$\begin{aligned}|J_2| &\leq t\kappa_2\sigma\left\|\nabla(|\boldsymbol{u}|^2\boldsymbol{w})\right\|_{\mathbb{L}^2}\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2} \\ &\leq Ct\left\|\boldsymbol{u}\right\|_{\mathbb{L}^6}^2\left\|\nabla\boldsymbol{u}\right\|_{\mathbb{L}^6}^2\left\|\boldsymbol{w}\right\|_{\mathbb{L}^6}^2 + Ct\left\|\boldsymbol{u}\right\|_{\mathbb{L}^6}^4\left\|\nabla\boldsymbol{w}\right\|_{\mathbb{L}^6}^2 + \frac{t\eta}{16}\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 \\ &\leq Ct\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^2\left\|\boldsymbol{u}\right\|_{\mathbb{H}^2}^2\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + Ct\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^4\left\|\boldsymbol{w}\right\|_{\mathbb{H}^2}^2 + \frac{t\eta}{16}\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2,\end{aligned}$$

where in the last step we used the Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$. Similarly, for the third term we have

$$|J_3| \leq Ct\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^1}^2\right)\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^2}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^2}^2\right)\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + Ct\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^4+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^1}^4\right)\left\|\boldsymbol{w}\right\|_{\mathbb{H}^2}^2 + \frac{t\eta}{16}\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2.$$

For the term J_4, we integrate by parts, then apply (2.21) with $p=q=6$ and Young's inequality to obtain

$$|J_4| \leq Ct\left(1+\left\|\boldsymbol{u}\right\|_{\mathbb{L}^\infty}^4+\left\|\boldsymbol{v}\right\|_{\mathbb{L}^\infty}^4\right)\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + Ct\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^1}^2\right)\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^2}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^2}^2\right)\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + \frac{t\eta}{16}\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2.$$

For the next term, integrating by parts once and applying (2.48) we have

$$|J_5| \leq Ct\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + \frac{t\eta}{16}\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2.$$

For the term J_6, by (2.14), Hölder's and Young's inequalities, we have

$$\begin{aligned}|J_6| &\leq Ct\varepsilon\left\|\nabla\boldsymbol{u}\right\|_{\mathbb{L}^6}^4\left\|\boldsymbol{w}\right\|_{\mathbb{L}^6}^2 + Ct\varepsilon\left\|\boldsymbol{u}\right\|_{\mathbb{L}^6}^2\left\|\Delta\boldsymbol{u}\right\|_{\mathbb{L}^6}^2\left\|\boldsymbol{w}\right\|_{\mathbb{L}^6}^2 + Ct\varepsilon\left\|\nabla\boldsymbol{w}\right\|_{\mathbb{L}^2}^2\left\|\boldsymbol{u}\right\|_{\mathbb{L}^\infty}^2\left\|\nabla\boldsymbol{u}\right\|_{\mathbb{L}^\infty}^2 \\ &\quad + Ct\varepsilon\left\|\boldsymbol{u}\right\|_{\mathbb{L}^6}^4\left\|\Delta\boldsymbol{w}\right\|_{\mathbb{L}^6}^2 + \frac{t\varepsilon}{16}\left\|\Delta^2\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 \\ &\leq Ct\varepsilon\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^2\left\|\boldsymbol{u}\right\|_{\mathbb{H}^3}^2\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + Ct\varepsilon\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^4\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 + \frac{t\varepsilon}{16}\left\|\Delta^2\boldsymbol{w}\right\|_{\mathbb{L}^2}^2,\end{aligned}$$

where in the last step we used the Gagliardo–Nirenberg inequalities. Similarly, for the term J_7 we have

$$|J_7| \leq Ct\varepsilon\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^1}^2\right)\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^3}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^3}^2\right)\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + Ct\varepsilon\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^4+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^1}^4\right)\left\|\nabla\Delta\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 + \frac{t\varepsilon}{16}\left\|\Delta^2\boldsymbol{w}\right\|_{\mathbb{L}^2}^2.$$

For the term J_8, by Young's inequality and (2.22),

$$\begin{aligned}|J_8| &\leq Ct\varepsilon\left\|\boldsymbol{w}\right\|_{\mathbb{L}^2}^2 + Ct\varepsilon\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^4+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^1}^4\right)\left\|\Delta\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 \\ &\quad + Ct\varepsilon\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^1}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^1}^2\right)\left(\left\|\boldsymbol{u}\right\|_{\mathbb{H}^3}^2+\left\|\boldsymbol{v}\right\|_{\mathbb{H}^3}^2\right)\left\|\boldsymbol{w}\right\|_{\mathbb{H}^1}^2 + \frac{t\varepsilon}{16}\left\|\Delta^2\boldsymbol{w}\right\|_{\mathbb{L}^2}^2.\end{aligned}$$

For the next term, by (2.48) and Young's inequality, we have

$$|J_9| \leq Ct\varepsilon \|\boldsymbol{w}\|_{\mathbb{H}^2}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2.$$

For the term J_{10}, by Young's inequality, Sobolev embedding, and the definition of $\boldsymbol{H}_1$,

$$\begin{aligned}|J_{10}| &\leq Ct\varepsilon^{-1} \|\boldsymbol{w}\|_{\mathbb{L}^6}^2 \|\boldsymbol{H}_1\|_{\mathbb{L}^3}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2 \\ &\leq Ct\varepsilon^{-1} \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 \left(\|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 \|\boldsymbol{u}\|_{\mathbb{L}^\infty}\right) + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2.\end{aligned}$$

Similarly, for the next two terms, applying Young's inequality and Sobolev embedding, we have

$$\begin{aligned}|J_{11}| &\leq Ct\varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{L}^4}^2 \|\boldsymbol{w}\|_{\mathbb{L}^4}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2 \leq Ct\varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{H}^1}^2 \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2, \\ |J_{12}| &\leq Ct\varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{u}\|_{\mathbb{L}^6}^4 \|\boldsymbol{w}\|_{\mathbb{L}^6}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2 \leq Ct\varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2.\end{aligned}$$

For the term J_{13}, using Young's inequality and (2.20) with $p = q = 6$, we obtain

$$\begin{aligned}|J_{13}| &\leq Ct\varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \|\Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v})\|_{\mathbb{L}^2}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2 \\ &\leq Ct\varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \|\boldsymbol{v}\|_{\mathbb{H}^1}^4\right) \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2.\end{aligned}$$

For the terms J_{14} and J_{15}, we integrate by parts and use Young's inequality and (2.48) to obtain

$$\begin{aligned}|J_{14}| &\leq Ct \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + Ct \|\boldsymbol{w}\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 + \frac{t\eta}{16} \|\nabla \Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2, \\ |J_{15}| &\leq Ct \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 + Ct \|\boldsymbol{w}\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \frac{t\eta}{16} \|\nabla \Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2.\end{aligned}$$

Finally, for the last two terms, by (2.32) and (2.44) we have

$$|J_{16}| + |J_{17}| \leq Ct\varepsilon^{-1} \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + \|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right) \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{t\varepsilon}{16} \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2.$$

Altogether, substituting these estimates into (4.21) (and applying Agmon's inequality), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \left(t \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2\right) + t\varepsilon \left\|\Delta^2 \boldsymbol{w}\right\|_{\mathbb{L}^2}^2 + t\eta \|\nabla \Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 &\leq Ct \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \|\boldsymbol{v}\|_{\mathbb{H}^1}^4\right) \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + \|\boldsymbol{v}\|_{\mathbb{H}^3}^2\right) \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 \\ &\quad + C \|\boldsymbol{w}\|_{\mathbb{H}^2}^2 + Ct \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \|\boldsymbol{v}\|_{\mathbb{H}^1}^4\right) \|\boldsymbol{w}\|_{\mathbb{H}^3}^2.\end{aligned}$$

Integrating both sides over $(0, t)$, then using Proposition 3.10, Theorem 3.11, and Lemma 4.8, we have

$$t \|\Delta \boldsymbol{w}(t)\|_{\mathbb{L}^2}^2 \leq C(1+t) e^{Ct^3} \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{H}^1}^2,$$

where C depends only on the coefficients of the equation, $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$ and $\|\boldsymbol{v}_0\|_{\mathbb{H}^1}$, as required. □

4.3. **Existence of global attractor.** In light of Theorem 3.11, for $k = 1$ or 2, the system (1.1) generates a strongly continuous semigroup

$$\boldsymbol{S}(t) : \mathbb{H}^k \to \mathbb{H}^k, \quad \boldsymbol{S}(t)\boldsymbol{u}_0 = \boldsymbol{u}(t) \quad \text{for } t \geq 0, \tag{4.22}$$

and thus $\left(\mathbb{H}^k, \{\boldsymbol{S}(t)\}_{t\geq 0}\right)$ is a semi-dynamical system.

The following theorem on the existence of global attractor for (4.22) is immediate.

Theorem 4.10. For $k = 1$ or 2, the semi-dynamical system $\left(\mathbb{H}^k, \{\boldsymbol{S}(t)\}_{t\geq 0}\right)$ generated by (1.1) has a connected global attractor $\mathcal{A}$ in the sense of Definition 4.1.

Proof. For $k = 1$ or 2, the existence of a compact absorbing set is furnished by (3.71). Noting that the embedding $\mathbb{H}^{k+1}(\mathscr{O}) \subset \mathbb{H}^k(\mathscr{O})$ is compact (for a regular bounded domain $\mathscr{O}$) and applying Theorem 4.2, we have the result. □

We now consider a special case of (1.1) where the spin current is not present and the only contribution to the effective magnetic field is the exchange field and the , i.e. $\mathcal{R}(\boldsymbol{u})$, $\Phi_{\mathrm{a}}(\boldsymbol{u})$, and $\Phi_{\mathrm{d}}(\boldsymbol{u})$ are all set to zero. In this case, we obtain that the set of fixed points of $S(t)$ defined by (4.22) is

$$\mathcal{N} = \big\{\boldsymbol{u} \in \mathrm{D}(\Delta) : \Delta \boldsymbol{u} + \kappa_1 \boldsymbol{u} - \kappa_2 |\boldsymbol{u}|^2 \boldsymbol{u} = \boldsymbol{0}\big\}. \tag{4.23}$$

This can be seen by formally setting $\partial_t \boldsymbol{u} = \boldsymbol{0}$, taking dot product of the first equation in (1.1) with $\boldsymbol{H}$, and integrating by parts. The set of fixed points (4.23) corresponds to solutions of the stationary vector-valued Allen–Cahn equation, the structure of which is still an area of active research [1, 50]. Next, we show that $\boldsymbol{S}(t)$ admits a global Lyapunov function.

Proposition 4.11. The continuous function $\mathcal{L} : \mathbb{H}^1 \to \mathbb{R}$ defined by

$$\mathcal{L}\big(\boldsymbol{u}(t)\big) := \frac{1}{2} \|\nabla \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \frac{\kappa_2}{4} \big\||\boldsymbol{u}(t)|^2 - \kappa_1/\kappa_2\big\|_{\mathbb{L}^2}^2$$

is a global Lyapunov function for $\boldsymbol{S}(t)$ in the sense of Definition 4.3.

Proof. Applying the same argument leading to (3.20) yields the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\boldsymbol{u}(t)) = -\langle \partial_t \boldsymbol{u}(t), \boldsymbol{H}(t)\rangle_{\mathbb{L}^2} = -\sigma \|\boldsymbol{H}(t)\|_{\mathbb{L}^2}^2 - \varepsilon \|\nabla \boldsymbol{H}(t)\|_{\mathbb{L}^2}^2 \leq 0, \tag{4.24}$$

which shows that the function $t \mapsto \mathcal{L}(\boldsymbol{S}(t)\boldsymbol{u}_0)$ is non-increasing. Furthermore, if $\mathcal{L}(\boldsymbol{S}(T)\boldsymbol{u}_0) = \mathcal{L}(\boldsymbol{u}_0)$, then integrating (4.24) over $(0,T)$ gives $\boldsymbol{H}(t) = 0$ for all $t \in [0,T]$. This implies $\partial_t \boldsymbol{u} = 0$, i.e. $\boldsymbol{u}(t) = \boldsymbol{u}_0$ for all $t \in [0,T]$, which shows $\boldsymbol{u}_0$ is a fixed point. This proves $\mathcal{L}$ is a global Lyapunov function for $\boldsymbol{S}(t)$. □

Theorem 4.12. Let $\mathcal{A}$ be the global attractor for $\boldsymbol{S}(t)$ and $\mathcal{F}$ be its set of fixed points (4.23). Then $\omega(\boldsymbol{u}_0) \subset \mathcal{F}$ for every $\boldsymbol{u}_0 \in \mathbb{H}^r$. Moreover, $\mathcal{A} = \mathcal{M}^{\mathrm{un}}(\mathcal{F})$.

Proof. This follows from Theorem 4.10 and Proposition 4.4. □

Furthermore, we will show that in fact $\boldsymbol{u}(t)$ converges to some function $\varphi \in \omega(\boldsymbol{u}) \subset \mathcal{F}$ as $t \to +\infty$ with some upper estimates on the rate. To this end, we will use some results on the Łojasiewicz–Simon gradient inequality applied to a gradient-like system [4, 10, 11].

Theorem 4.13. There exists $\boldsymbol{\varphi} \in \omega(\boldsymbol{u})$ such that

$$\lim_{t\to+\infty} \|\boldsymbol{u}(t) - \boldsymbol{\varphi}\|_{\mathbb{H}^1} = 0.$$

Moreover, as $t \to +\infty$,

$$\|\boldsymbol{u}(t) - \boldsymbol{\varphi}\|_{\mathbb{H}^{-1}} = \begin{cases} O(e^{-ct}) & \text{if } \theta = \frac{1}{2}, \\ O\big(t^{-\theta/(1-2\theta)}\big) & \text{if } \theta = (0, \frac{1}{2}), \end{cases}$$

where θ is the Łojasiewicz exponent of $\mathcal{E}$ and c is a constant.

Proof. We will apply [11, Theorem 1 and 2]. The global Lyapunov function $\mathcal{L}$ satisfies the Łojasiewicz–Simon gradient inequality, as shown in [10] and [54, Section 3.6], so it remains to verify the so-called angle and comparability conditions [4, conditions (AC+C)]. In our context, we will show that there is a constant $C > 0$ such that

$$\big\langle \mathcal{L}'(\boldsymbol{u}), -\sigma \boldsymbol{H} + \varepsilon \Delta \boldsymbol{H} + \gamma \boldsymbol{u} \times \boldsymbol{H} \big\rangle_{(\mathbb{H}^1)^*, \mathbb{H}^1} \geq C \left(\big\|\mathcal{L}'(\boldsymbol{u})\big\|_{\mathbb{H}^{-1}}^2 + \|-\sigma \boldsymbol{H} + \varepsilon \Delta \boldsymbol{H} + \gamma \boldsymbol{u} \times \boldsymbol{H}\|_{\mathbb{H}^{-1}}^2 \right) \tag{4.25}$$

for all $\boldsymbol{u}$ in a neighbourhood of $\boldsymbol{\varphi} \in \omega(\boldsymbol{u})$, i.e. $\boldsymbol{u} \in N_R(\boldsymbol{\varphi}) := \{\boldsymbol{z} \in \omega(\boldsymbol{u}) : \|\boldsymbol{z} - \boldsymbol{\varphi}\|_{\mathbb{H}^1} \leq R\}$. We have $\mathcal{L}'(\boldsymbol{u}) = -\boldsymbol{H}$ in $\mathbb{H}^{-1}$. Therefore, noting that

$$\big\langle \mathcal{L}'(\boldsymbol{u}), -\sigma \boldsymbol{H} + \varepsilon \Delta \boldsymbol{H} + \gamma \boldsymbol{u} \times \boldsymbol{H} \big\rangle_{(\mathbb{H}^1)^*, \mathbb{H}^1} = \sigma \|\boldsymbol{H}\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \boldsymbol{H}\|_{\mathbb{L}^2}^2, \tag{4.26}$$

we obtain

$$\begin{aligned} \big\|\mathcal{L}'(\boldsymbol{u})\big\|_{\mathbb{H}^{-1}}^2 + \|-\sigma \boldsymbol{H} + \varepsilon \Delta \boldsymbol{H} + \gamma \boldsymbol{u} \times \boldsymbol{H}\|_{\mathbb{H}^{-1}}^2 &\leq \|\boldsymbol{H}\|_{\mathbb{H}^{-1}}^2 + \sigma \|\boldsymbol{H}\|_{\mathbb{H}^{-1}}^2 + \varepsilon \|\Delta \boldsymbol{H}\|_{\mathbb{H}^{-1}}^2 + \gamma \|\boldsymbol{u} \times \boldsymbol{H}\|_{\mathbb{H}^{-1}}^2 \\ &\leq C \|\boldsymbol{H}\|_{\mathbb{L}^2}^2 + \varepsilon \|\boldsymbol{H}\|_{\mathbb{H}^1}^2 + C \|\boldsymbol{u}\|_{\mathbb{L}^3}^2 \|\boldsymbol{H}\|_{\mathbb{L}^2}^2 \\ &\leq C \left(\sigma \|\boldsymbol{H}\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \boldsymbol{H}\|_{\mathbb{L}^2}^2 \right), \end{aligned}$$

where we used the embedding $\mathbb{L}^{6/5} \subset \mathbb{H}^{-1}$, Hölder's inequality, the fact that $\boldsymbol{u} \in N_R(\boldsymbol{\varphi})$, and (3.23). This proves (4.25), thus completing the proof of the theorem. ☐

4.4. **Existence of exponential attractor.** It is known that, while the global attractor has many desirable properties as an appropriate object to study when considering long-time behaviour, it may attract trajectories at a very slow rate and is sensitive to perturbation. Furthermore, it is in general very difficult to express the convergence rate only in terms of the physical parameters of the problem. As such, it is argued in [17] that one should consider a larger object which are more robust under perturbation, attract trajectories at a fast rate, but are still finite dimensional. Such an object is called an exponential attractor, whose construction is explained in [40].

Definition 4.14 (Exponential attractor)**.** A subset $\mathcal{M} \subseteq X$ is an *exponential attractor* for $S(t)$ if

(1) it is compact in X,
(2) it has finite fractal dimension, $\dim_{\mathrm{F}}\mathcal{M} < +\infty$,
(3) it is semi-invariant, i.e. $S(t)\mathcal{M} \subseteq \mathcal{M}$, $\forall t \geq 0$,
(4) it attracts exponentially fast bounded subsets of X in the following sense: for all bounded set $B \subset X$, there exists a constant c depending on B and $\alpha \geq 0$ such that
$$\operatorname{dist}\big(S(t)B, \mathcal{M}\big) \leq ce^{-\alpha t}, \quad \forall t \geq 0.$$

Therefore, an exponential attractor, if it exists, contains the global attractor and implies the finite dimensionality of the global attractor. The existence of an exponential attractor has been shown for various models of physical significance. To show the existence of an exponential attractor, we follow the general ideas from [40] (also see [41]).

Theorem 4.15 ([40])**.** Let X and H be two Hilbert spaces such that $X \hookrightarrow H$ is a compact embedding, and let $S(t) : E \to E$ be a strongly continuous semigroup acting on a subset $E \subseteq X$. For a fixed $R > 0$, let
$$B_R := \{x \in E : \|x\|_H \leq R\}.$$
Suppose that

(1) the smoothing property holds, i.e. for all $x_1, x_2 \in B_R$, and $t > 0$,
$$\|S(t)x_1 - S(t)x_2\|_X \leq h(t) \|x_1 - x_2\|_H ,$$
where h is a continuous function of t, which may depend on R.
(2) for any $T > 0$ and $x \in B_R$, the map $t \mapsto S(t)x$ is Hölder continuous on $[0, T]$,
(3) for any $t \in [0, T]$, the map $x \mapsto S(t)x$ is Lipschitz continuous on B_R.

Then $S(t)$ possesses an exponential attractor $\mathcal{M}$ on H.

Theorem 4.16. The semi-dynamical system generated by (1.1) has an exponential attractor $\mathcal{M}$ on $\mathbb{H}^1$ in the sense of Definition 4.14.

Proof. It remains to verify the conditions in Theorem 4.15, where $X = \mathbb{H}^2$ and $E = H = \mathbb{H}^1$. Smoothing property is inferred from Lemma 4.9. Lipschitz continuity on B_R is given by Lemma 4.8. Hölder continuity in time can be shown as in [48, Theorem 2.3]. This completes the proof of the theorem. ☐

5. Further properties of the attractor

In this section, we will study further properties of the attractor, such as its fractal dimension as well as the existence of a family of exponential attractors for (1.1) which are continuous with respect to the parameter ε. The latter also implies the existence of an exponential attractor, hence also global attractor with finite fractal dimension, for the Landau–Lifshitz–Bloch equation with spin torque terms. Throughout this section, we will write $\boldsymbol{S}_\varepsilon(t)$ for the semigroup generated by the system (1.1) to highlight the dependence on ε.

5.1. **Fractal dimension of the global attractor.** We aim find an upper bound for the fractal dimension of the global attractor. First, we need the following lemma.

Lemma 5.1. Let $\boldsymbol{S}_\varepsilon(t)$ be the semigroup generated by (1.1) with global attractor $\mathcal{A}$. Let $\boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathcal{A}$, and write $\boldsymbol{u}(t) := \boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0$ and $\boldsymbol{v}(t) := \boldsymbol{S}_\varepsilon(t)\boldsymbol{v}_0$. For any $t > 0$,

$$\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_{\mathbb{L}^2}^2 + \varepsilon \int_0^t \|\Delta \boldsymbol{u}(s) - \Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \int_0^t \|\nabla \boldsymbol{u}(s) - \nabla \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \leq C e^{C\varepsilon^{-5} t} \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{L}^2}^2 . \tag{5.1}$$

Moreover,

$$\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_{\mathbb{H}^1}^2 \leq C t^{-1} \left(1 + t\varepsilon^{-8}\right) e^{C\varepsilon^{-5} t} \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{L}^2}^2 , \tag{5.2}$$

where C is independent of $\boldsymbol{u}_0$, ε, and t.

Proof. Let $\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t)$. We have, as in (4.10),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 + \varepsilon \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + \|\nabla \boldsymbol{w}\|_{\mathbb{L}^2}^2 &\leq C \left(1 + \varepsilon^{-1} \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^2 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 \\ &\leq C \left(1 + \varepsilon^{-5}\right) \|\boldsymbol{w}\|_{\mathbb{L}^2}^2 , \end{aligned}$$

where in the last step we used the Gagliardo–Nirenberg inequalities, (3.23), and (3.42), noting that $\boldsymbol{u}(t), \boldsymbol{v}(t) \in \mathcal{A}$. An application of the Gronwall inequality gives (5.1).

By similar argument as in the proof of Lemma 4.8, but instead successively taking the inner product with $-t\Delta \boldsymbol{w}$ in (4.11), with $-\sigma t \Delta \boldsymbol{w}$ in (4.12), and with $\varepsilon t \Delta^2 \boldsymbol{w}$ in (4.13), we obtain an inequality analogous to (4.16):

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(t \|\boldsymbol{w}\|_{\mathbb{H}^1}^2\right) + \varepsilon t \|\nabla \Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 + t \|\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2 \leq \frac{\mathrm{d}}{\mathrm{d}t} \left(t \|\boldsymbol{w}\|_{\mathbb{L}^2}^2\right) + \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + Ct\, \mathcal{B}(\boldsymbol{u}, \boldsymbol{v}) \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + Ct\varepsilon \|\boldsymbol{w}\|_{\mathbb{H}^2}^2 ,$$

where

$$\mathcal{B}(\boldsymbol{u}, \boldsymbol{v}) := 1 + \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^4 + \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \|\boldsymbol{v}\|_{\mathbb{H}^1}^4\right) \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + \|\boldsymbol{v}\|_{\mathbb{H}^2}^2\right).$$

Integrating over $(0, t)$, noting (3.67) and using (5.1), we obtain (5.2). ☐

The following theorem shows that the fractal dimension of the global attractor is finite.

Theorem 5.2. Let $\mathcal{A}$ be the global attractor of the semi-dynamical system generated by (1.1). Then

$$\dim_{\mathrm{F}} \mathcal{A} \leq C \varepsilon^{-4d},$$

where C may depend on the coefficients of (1.1), but is independent of ε. In particular, $\mathcal{A}$ has a finite fractal dimension.

Proof. We take $t = \varepsilon^8$ in (5.2) to obtain

$$\left\|\boldsymbol{S}_\varepsilon(\varepsilon^8)\boldsymbol{u}_0 - \boldsymbol{S}_\varepsilon(\varepsilon^8)\boldsymbol{v}_0\right\|_{\mathbb{H}^1} \leq C\varepsilon^{-4} \|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{L}^2} , \quad \forall \boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathcal{A}$$

where C is a constant independent of ε (but may depend on other parameters of (1.1)). Theorem 4.6 and Theorem A.3 then imply the required result. ☐

5.2. **A family of robust exponential attractors.** It is known that in general, the global attractor is sensitive to a perturbation of parameter. There exist abstract conditions that guarantee continuous dependence of the global attractor on a parameter, however it is difficult to verify in practice [57].

Here, we show the robustness (continuity) of the exponential attractor with respect to the parameter ε for the case $d \leq 2$ and $\lambda_2 = 0$, i.e. the higher-order term of the anisotropy field Φ_{a} is assumed to be negligible (which is physically reasonable). To this end, we need to obtain estimates for the solution and the difference of two solutions which are *uniform in* ε. First, we derive uniform estimates analogous to Proposition 3.1, Proposition 3.2, and Proposition 3.3 in the following lemma. We write $\eta := \sigma - \kappa_1 \varepsilon > 0$ as before.

Lemma 5.3. Let $\boldsymbol{u}$ be the solution of (1.1) with initial data $\boldsymbol{u}_0 \in \mathbb{H}^1$, whose corresponding effective field is $\boldsymbol{H}$. The following statements hold:

(1) For all $t \geq 0$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 \leq C \|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2 , \tag{5.3}$$

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2 \leq Ce^{Ct} \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2 , \tag{5.4}$$

where C is independent of ε, t, and $\boldsymbol{u}_0$.

(2) For all $t \geq 0$,

$$\int_0^t \|\boldsymbol{u}(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s + \int_0^t \|\nabla \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \varepsilon \int_0^t \||\boldsymbol{u}(s)||\nabla \boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \leq C(1+t) \|\boldsymbol{u}_0\|_{\mathbb{L}^2}^2 , \tag{5.5}$$

$$\int_0^t \|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2 + \int_0^t \|\Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \varepsilon \int_0^t \|\nabla \Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \int_0^t \||\boldsymbol{u}(s)||\nabla \boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \leq Ce^{Ct} \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2 , \tag{5.6}$$

where C is independent of ε, t, and $\boldsymbol{u}_0$.

(3) There exists t_1 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ (but is independent of ε) such that for all $t \geq t_1$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2 + \int_t^{t+1} \left(\|\Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}(s)||\nabla \boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2 \right) \mathrm{d}s \leq \rho_1, \tag{5.7}$$

where ρ_1 is independent of $\boldsymbol{u}_0$, ε, and t.

(4) Let $\delta > 0$ be arbitrary. For all $t \geq \delta$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2 + \int_t^{t+\delta} \left(\|\Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}(s)||\nabla \boldsymbol{u}(s)|\|_{\mathbb{L}^2}^2 \right) \mathrm{d}s \leq \mu_1, \tag{5.8}$$

where μ_1 depends on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of ε and t.

(5) For all $t > 0$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^1}^2 \leq M_0 \left(1 + t + t^{-1}\right), \tag{5.9}$$

where M_0 is independent of ε and t, but may depend on other coefficients of (1.1), $|\mathscr{O}|$, and $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$.

Proof. The proof of (5.3) and (5.5) is established in Proposition 3.1, noting that the estimates in Proposition 3.1 is already uniform in ε. Next, we show (5.4) for $k = 1$. Taking the inner product of both equations in (1.1) with $-\Delta \boldsymbol{u}$ and rearranging the terms, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \eta \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + 2\kappa_2 \|\boldsymbol{u} \cdot \nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \kappa_2 \||\boldsymbol{u}||\nabla \boldsymbol{u}|\|_{\mathbb{L}^2}^2 \\
&= -\sigma \langle \Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta \boldsymbol{u} \rangle_{\mathbb{L}^2} - \varepsilon\kappa_2 \left\langle \nabla(|\boldsymbol{u}|^2 \boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} + \varepsilon \langle \Delta \Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta \boldsymbol{u} \rangle_{\mathbb{L}^2} \\
&\quad + \gamma \langle \boldsymbol{u} \times \Phi_{\mathrm{a}}(\boldsymbol{u}), \Delta \boldsymbol{u} \rangle_{\mathbb{L}^2} + \gamma \langle \boldsymbol{u} \times \Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta \boldsymbol{u} \rangle_{\mathbb{L}^2} - \langle \mathcal{R}(\boldsymbol{u}), \Delta \boldsymbol{u} \rangle_{\mathbb{L}^2} - \langle \mathcal{S}(\boldsymbol{u}), \Delta \boldsymbol{u} \rangle_{\mathbb{L}^2} \\
&=: I_1 + I_2 + \cdots + I_7.
\end{aligned} \tag{5.10}$$

We will estimate each term on the last line. For the first and the third terms, by Young's inequality and (2.48), we have

$$|I_1| \leq C \|\boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{6} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 ,$$

$$|I_3| \leq C\varepsilon^2 \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + \frac{\eta}{6} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 .$$

For the second term, by (2.12), Young's and Agmon's inequalities (for $d \leq 2$), we have

$$|I_2| \leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{4} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 \leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{4} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 .$$

For the term I_4 and I_5 (noting that we assumed $\lambda_2 = 0$), similarly we have

$$|I_4| + |I_5| \leq C \|\boldsymbol{u}\|_{\mathbb{L}^4}^4 + \frac{\eta}{6} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 .$$

Finally, for the terms I_6 and I_7, we apply (2.37) and (2.41) respectively, to obtain

$$|I_6| \leq C \left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^4}^4\right) \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{6} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 ,$$

$$|I_7| \le C \left(\|\boldsymbol{u}\|_{\mathbb{L}^2}^2 + \|\boldsymbol{u}\|_{\mathbb{L}^4}^4 \right) + \frac{\eta}{6} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 .$$

Substituting these estimates into (5.10), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \varepsilon \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 \\ &\le C \|\boldsymbol{u}\|_{\mathbb{L}^2}^2 + C \|\boldsymbol{u}\|_{\mathbb{L}^4}^4 + C\varepsilon^2 \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + C \left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^4}^4 + \varepsilon \|\boldsymbol{u}\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 \right) \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 . \end{aligned} \tag{5.11}$$

Integrating this over $(0,t)$, and applying the Gronwall inequality (noting Proposition 3.1), we obtain (5.4). The estimate (5.6) then follows by noting the definition of H and applying Hölder's inequality. Inequalities (5.7) and (5.8) also follow from (5.11) and the uniform Gronwall inequality.

Finally, by the same argument as in Proposition 3.2, we infer inequality (5.9) from the uniform Gronwall inequality and (5.7). This completes the proof of the lemma. ☐

Furthermore, we need another estimate for $\|\boldsymbol{u}(t)\|_{\mathbb{H}^2}$ which is uniform in ε.

Lemma 5.4. Let $\boldsymbol{u}$ be the solution of (1.1) with initial data $\boldsymbol{u}_0$, whose corresponding effective field is $\boldsymbol{H}$. The following statements hold:

(1) For all $t \ge 0$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2 + \int_0^t \|\nabla \Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 \, \mathrm{d}s + \varepsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \le \kappa_2(t), \tag{5.12}$$

where $\kappa_2(t)$ is an increasing function of t which also depends on $\|\boldsymbol{u}_0\|_{\mathbb{H}^2}$, but is independent of ε.

(2) There exists t_2 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ (but is independent of ε) such that for all $t \ge t_2$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2 + \int_t^{t+1} \left(\|\nabla \Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 + \varepsilon \left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \right) \mathrm{d}s \le \rho_2, \tag{5.13}$$

where ρ_2 is independent of $\boldsymbol{u}_0$, ε, and t.

(3) For all $t > 0$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2 \le C \left(1 + t^{-1}\right) \exp(e^{Ct}), \tag{5.14}$$

where C is independent of ε and t, but may depend on other coefficients of (1.1), $|\mathscr{O}|$, and $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$.

Proof. Taking the inner product of both equations in (1.1) with $\Delta^2 \boldsymbol{u}$ and integrating by parts whenever appropriate, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \eta \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \varepsilon \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \\ &= \kappa_1 \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \kappa_2 \left\langle \nabla\left(|\boldsymbol{u}|^2 \boldsymbol{u}\right), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} - \sigma \left\langle \nabla \Phi_{\mathrm{d}}(\boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} \\ &\quad + \kappa_2 \varepsilon \left\langle \Delta\left(|\boldsymbol{u}|^2 \boldsymbol{u}\right), \Delta^2 \boldsymbol{u} \right\rangle_{\mathbb{L}^2} - \varepsilon \left\langle \Delta \Phi_{\mathrm{a}}(\boldsymbol{u}), \Delta^2 \boldsymbol{u} \right\rangle_{\mathbb{L}^2} - \varepsilon \left\langle \Delta \Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta^2 \boldsymbol{u} \right\rangle_{\mathbb{L}^2} \\ &\quad + \gamma \left\langle \nabla \boldsymbol{u} \times \Delta \boldsymbol{u}, \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} + \gamma \left\langle \nabla \boldsymbol{u} \times \Phi_{\mathrm{a}}(\boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} + \gamma \left\langle \boldsymbol{u} \times \nabla \Phi_{\mathrm{a}}(\boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} \\ &\quad + \gamma \left\langle \nabla \boldsymbol{u} \times \Phi_{\mathrm{d}}(\boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} + \gamma \left\langle \boldsymbol{u} \times \nabla \Phi_{\mathrm{d}}(\boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} - \left\langle \nabla \mathcal{R}(\boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} - \left\langle \nabla \mathcal{S}(\boldsymbol{u}), \nabla \Delta \boldsymbol{u} \right\rangle_{\mathbb{L}^2} \\ &= I_1 + I_2 + \cdots + I_{13}. \end{aligned} \tag{5.15}$$

We will estimate each term on the last line. The first term is left as is. For the terms I_2 and I_3, by Young's inequality and Sobolev embedding,

$$\begin{aligned} |I_2| &\le 3\kappa_2 \|\boldsymbol{u}\|_{\mathbb{L}^6}^2 \|\nabla \boldsymbol{u}\|_{\mathbb{L}^6} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2} \le C \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 , \\ |I_3| &\le \sigma \|\nabla \Phi_{\mathrm{d}}(\boldsymbol{u})\|_{\mathbb{L}^2} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2} \le C \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 . \end{aligned}$$

For the next term, by (2.14), Hölder's and Young's inequalities, and Agmon's inequality in 2D, we have

$$\begin{aligned} |I_4| &\le C\varepsilon \left\|\Delta\left(|\boldsymbol{u}|^2 \boldsymbol{u}\right)\right\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{8} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \\ &\le C\varepsilon \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^2 + C\varepsilon \|\boldsymbol{u}\|_{\mathbb{L}^\infty}^4 \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{8} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \end{aligned}$$

$$\leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^1}^3 \|\boldsymbol{u}\|_{\mathbb{H}^3} \|\boldsymbol{u}\|_{\mathbb{L}^2} \|\boldsymbol{u}\|_{\mathbb{H}^2} + C\varepsilon \|\boldsymbol{u}\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + \frac{\varepsilon}{8} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2$$
$$\leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + \frac{\varepsilon}{8} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 .$$

By similar argument, we also have

$$|I_5| \leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + \frac{\varepsilon}{8} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 ,$$

and

$$|I_6| \leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + \frac{\varepsilon}{8} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 .$$

For the next three terms, by the Young and the Gagliardo–Nirenberg inequalities, and the Sobolev embedding, we have

$$\begin{aligned}
|I_7| &\leq \gamma \|\nabla \boldsymbol{u}\|_{\mathbb{L}^4} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^4} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2} \\
&\leq C \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^{1/2} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^{3/2} \leq C \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^4 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 , \\
|I_8| &\leq \gamma \|\nabla \boldsymbol{u}\|_{\mathbb{L}^4} \|\Phi_{\mathrm{a}}(\boldsymbol{u})\|_{\mathbb{L}^4} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2} \\
&\leq C \left(1 + \|\boldsymbol{u}\|_{\mathbb{L}^4}^2\right) \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 \leq C \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^2\right) \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 , \\
|I_9| &\leq \gamma \|\boldsymbol{u}\|_{\mathbb{L}^4} \|\nabla \Phi_{\mathrm{a}}(\boldsymbol{u})\|_{\mathbb{L}^4} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2} \\
&\leq C \|\boldsymbol{u}\|_{\mathbb{L}^4}^2 \|\nabla \boldsymbol{u}\|_{\mathbb{L}^4}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 \leq C \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^2\right) \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 .
\end{aligned}$$

For the terms I_{10} and I_{11}, using (2.48), similarly we obtain

$$|I_{10}| \leq \gamma \|\nabla \boldsymbol{u}\|_{\mathbb{L}^4} \|\Phi_{\mathrm{d}}(\boldsymbol{u})\|_{\mathbb{L}^4} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2} \leq C \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 ,$$
$$|I_{11}| \leq \gamma \|\boldsymbol{u}\|_{\mathbb{L}^4} \|\nabla \Phi_{\mathrm{d}}(\boldsymbol{u})\|_{\mathbb{L}^4} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2} \leq C \|\boldsymbol{u}\|_{\mathbb{H}^1}^2 \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 .$$

Finally, for the last two terms we apply (2.38) and (2.42) to obtain

$$|I_{12}| \leq C\nu_\infty \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^4\right) + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 ,$$
$$|I_{13}| \leq C \left(\|\nabla \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \||\boldsymbol{u}||\nabla \boldsymbol{u}|\|_{\mathbb{L}^2}^2\right) + \frac{\eta}{12} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 .$$

Altogether, we obtain from (5.15),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \varepsilon \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 &\leq C \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4\right) \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 \\
&\quad + C \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \||\boldsymbol{u}||\nabla \boldsymbol{u}|\|_{\mathbb{L}^2}^2 + \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^4\right) . \qquad (5.16)
\end{aligned}$$

Integrating over $(0,t)$ and applying Lemma 5.3, we obtain

$$\|\Delta \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \int_0^t \|\nabla \Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \varepsilon \int_0^t \left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \leq C \|\boldsymbol{u}_0\|_{\mathbb{H}^2}^2 + Ce^{Ct} \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^6 + C \int_0^t \|\Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^4 \,\mathrm{d}s.$$

The inequality (5.12) then follows from the Gronwall inequality.

If we take the inner product of (1.1) with $t\Delta^2 \boldsymbol{u}$ and follow the same argument as before, instead of (5.16) we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left(t \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2\right) + t \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + t\varepsilon \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 &\leq \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + Ct \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4\right) \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + Ct\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 \\
&\quad + Ct \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^1}^4 + \||\boldsymbol{u}||\nabla \boldsymbol{u}|\|_{\mathbb{L}^2}^2 + \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^4\right) ,
\end{aligned}$$

which upon integration over $(0,t)$ yields

$$t \|\Delta \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 \leq Ce^{Ct} \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^2 + Cte^{Ct} \|\boldsymbol{u}_0\|_{\mathbb{H}^1}^6 + C \int_0^t s \|\Delta \boldsymbol{u}(s)\|_{\mathbb{L}^2}^4 \,\mathrm{d}s.$$

Thus, (5.14) follows from the Gronwall inequality (noting Lemma 5.3).

Next, from (5.16) and (5.7) we obtain for $t \geq t_1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \varepsilon \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \leq C\left(1+\rho_1^3\right) \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + C\varepsilon \rho_1^2 \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 + C\left(1+\rho_1^2 + \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^4\right). \tag{5.17}$$

Note that by (5.7), we have

$$\int_t^{t+1} C\left(1+\rho_1^3\right) \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \int_t^{t+1} C\varepsilon\rho_1^2 \|\boldsymbol{u}\|_{\mathbb{H}^3}^2 \,\mathrm{d}s \leq C\left(1+\rho_1^4\right) \quad \text{and} \quad \int_t^{t+1} C \|\Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \leq C\rho_1,$$

where C is independent of ε. Therefore, by the uniform Gronwall inequality,

$$\|\Delta \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 \leq C\left(1+\rho_1^4\right) \exp\left(\rho_1^2\right), \quad \forall t \geq t_1 + 1. \tag{5.18}$$

Finally, integrating (5.17) over $(t, t+1)$, then applying (5.7) and (5.18) yield (5.13). This completes the proof of the lemma. ☐

Lemma 5.5. Let $\boldsymbol{u}$ be the solution of (1.1) with initial data $\boldsymbol{u}_0$, whose corresponding effective field is $\boldsymbol{H}$. The following statements hold:

(1) For all $t \geq 0$,

$$\|\nabla \Delta \boldsymbol{u}(t)\|_{\mathbb{L}^2}^2 + \int_0^t \left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s + \varepsilon \int_0^t \left\|\nabla \Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \leq \kappa_3(t), \tag{5.19}$$

where $\kappa_3(t)$ is an increasing function of t which also depends on $\|\boldsymbol{u}_0\|_{\mathbb{H}^3}$, but is independent of ε.

(2) There exists t_3 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ (but is independent of ε) such that for all $t \geq t_3$,

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^3}^2 + \int_t^{t+1} \left(\left\|\Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 + \varepsilon \left\|\nabla \Delta^2 \boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2\right) \mathrm{d}s \leq \rho_3, \tag{5.20}$$

where ρ_3 is independent of $\boldsymbol{u}_0$, ε, and t.

Proof. We apply the operator $-\Delta$, then take the inner product of both equations in (1.1) with $\Delta^2 \boldsymbol{u}$, and integrate by parts as necessary (whenever allowed) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \eta \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 + \varepsilon \left\|\nabla \Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \\
&= \kappa_2 \sigma \left\langle \Delta(|\boldsymbol{u}|^2 \boldsymbol{u}), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} - \sigma \left\langle \Delta \Phi_{\mathrm{a}}(\boldsymbol{u}), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} - \sigma \left\langle \Delta \Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} \\
&\quad - \kappa_2 \varepsilon \left\langle \nabla \Delta(|\boldsymbol{u}|^2 \boldsymbol{u}), \nabla \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} - \kappa_2 \varepsilon \left\langle \nabla \Delta \Phi_{\mathrm{a}}(\boldsymbol{u}), \nabla \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} + \varepsilon \left\langle \Delta^2 \Phi_{\mathrm{d}}(\boldsymbol{u}), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} \\
&\quad + \gamma \left\langle \Delta\big(\boldsymbol{u} \times \Phi_{\mathrm{a}}(\boldsymbol{u})\big), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} + \gamma \left\langle \Delta\big(\boldsymbol{u} \times \Phi_{\mathrm{d}}(\boldsymbol{u})\big), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} - \left\langle \Delta \mathcal{R}(\boldsymbol{u}), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} - \left\langle \Delta \mathcal{S}(\boldsymbol{u}), \Delta^2 \boldsymbol{u}\right\rangle_{\mathbb{L}^2} \\
&= I_1 + I_2 + \cdots + I_{10}.
\end{aligned} \tag{5.21}$$

By using Young's inequality, (2.12), (2.14), (2.17), (2.23), (2.39), (2.43), and (2.48), without elaborating further, we obtain the following inequalities:

$$\begin{aligned}
|I_1| &\leq C \|\boldsymbol{u}\|_{\mathbb{H}^2}^6 + \frac{\eta}{9} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2, \\
|I_2| &\leq C \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + \frac{\eta}{9} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2, \\
|I_3| &\leq C \|\boldsymbol{u}\|_{\mathbb{H}^2}^2 + \frac{\eta}{9} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2, \\
|I_4| &\leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^2}^4 \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{4} \left\|\nabla \Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \\
|I_5| &\leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^2}^4 \|\nabla \Delta \boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{4} \left\|\nabla \Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \\
|I_6| &\leq C\varepsilon \|\boldsymbol{u}\|_{\mathbb{H}^4}^2 + \frac{\eta}{9} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2, \\
|I_7| &\leq C\left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^2}^4\right) + \frac{\eta}{9} \left\|\Delta^2 \boldsymbol{u}\right\|_{\mathbb{L}^2}^2,
\end{aligned}$$

$$|I_8| \le C\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}^4\right) + \frac{\eta}{9}\left\|\Delta^2\boldsymbol{u}\right\|_{\mathbb{L}^2}^2,$$
$$|I_9| \le C\nu_\infty\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}^4\right) + C\nu_\infty\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}^2\right)\|\nabla\Delta\boldsymbol{u}\|_{\mathbb{L}^2}^2 + \frac{\eta}{9}\left\|\Delta^2\boldsymbol{u}\right\|_{\mathbb{L}^2}^2,$$
$$|I_{10}| \le C\|\boldsymbol{u}\|_{\mathbb{H}^2}^4 + \frac{\eta}{9}\left\|\Delta^2\boldsymbol{u}\right\|_{\mathbb{L}^2}^2.$$

Altogether, substituting these into (5.21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta\boldsymbol{u}\|_{\mathbb{L}^2}^2 + \left\|\Delta^2\boldsymbol{u}\right\|_{\mathbb{L}^2}^2 + \varepsilon\left\|\nabla\Delta^2\boldsymbol{u}\right\|_{\mathbb{L}^2}^2 \le C\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}^4\right) + C\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^2}^2\right)\|\nabla\Delta\boldsymbol{u}\|_{\mathbb{L}^2}^2 + C\varepsilon\|\boldsymbol{u}\|_{\mathbb{H}^4}^2.$$

Integrating this over $(0,t)$ and applying (5.12) yields (5.19). Applying the same argument as in the proof of (5.13) then gives (5.20). This completes the proof of the lemma. □

Lemma 5.6. Let $\boldsymbol{u}$ be the solution of (1.1) with initial data $\boldsymbol{u}_0$, whose corresponding effective field is $\boldsymbol{H}$. The following statements hold:

(1) For all $t \ge 0$,
$$\left\|\Delta^2\boldsymbol{u}(t)\right\|_{\mathbb{L}^2}^2 + \int_0^t \left\|\nabla\Delta^2\boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s + \varepsilon\int_0^t \left\|\Delta^3\boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 \mathrm{d}s \le \kappa_4(t), \tag{5.22}$$
where $\kappa_4(t)$ is an increasing function of t which also depends on $\|\boldsymbol{u}_0\|_{\mathbb{H}^4}$, but is independent of ε.

(2) There exists t_4 depending on $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$ (but is independent of ε) such that for all $t \ge t_4$,
$$\|\boldsymbol{u}(t)\|_{\mathbb{H}^4}^2 + \int_t^{t+1}\left(\left\|\nabla\Delta^2\boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2 + \varepsilon\left\|\Delta^3\boldsymbol{u}(s)\right\|_{\mathbb{L}^2}^2\right)\mathrm{d}s \le \rho_4, \tag{5.23}$$
where ρ_4 is independent of $\boldsymbol{u}_0$, ε, and t.

Proof. The proof is similar to that of Lemma 5.5, but instead we apply the operator Δ^2, then take the inner product of both equations in (1.1) with $\Delta^2\boldsymbol{u}$. □

We will derive some estimates for the difference of two solutions which are uniform in ε. Recall that $\boldsymbol{S}_\varepsilon(t)$ is the semigroup generated by the system (1.1). An analogue of Lemma 4.8 is stated below.

Lemma 5.7. Let $\varepsilon \in [0, \sigma/\kappa_1]$, and let $\mathbb{B} \subset \mathbb{H}^1$ be a semi-invariant absorbing set for $\boldsymbol{S}_\varepsilon(t)$ furnished by Lemma 5.4 and Lemma 5.5. There exists a constant C such that for all $t > 0$,

$$\|\boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0 - \boldsymbol{S}_\varepsilon(t)\boldsymbol{v}_0\|_{\mathbb{H}^1} + \varepsilon\int_0^t \|\boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0 - \boldsymbol{S}_\varepsilon(t)\boldsymbol{v}_0\|_{\mathbb{H}^3}^2 \mathrm{d}s + \int_0^t \|\boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0 - \boldsymbol{S}_\varepsilon(t)\boldsymbol{v}_0\|_{\mathbb{H}^2}^2 \mathrm{d}s$$
$$\le \beta_1(t)\|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{H}^1}^2, \quad \forall \boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathbb{B},$$

where $\beta_1(t)$ is an increasing function of t which is independent of ε.

Proof. The proof is similar to that of Lemma 4.8, noting that we now have (5.13) for $\boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathbb{B}$. Further details are omitted. □

Lemma 5.8. Let $\varepsilon \in [0, \sigma/\kappa_1]$, and let $\mathbb{B} \subset \mathbb{H}^1$ be a semi-invariant absorbing set for $\boldsymbol{S}_\varepsilon(t)$ furnished by Lemma 5.4 and Lemma 5.5. There exists a constant C such that for all $t > 0$,

$$\|\boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0 - \boldsymbol{S}_\varepsilon(t)\boldsymbol{v}_0\|_{\mathbb{H}^2} \le C\left(1+t^{-1}\right)\beta_1(t)\|\boldsymbol{u}_0 - \boldsymbol{v}_0\|_{\mathbb{H}^1}, \quad \forall \boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathbb{B}, \tag{5.24}$$

where $\beta_1(t)$ is an increasing function of t in Lemma 5.7 (which is independent of ε).

Proof. We repeat the steps of the proof of Lemma 4.9 with $\boldsymbol{w}(t) := \boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0 - \boldsymbol{S}_\varepsilon(t)\boldsymbol{v}_0$ to obtain (4.21). Each term J_i, where $i = 1, 2, \dots, 16$, is then estimated as before, except for the terms J_{10}, J_{11}, J_{12}, J_{13}, J_{16}, and J_{17}. The estimates for these terms in the proof of Lemma 4.9 still depend on ε^{-1}, thus they need to be estimated differently here.

For the term J_{10}, by the definition of $\boldsymbol{H}_1$ and (5.12), after integrating by parts we have

$$|J_{10}| \le t\gamma\|\nabla\boldsymbol{w}\|_{\mathbb{L}^4}\|\boldsymbol{H}_1\|_{\mathbb{L}^4}\|\nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2} + t\gamma\|\boldsymbol{w}\|_{\mathbb{L}^\infty}\|\nabla\boldsymbol{H}_1\|_{\mathbb{L}^2}\|\nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}$$
$$\le Ct\left(1+\|\boldsymbol{u}\|_{\mathbb{H}^1}^4\right)\|\boldsymbol{u}\|_{\mathbb{H}^3}^2\|\boldsymbol{w}\|_{\mathbb{H}^2}^2 + \frac{t\eta}{16}\|\nabla\Delta\boldsymbol{w}\|_{\mathbb{L}^2}^2.$$

Similarly, for the terms J_{11} and J_{12}, we have

$$\begin{aligned}
|J_{11}| &\le t\kappa_1\gamma \|\nabla \boldsymbol{v}\|_{\mathbb{L}^4} \|\boldsymbol{w}\|_{\mathbb{L}^4} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2} + t\kappa_1\gamma \|\boldsymbol{v}\|_{\mathbb{L}^4} \|\nabla \boldsymbol{w}\|_{\mathbb{L}^4} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2} \\
&\le Ct \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 \|\boldsymbol{w}\|_{\mathbb{H}^2}^2 + \frac{t\eta}{16} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2, \\
|J_{12}| &\le t\kappa_2\gamma \|\nabla \boldsymbol{v}\|_{\mathbb{L}^8} \|\boldsymbol{u}\|_{\mathbb{L}^8}^2 \|\boldsymbol{w}\|_{\mathbb{L}^8} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2} + 2t\kappa_2\gamma \|\boldsymbol{v}\|_{\mathbb{L}^8} \|\boldsymbol{u}\|_{\mathbb{L}^8} \|\nabla \boldsymbol{u}\|_{\mathbb{L}^8} \|\boldsymbol{w}\|_{\mathbb{L}^8} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2} \\
&\quad + t\kappa_2\gamma \|\boldsymbol{v}\|_{\mathbb{L}^8} \|\boldsymbol{u}\|_{\mathbb{L}^8}^2 \|\nabla \boldsymbol{w}\|_{\mathbb{L}^8} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2} \\
&\le Ct \|\boldsymbol{u}\|_{\mathbb{H}^2}^4 \|\boldsymbol{v}\|_{\mathbb{H}^2}^2 \|\boldsymbol{w}\|_{\mathbb{H}^2}^2 + \frac{t\eta}{16} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2.
\end{aligned}$$

For the term J_{13}, we integrate by parts and apply (2.21) with $p = q = 6$ to obtain

$$\begin{aligned}
|J_{13}| &\le t\gamma \|\nabla \boldsymbol{v}\|_{\mathbb{L}^4} \|\Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v})\|_{\mathbb{L}^4} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2} + t\gamma \|\boldsymbol{v}\|_{\mathbb{L}^\infty} \|\Phi_{\mathrm{a}}(\boldsymbol{u}) - \Phi_{\mathrm{a}}(\boldsymbol{v})\|_{\mathbb{L}^2} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2} \\
&\le Ct \left(1 + \|\boldsymbol{u}\|_{\mathbb{H}^2}^6 + \|\boldsymbol{v}\|_{\mathbb{H}^2}^6\right) \|\boldsymbol{w}\|_{\mathbb{H}^1}^2 + \frac{t\eta}{16} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2.
\end{aligned}$$

For the terms J_{16} and J_{17}, we infer from (2.40) and (2.45) that

$$|J_{16}| + |J_{17}| \le Ct \|\boldsymbol{w}\|_{\mathbb{H}^2}^2 + \frac{t\eta}{16} \|\nabla\Delta \boldsymbol{w}\|_{\mathbb{L}^2}^2.$$

These estimates, together with estimates for the other terms derived in Lemma 4.9 and Lemma 5.7, yield

$$\begin{aligned}
t \|\boldsymbol{w}(t)\|_{\mathbb{H}^2}^2 &\le C \int_0^t \|\boldsymbol{w}(s)\|_{\mathbb{H}^2}^2 \,\mathrm{d}s + C \int_0^t s \left(1 + \|\boldsymbol{u}(s)\|_{\mathbb{H}^2}^4 + \|\boldsymbol{v}(s)\|_{\mathbb{H}^2}^4\right) \|\boldsymbol{w}(s)\|_{\mathbb{H}^2}^2 \,\mathrm{d}s \\
&\quad + C \int_0^t s \|\boldsymbol{u}(s)\|_{\mathbb{H}^1}^4 \|\boldsymbol{u}(s)\|_{\mathbb{H}^3}^2 \|\boldsymbol{w}(s)\|_{\mathbb{H}^2}^2 \,\mathrm{d}s \\
&\quad + C\varepsilon \int_0^t s \left(\|\boldsymbol{u}(s)\|_{\mathbb{H}^1}^2 + \|\boldsymbol{v}(s)\|_{\mathbb{H}^1}^2\right) \left(\|\boldsymbol{u}(s)\|_{\mathbb{H}^3}^2 + \|\boldsymbol{v}(s)\|_{\mathbb{H}^3}^2\right) \|\boldsymbol{w}(s)\|_{\mathbb{H}^1}^2 \,\mathrm{d}s \\
&\le C\beta_1(t) \|\boldsymbol{w}_0\|_{\mathbb{H}^1}^2 + C\left(1 + \rho_2^2\right) t\beta_1(t) \|\boldsymbol{w}_0\|_{\mathbb{H}^1}^2 + C\left(1 + \rho_1^2\right) \rho_3 t\beta_1(t) \|\boldsymbol{w}_0\|_{\mathbb{H}^1}^2 \\
&\le C(1+t)\beta_1(t) \|\boldsymbol{w}_0\|_{\mathbb{H}^1}^2,
\end{aligned}$$

where in the last step we used (5.7), (5.13), (5.20), and Lemma 5.7. This implies the inequality (5.24). ☐

The following lemma shows a continuous dependence estimate on the parameter ε.

Lemma 5.9. Let $\varepsilon \in [0, \sigma/\kappa_1]$, and let $\mathbb{B} \subset \mathbb{H}^1$ be a semi-invariant absorbing set for $\boldsymbol{S}_\varepsilon(t)$ furnished by Lemma 5.4 and Lemma 5.5. There exists a constant C such that for all $t \ge 0$,

$$\|\boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0 - \boldsymbol{S}_0(t)\boldsymbol{u}_0\|_{\mathbb{H}^1} \le C\varepsilon e^{Ct}, \quad \forall \boldsymbol{u}_0 \in \mathbb{B},$$

where C is independent of ε and t.

Proof. Let $\boldsymbol{u}^\varepsilon(t) := \boldsymbol{S}_\varepsilon(t)\boldsymbol{u}_0$ and $\boldsymbol{u}^0(t) := \boldsymbol{S}_0(t)\boldsymbol{u}_0$. Let $\boldsymbol{v}(t) := \boldsymbol{u}^\varepsilon(t) - \boldsymbol{u}^0(t)$. Then $\boldsymbol{v}$ solves

$$\begin{aligned}
\partial_t \boldsymbol{v} &= \sigma\left(\Psi(\boldsymbol{u}^\varepsilon) - \Psi(\boldsymbol{u}^0)\right) + \sigma\left(\Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{a}}(\boldsymbol{u}^0)\right) + \sigma\left(\Phi_{\mathrm{d}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{d}}(\boldsymbol{u}^0)\right) \\
&\quad - \varepsilon\Delta\Psi(\boldsymbol{u}^\varepsilon) - \varepsilon\Delta\Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \varepsilon\Delta\Phi_{\mathrm{d}}(\boldsymbol{u}^\varepsilon) \\
&\quad - \gamma\left(\boldsymbol{u}^\varepsilon \times \left(\Psi(\boldsymbol{u}^\varepsilon) + \Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) + \Phi_{\mathrm{d}}(\boldsymbol{u}^\varepsilon)\right) - \boldsymbol{u}^0 \times \left(\Psi(\boldsymbol{u}^0) + \Phi_{\mathrm{a}}(\boldsymbol{u}^0) + \Phi_{\mathrm{d}}(\boldsymbol{u}^0)\right)\right) \\
&\quad + \mathcal{R}(\boldsymbol{u}^\varepsilon) - \mathcal{R}(\boldsymbol{u}^0) + \mathcal{S}(\boldsymbol{u}^\varepsilon) - \mathcal{S}(\boldsymbol{u}^0) \\
&= \eta\Delta \boldsymbol{v} + \sigma\kappa_1 \boldsymbol{v} - \sigma\kappa_2 |\boldsymbol{u}^\varepsilon|^2 \boldsymbol{v} - \sigma\kappa_2 \left((\boldsymbol{u}^\varepsilon + \boldsymbol{u}^0)\cdot \boldsymbol{v}\right) \boldsymbol{u}^0 + \sigma\left(\Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{a}}(\boldsymbol{u}^0)\right) + \sigma\Phi_{\mathrm{d}}(\boldsymbol{v}) \\
&\quad - \varepsilon\Delta^2 \boldsymbol{u}^\varepsilon - \varepsilon\kappa_2\Delta\left(|\boldsymbol{u}^\varepsilon|^2 \boldsymbol{u}^\varepsilon\right) - \varepsilon\Delta\Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \varepsilon\Delta\Phi_{\mathrm{d}}(\boldsymbol{v}) - \gamma \boldsymbol{v} \times \Delta \boldsymbol{u}^\varepsilon - \gamma \boldsymbol{u}^0 \times \Delta \boldsymbol{v} - \gamma \boldsymbol{v} \times \Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) \\
&\quad - \gamma \boldsymbol{u}^0 \times \left(\Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{a}}(\boldsymbol{u}^0)\right) - \gamma \boldsymbol{v} \times \Phi_{\mathrm{d}}(\boldsymbol{u}^\varepsilon) - \gamma \boldsymbol{u}^0 \times \Phi_{\mathrm{d}}(\boldsymbol{v}) \\
&\quad + \mathcal{R}(\boldsymbol{u}^\varepsilon) - \mathcal{R}(\boldsymbol{u}^0) + \mathcal{S}(\boldsymbol{u}^\varepsilon) - \mathcal{S}(\boldsymbol{u}^0).
\end{aligned} \tag{5.25}$$

Taking the inner product of (5.25) with $\boldsymbol{v}$ and integrating by parts as necessary, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \eta \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + \sigma\kappa_2 \||\boldsymbol{u}^\varepsilon||\boldsymbol{v}|\|_{\mathbb{L}^2}^2 + \sigma\kappa_2 \left\|\boldsymbol{u}^0 \cdot \boldsymbol{v}\right\|_{\mathbb{L}^2}^2$$

$$
\begin{aligned}
&= \sigma\kappa_1 \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 - \sigma\kappa_2 \left\langle (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v})\boldsymbol{u}^0, \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \sigma \left\langle \Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{a}}(\boldsymbol{u}^0), \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \sigma \left\langle \Phi_{\mathrm{d}}(\boldsymbol{v}), \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&\quad + \varepsilon \left\langle \nabla\Delta \boldsymbol{u}^\varepsilon, \nabla \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \varepsilon\kappa_2 \left\langle \nabla\big(|\boldsymbol{u}^\varepsilon|^2 \boldsymbol{u}^\varepsilon\big), \nabla \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \varepsilon \left\langle \nabla \Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon), \nabla \boldsymbol{v}\right\rangle_{\mathbb{L}^2} - \varepsilon \left\langle \Delta \Phi_{\mathrm{d}}(\boldsymbol{u}^\varepsilon), \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&\quad - \gamma \left\langle \nabla \boldsymbol{u}^0 \times \boldsymbol{v}, \nabla \boldsymbol{v}\right\rangle_{\mathbb{L}^2} - \gamma \left\langle \boldsymbol{u}^0 \times \left(\Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{a}}(\boldsymbol{u}^0)\right), \boldsymbol{v}\right\rangle_{\mathbb{L}^2} - \gamma \left\langle \boldsymbol{u}^0 \times \Phi_{\mathrm{d}}(\boldsymbol{v}), \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&\quad + \left\langle \mathcal{R}(\boldsymbol{u}^\varepsilon) - \mathcal{R}(\boldsymbol{u}^0), \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \left\langle \mathcal{S}(\boldsymbol{u}^\varepsilon) - \mathcal{S}(\boldsymbol{u}^0), \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&=: I_1 + I_2 + \cdots + I_{13}.
\end{aligned} \tag{5.26}
$$

We estimate each term I_j, where $j = 2, 3, \ldots, 13$, by using Hölder's and Young's inequalities in a standard way and noting that $\boldsymbol{u}^\varepsilon \in \mathbb{B}$ for $\varepsilon \in [0, \sigma/\kappa_1]$. Applying (2.20) and (2.48) as necessary, we obtain

$$
\begin{aligned}
|I_2| &\le \frac{\sigma\kappa_2}{2} \||\boldsymbol{u}^\varepsilon||\boldsymbol{v}|\|_{\mathbb{L}^2}^2 + \frac{\sigma\kappa_2}{2} \left\|\boldsymbol{u}^0 \cdot \boldsymbol{v}\right\|_{\mathbb{L}^2}^2, \\
|I_3| &\le C \left(1 + \|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^\infty}^4 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C(1+\rho_2^2) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_4| &\le C \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \\
|I_5| &\le C\varepsilon^2 \|\nabla\Delta \boldsymbol{u}^\varepsilon\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_3 \varepsilon^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_6| &\le C\varepsilon^2 \|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^\infty}^4 \|\nabla \boldsymbol{u}^\varepsilon\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2^3 \varepsilon^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_7| &\le C\varepsilon^2 \|\boldsymbol{u}^\varepsilon\|_{\mathbb{H}^1}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_1 \varepsilon^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_8| &\le C\varepsilon^2 \|\boldsymbol{u}^\varepsilon\|_{\mathbb{H}^2}^2 + \frac{\eta}{8} \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2 \varepsilon^2 + \frac{\eta}{8} \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_9| &\le C \left\|\nabla \boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_3^2 \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_{10}| &\le C \left(1 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^2\right) \left(1 + \|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^\infty}^4 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^4\right) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C(1+\rho_2^3) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_{11}| &\le C \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^2 \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2 \|\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_{12}| &\le C \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{O};\mathbb{R}^d)} \left(1 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^2\right) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C(1+\rho_2) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|I_{13}| &\le C \left(1 + \|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^\infty}^2 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^2\right) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C(1+\rho_2) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{8} \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2,
\end{aligned}
$$

where we also used (2.31) and (2.44) to estimate I_{12} and I_{13} respectively. Moreover, we used the inequalities (5.13), (5.20), and the Sobolev embedding in the second step for each inequality (if there is any). Altogether, substituting these into (5.26) and applying the Gronwall inequality, we obtain

$$
\|\boldsymbol{v}(t)\|_{\mathbb{L}^2}^2 + \int_0^t \|\nabla \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s + \int_0^t \varepsilon \|\Delta \boldsymbol{v}(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s \le C\varepsilon^2 e^{Ct}. \tag{5.27}
$$

Next, we take the inner product of (5.25) with $-\Delta \boldsymbol{v}$ and integrate by parts as necessary to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \eta \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \sigma\kappa_1 \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + \sigma\kappa_2 \left\langle |\boldsymbol{u}^\varepsilon|^2 \boldsymbol{v}, \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \sigma\kappa_2 \left\langle \big((\boldsymbol{u}^\varepsilon + \boldsymbol{u}^0) \cdot \boldsymbol{v}\big)\boldsymbol{u}^0, \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&\quad - \sigma \left\langle \Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{a}}(\boldsymbol{u}^0), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} - \sigma \left\langle \Phi_{\mathrm{d}}(\boldsymbol{v}), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \varepsilon \left\langle \Delta^2 \boldsymbol{u}^\varepsilon, \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \varepsilon\kappa_2 \left\langle \Delta\big(|\boldsymbol{u}^\varepsilon|^2 \boldsymbol{u}^\varepsilon\big), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&\quad + \varepsilon \left\langle \Delta \Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \varepsilon \left\langle \Delta \Phi_{\mathrm{d}}(\boldsymbol{v}), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \gamma \left\langle \boldsymbol{v} \times \Delta \boldsymbol{u}^\varepsilon, \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \gamma \left\langle \boldsymbol{v} \times \Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&\quad + \gamma \left\langle \boldsymbol{u}^0 \times \left(\Phi_{\mathrm{a}}(\boldsymbol{u}^\varepsilon) - \Phi_{\mathrm{a}}(\boldsymbol{u}^0)\right), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \gamma \left\langle \boldsymbol{v} \times \Phi_{\mathrm{d}}(\boldsymbol{u}^\varepsilon), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} + \gamma \left\langle \boldsymbol{u}^0 \times \Phi_{\mathrm{d}}(\boldsymbol{v}), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&\quad - \left\langle \mathcal{R}(\boldsymbol{u}^\varepsilon) - \mathcal{R}(\boldsymbol{u}^0), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} - \left\langle \mathcal{S}(\boldsymbol{u}^\varepsilon) - \mathcal{S}(\boldsymbol{u}^0), \Delta \boldsymbol{v}\right\rangle_{\mathbb{L}^2} \\
&=: J_1 + J_2 + \cdots + J_{17}.
\end{aligned} \tag{5.28}
$$

We estimate each term J_k, where $k = 3, 4, \ldots, 17$, in the usual manner as follows:

$$
|J_3| \le C \|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^\infty}^4 \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18} \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2^2 \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18} \|\Delta \boldsymbol{v}\|_{\mathbb{L}^2}^2,
$$

$$
\begin{aligned}
|J_4| &\le C\left(\|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^\infty}^4 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^4\right)\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2^2\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_5| &\le C\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_6| &\le C\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_7| &\le C\varepsilon^2\left\|\Delta^2\boldsymbol{u}^\varepsilon\right\|_{\mathbb{L}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_4\varepsilon^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_8| &\le C\varepsilon^2\|\boldsymbol{u}^\varepsilon\|_{\mathbb{H}^2}^6 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2^3\varepsilon^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_9| &\le C\varepsilon^2\|\boldsymbol{u}^\varepsilon\|_{\mathbb{H}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2\varepsilon^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_{10}| &\le C\varepsilon^2\|\boldsymbol{v}\|_{\mathbb{H}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_2\varepsilon^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_{11}| &\le C\|\Delta\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^4}^2\|\boldsymbol{v}\|_{\mathbb{L}^4}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_3\|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_{12}| + |J_{14}| &\le C\|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^4}^2\|\boldsymbol{v}\|_{\mathbb{L}^4}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_1\|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_{13}| + |J_{15}| &\le C\left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^4}^2\|\boldsymbol{v}\|_{\mathbb{L}^4}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C\rho_1\|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_{16}| &\le C\left(1 + \|\boldsymbol{u}^\varepsilon\|_{\mathbb{H}^2}^2 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{H}^2}^2\right)\|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C(1+\rho_2)\|\boldsymbol{v}\|_{\mathbb{H}^1}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2, \\
|J_{17}| &\le C\left(1 + \|\boldsymbol{u}^\varepsilon\|_{\mathbb{L}^\infty}^2 + \left\|\boldsymbol{u}^0\right\|_{\mathbb{L}^\infty}^2\right)\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2 \le C(1+\rho_2)\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \frac{\eta}{18}\|\Delta\boldsymbol{v}\|_{\mathbb{L}^2}^2.
\end{aligned}
$$

Substituting these into (5.28), integrating over $(0,t)$, and using (5.27), we obtain the required result. □

We can finally state the main theorem of this section on the existence of a family of exponential attractors $\{\mathcal{M}_\varepsilon : \varepsilon \in [0,\sigma/\kappa_1]\}$ for the semigroup $\boldsymbol{S}_\varepsilon(t)$ generated by (1.1).

Theorem 5.10. Let $d \le 2$, and let $\boldsymbol{S}_\varepsilon(t)$ be the semigroup generated by (1.1) with $\lambda_2 = 0$. There exists a robust family of exponential attractors $\{\mathcal{M}_\varepsilon : \varepsilon \in [0,\sigma/\kappa_1]\}$ for $\boldsymbol{S}_\varepsilon(t)$ such that

(1) the fractal dimension of $\mathcal{M}_\varepsilon$ is bounded, uniformly with respect to ε,

(2) for every bounded subsets $D \subset \mathbb{H}^1$, there exists a constant c depending on D such that

$$\operatorname{dist}\left(\boldsymbol{S}_\varepsilon(t)D, \mathcal{M}_\varepsilon\right) \le ce^{-\alpha t}, \quad \forall t \ge 0,$$

where the positive constant c and α are independent of ε,

(3) the family $\{\mathcal{M}_\varepsilon : \varepsilon \in [0,\sigma/\kappa_1]\}$ is Hölder continuous at 0, namely

$$\operatorname{dist}_{\mathrm{sym}}\left(\mathcal{M}_\varepsilon, \mathcal{M}_0\right) \le c\varepsilon^k,$$

where $c \ge 0$, $k \in (0,1)$ are independent of ε, and $\operatorname{dist}_{\mathrm{sym}}$ is the symmetric Hausdorff distance defined in (A.5).

Proof. We apply Theorem A.4 with $X = \mathbb{H}^2$, $H = \mathbb{H}^1$, and $B = \mathbb{B}$. The hypotheses of Theorem A.4 are verified in Lemma 5.8 and Lemma 5.9. Lipschitz continuity on $\mathbb{H}^1$ and Hölder continuity in time have been verified before in Theorem 4.16. This completes the proof of the theorem. □

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Appendix A. Auxiliary results

In this section, we collect some inequalities and results which are extensively used in this paper.

Theorem A.1 (The uniform Gronwall inequality)**.** Let g, h, and y be non-negative locally integrable functions on (t_0, ∞) such that $\mathrm{d}y/\mathrm{d}t$ is locally integrable on (t_0, ∞), and which satisfy

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leq g(t)y(t) + h(t), \quad \forall t \geq t_0. \tag{A.1}$$

Let $r > 0$. Suppose that there exist a_1, a_2, $a_3 \geq 0$ such that

$$\int_t^{t+r} g(s)\,\mathrm{d}s \leq a_1, \quad \int_t^{t+r} h(s)\,\mathrm{d}s \leq a_2, \quad \int_t^{t+r} y(s)\,\mathrm{d}s \leq a_3, \quad \forall t \geq t_0.$$

Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right)\exp(a_1), \quad \forall t \geq t_0.$$

Corollary A.2. Suppose that the assumptions of Theorem A.1 hold with $r = 1$. Furthermore, suppose that for all $t > t_0$,

$$\int_{t_0}^t g(s)y(s) + h(s)\,\mathrm{d}s \leq P(t), \tag{A.2}$$

where $P(t)$ is a non-negative function of t. Then for all $t > t_0$,

$$y(t) \leq \left(\frac{a_3}{t-t_0} + 2a_2 + a_3\right)\exp(a_1) + P(t).$$

Proof. By the uniform Gronwall inequality, the assumptions imply that for any $\delta \in (0, 1]$,

$$y(t_0 + \delta) \leq \left(\frac{a_3}{\delta} + a_2\right)\exp(a_1),$$

or equivalently, for any $t \in (t_0, t_0 + 1]$,

$$y(t) \leq \left(\frac{a_3}{t-t_0} + a_2\right)\exp(a_1) \tag{A.3}$$

Integrating (A.1) over $(t_0 + 1, t)$, using (A.2) and (A.3), we obtain for all $t \geq t_0 + 1$,

$$y(t) \leq y(t_0+1) + \int_{t_0+1}^t g(s)y(s) + h(s)\,\mathrm{d}s \leq (a_3 + a_2)\exp(a_1) + P(t). \tag{A.4}$$

Combining (A.3) and (A.4) yields the required inequality. □

Theorem A.3 (Theorem 2, Section 3.3.3 in [19])**.** Let $\mathscr{O}$ be a regular bounded domain. Let $M := \mathbb{W}^{s_1,p_1}(\mathscr{O})$ and let X be the unit ball of the space $\mathbb{W}^{s_2,p_2}(\mathscr{O})$ with

$$\frac{1}{p_1} - \frac{s_1}{d} > \frac{1}{p_2} - \frac{s_2}{d}.$$

Then X is pre-compact in M, and thus the Kolmogorov ϵ-entropy of X (considered as a compact subset of M), denoted by $\mathcal{H}_\epsilon(X)$, is well-defined and satisfies

$$C_1\epsilon^{-d/(s_2-s_1)} \leq \mathcal{H}_\epsilon(X) \leq C_2\epsilon^{-d/(s_2-s_1)},$$

where C_1 and C_2 are independent of ϵ.

Theorem A.4 ([22, 40])**.** Let X and H be two Hilbert spaces such that $X \hookrightarrow H$ is a compact embedding. Suppose that for every $\epsilon \in [0, \epsilon_0]$, $S_\epsilon(t) : B \to B$ is a strongly continuous semigroup acting on a semi-invariant absorbing set $B \subset X$. Suppose further that

(1) there exists a non-negative function $\kappa(t)$, independent of ϵ, such that for every $\epsilon \in [0, \epsilon_0]$ and every $u, v \in B$,

$$\|S_\epsilon(t)u - S_\epsilon(t)v\|_X \leq \kappa(t)\,\|u - v\|_H\,,$$

(2) there exists a constant c, independent of ϵ, such that for every $t \geq 0$ and every $u \in B$,

$$\|S_\epsilon(t)u - S_0(t)u\|_H \leq c\epsilon e^{ct},$$

(3) for every $\epsilon \in [0, \epsilon_0]$, $T > 0$, and $u \in B$, the map $t \mapsto S_\epsilon(t)u$ is Hölder continuous on $[0, T]$;
(4) for every $\epsilon \in [0, \epsilon_0]$ and $t \geq 0$, the map $u \mapsto S_\epsilon(t)u$ is Lipschitz continuous on B.

Then there exists a family of robust exponential attractors $\{\mathcal{M}_\epsilon : \epsilon \in [0, \epsilon_0]\}$ on H such that

(1) the fractal dimension of $\mathcal{M}_\epsilon$ is bounded, uniformly with respect to ϵ,

(2) $\mathcal{M}_\epsilon$ attracts bounded subsets $D \subset H$ exponentially fast, uniformly with respect to ϵ, namely there exists a constant c depending on D such that

$$\operatorname{dist}(S_\epsilon(t)D, \mathcal{M}_\epsilon) \le ce^{-\alpha t}, \quad \forall t \ge 0,$$

where the positive constant c and α are independent of ϵ,

(3) the family $\{\mathcal{M}_\epsilon : \epsilon \in [0, \epsilon_0]\}$ is Hölder continuous at 0, namely

$$\operatorname{dist}_{\text{sym}}(\mathcal{M}_\epsilon, \mathcal{M}_0) \le c\epsilon^k,$$

where $c \ge 0$ and $k \in (0, 1)$ are independent of ϵ. Here, $\operatorname{dist}_{\text{sym}}$ denotes the symmetric Hausdorff distance between sets defined by

$$\operatorname{dist}_{\text{sym}}(A, B) := \max\big(\operatorname{dist}(A, B), \operatorname{dist}(B, A)\big). \tag{A.5}$$

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