OPERATOR SEMIGROUPS IN THE MIXED TOPOLOGY AND THE INFINITESIMAL DESCRIPTION OF MARKOV PROCESSES

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ABSTRACT. We define a class of not necessarily linear C_0-semigroups $(P_t)_{t\geq 0}$ on $C_b(E)$ (more generally, on $C_\kappa(E) := \frac{1}{\kappa}C_b(E)$, for some growth bounding continuous function κ) equipped with the mixed topology $\tau_1^{\mathscr{M}}$ for a large class of topological state spaces E. In the linear case we prove that such $(P_t)_{t\geq 0}$ can be characterized as integral operators given by measure kernels satisfying certain properties. We prove that the strong and weak infinitesimal generators of such C_0-semigroups coincide. As a main result we prove that transition semigroups of Markov processes are C_0-semigroups on $(C_b(E), \tau_1^{\mathscr{M}})$, if they leave $C_b(E)$ invariant and they are jointly weakly continuous in space and time. In particular, they are infinitesimally generated by their generator $(L, D(L))$ and thus reconstructable through an Euler formula from their strong derivative at zero in $(C_b(E), \tau_1^{\mathscr{M}})$. This solves a long standing open problem on Markov processes. Our results apply to a large number of Markov processes given as the laws of solutions to SDEs and SPDEs, including the stochastic 2D Navier-Stokes equations and the stochastic fast and slow diffusion porous media equations. Furthermore, we introduce the notion of a Markov core operator $(L_0, D(L_0))$ for the above generators $(L, D(L))$ and prove that uniqueness of the Fokker-Planck-Kolmogorov equations corresponding to $(L_0, D(L_0))$ for all Dirac initial conditions implies that $(L_0, D(L_0))$ is a Markov core operator for $(L, D(L))$. As a consequence we can identify the Kolmogorov operator of a large number of SDEs on finite and infinite dimensional state spaces as Markov core operators for the infinitesimal generators of the C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ given by their transition semigroups. Furthermore, if each P_t is merely convex, we prove that $(P_t)_{t\geq 0}$ gives rise to viscosity solutions to the Cauchy problem of its associated (non linear) infinitesimal generators. We also show that value functions of optimal control problems, both, in finite and infinite dimensions are particular instances of convex C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$.

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CONTENTS

1. Introduction

This paper addresses a longstanding open problem in the theory of Markov processes. The literature on Markov processes is huge. We here only refer to a selection from pioneering and/or fundamental books on the subject and to the references therein, as, e.g., [3][4][23][28][32][35][39][48][63][66][67][69]. Let us briefly recall the definition of a Markov process: Let $(E, \mathcal{B})$ be a measurable space and, for each $x \in E$, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ be a filtered probability space and $X(t) \colon \Omega \to E$ $\mathcal{F}_t/\mathcal{B}$-measurable maps, $t \geq 0$, such that $\mathbb{P}_x[X(0) = x] = 1$. Then the tuple $\mathbb{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X(t))_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$ is called a (time-homogeneous) *Markov process* with state space E, if it satisfies the *Markov property*, i.e., for all $x \in E, A \in \mathcal{B}, t, s \geq 0$,

$$(1.1) \qquad \mathbb{P}_x[X(s + t) \in A | \mathcal{F}_s] = \mathbb{P}_{X(s)}[X(t) \in A] \quad \mathbb{P}_x\text{-a.s.,}$$

where $\mathbb{P}_x[\,\cdot\,|\mathcal{F}_s]$ denotes the conditional probability of $\mathbb{P}_x$ given $\mathcal{F}_s$. Its corresponding transition semigroup of probability kernels is defined by the time marginal laws of $\mathbb{P}_x$ under $X(t), t \geq 0$, i.e.,

$$(1.2) \qquad p_t(x, dy) := (\mathbb{P}_x \circ X(t)^{-1})(dy), \quad x \in E, t \geq 0.$$

Usually, one also assumes some path regularity on $X(t), t \geq 0$, by considering topological state spaces E together with the corresponding Borel σ-algebra $\mathcal{B} := \mathcal{B}(E)$ and assuming that, for all $x \in E$, the map $[0, \infty) \ni t \mapsto X(t) \in E$ is right-continuous $\mathbb{P}_x$-a.s.. Define for $f \colon E \to \mathbb{R}$, bounded, $\mathcal{B}$-measurable,

$$(1.3) \qquad P_t f(x) := \int_E f(y)\, p_t(x, dy) = \mathbb{E}_x[f(X(t))], \quad x \in E, t \geq 0,$$

where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$. Then the Markov property (1.1) implies the semigroup property

$$(1.4) \qquad P_{t+s}f(x) = P_t \, (P_s \, f)(x), \quad x \in E, t, s \geq 0.$$

A common very natural assumption, which is fulfilled in many situations (in particular, where $\mathbb{P}_x, x \in E$, are the laws of the solutions of a stochastic differential equation (SDE) with respective initial data $x \in E$ and where E is, say a Banach space or just $\mathbb{R}^d$) is the so-called *Feller property*, i.e.,

$$(1.5) \qquad P_t f \in C_b(E), \text{ if } f \in C_b(E), \quad t \geq 0.$$

Here $C_b(E)$ denotes the set of all bounded real-valued continuous functions on E. Let $\mathcal{P}(E)$ denote the set of all probability measures an $(E, \mathcal{B}(E))$. Then (1.5) means:

$$(1.6) \qquad E \ni x \to p_t(x, dy) \in \mathcal{P}(E) \text{ is continuous in the weak topology}$$
$$\text{on } \mathcal{P}(E) \text{ for all } t \geq 0.$$

By the assumed right continuity of sample paths and by (1.5) we also have

$$(1.7) \qquad [0, \infty) \ni t \to p_t(x, dy) \in \mathcal{P}(E) \text{ is right continuous in the weak topology on}$$
$$\mathcal{P}(E) \text{ for all } x \in E.$$

It is well-known that, if we consider $C_b(E)$ with its supremum norm $\| \cdot \|_\infty$, then $t \mapsto P_t f$ is (in general) not continuous at $t = 0$ for all $f \in C_b(E)$, i.e., $(P_t)_{t \geq 0}$ is not a C_0-semigroup on $(C_b(E), \| \cdot \|_\infty)$.

If E is metric space, then the next natural choice is the space $UC_b(E)$ of bounded uniformly continuous functions which, when endowed with the the norm $\| \cdot \|_\infty$, is a closed subspace of $C_b(E)$. It turns out that the gain is very limited. It can be shown that if E is a separable Hilberts space and $(P_t)_{t \geq 0}$ is a transiton semigroup of an E-valued Wiener process that $(P_t)_{t \geq 0}$ is a C_0-semigroup on $UC_b(E)$, see Proposition 3.5.1 in [17]. This result can be easily extended to a general Lévy process. However, the transition semigroup of an Ornstein-Uhlenbeck process in $E = \mathbb{R}$, while it turns out to leave the space $UC_b(\mathbb{R})$ invariant, is not strongly continuous there, see Example 6.1 in [14] and Theorem 2.1 in [70]. The latter result also implies that the transition semigroup of a general Ornstein-Uhlenbeck process with non-zero drift is never strongly continuous on $C_b(E)$.

Hence the theory of C_0-semigroups on Banach spaces (see e.g. [56], [25]) does not apply. If it did, $P_t, t \geq 0$, would be uniquely determined by its derivative at $t = 0$, i.e.,

$$(1.8) \qquad Lf := \frac{d}{dt}_{\restriction t=0} P_t f = \| \cdot \|_\infty - \lim_{t \to 0} \frac{1}{t}(P_t f - f), \quad f \in D(L),$$

which defines a linear operator $L \colon D(L) \subset C_b(E) \to C_b(E)$ with $D(L)$ being the set of all $f \in C_b(E)$ for which the limit in (1.8) exists. In this case $P_t, t \geq 0$, can be recalculated from the operator $(L, D(L))$, called *infinitesimal generator* of $(P_t)_{t \geq 0}$, through Euler's formula. But as said, this is in general not possible on $(C_b(E), \| \cdot \|_\infty)$.

A way out of this, which only works if E is locally compact (hence excludes, e.g., that E is an infinite dimensional Banach space, which in turn are the typical state spaces for solutions

$X(t), t \geq 0$, to stochastic partial differential equations (SPDEs) or measure-valued Markov processes) is to replace (1.5) by

(1.9) $P_t f \in C_\infty(E)$, if $f \in C_\infty(E)$, $t \geq 0$,

where $C_\infty(E)$ denotes the subset of all elements in $C_b(E)$ which vanish at infinity. $(P_t)_{t \geq 0}$, satisfying (1.9) are called *Feller semigroups* in the literature, which sometimes leads to confusion, since the much weaker property (1.5) is usually called Feller property and the latter makes sense on general topological spaces (see, e.g., [59]). But, if E is locally compact and (1.9) holds, there are a large number of examples, for which $(P_t)_{t \geq 0}$ is a C_0-semigroup on $(C_\infty(E), \| \cdot \|_\infty)$ and thus uniquely determined by and reconstructable from its infinitesimal generator $(L, D(L))$, i.e., from its strong derivative at zero (see e.g. [25]). This is usually expressed by the symbolic writing $P_t = e^{tL}, t \geq 0$. On the other hand, condition (1.9) is very strong and in general, of course, not fulfilled, even if $E = \mathbb{R}^d$.

Another approach is to avoid the C_0- (i.e., strong continuity) property and associate to $(P_t)_{t \geq 0}$ an operator $(L, D(L))$, also called *generator* of $(P_t)_{t \geq 0}$, which is obtained by inverting the resolvent of $(P_t)_{t \geq 0}$, which in turn is given by the Laplace transform of $(P_t)_{t \geq 0}$ (see, e.g., [59]). But this definition of generator uses the whole semigroup $(P_t)_{t \geq 0}$ and is thus definitely not an *infinitesimal* generator of $(P_t)_{t \geq 0}$.

Finally, another way out is to replace $C_b(E)$ by an $L^p(E, \mu)$-space, $p \in [1, \infty)$, for some suitable reference measure μ on $(E, \mathcal{B}(E))$ (e.g., an invariant measure for $(P_t)_{t \geq 0}$). Then $(P_t)_{t \geq 0}$ extends to a C_0-semigroup on $L^p(E, \mu)$, which has a true *infinitesimal* generator there (see, e.g., [61] and , [8, Section 4]] and also [35] for symmetrizing measures μ). Clearly, a symmetrizing or invariant measure does not exist in general for $(P_t)_{t \geq 0}$. In [62, Proposition 2.4], however, it was proved that a natural reference measure μ always exists so that the transition semigroup $(P_t)_{t \geq 0}$ of a Markov process $\mathbb{M}$ as above extends to a C_0-semigroup on $L^p(E, \mu)$. But this measure μ again is contructed through the resolvent of $(P_t)_{t \geq 0}$, hence again uses the whole semigroup $(P_t)_{t \geq 0}$. So, the infinitesimal generator of $(P_t)_{t \geq 0}$, extended to a C_0-semigroup on $L^p(E, \mu)$, is not really "infinitesimal". In addition, the analysis of this extension of $(P_t)_{t \geq 0}$, depends on the measure μ and statements can always be only made μ-a.e., and the measure μ is in no sense unique.

So, concluding it can be said that it has been an open problem whether the transition semigroup of a general Markov process $\mathbb{M}$ as above, which has the Feller property (1.5), is infinitesimally generated by its strong derivative at zero in a "suitable" topology on $C_b(E)$.

The first main contribution of this paper concerning the above open problem is to prove that such a "suitable" topology is the well-known mixed topology $\tau_1^{\mathscr{M}}$ on $C_b(E)$, i.e., the strongest locally convex topology on $C_b(E)$ which on $\| \cdot \|_\infty$-bounded subsets of $C_b(E)$ coincides with the topology of uniform convergence on compact subsets of E (see Section 2 and Appendix A for details), provided $(P_t)_{t \geq 0}$, satisfies the following very general condition (cf. (1.6) and (1.7) above):

(1.10) $[0, \infty) \times K \ni (t, x) \to p_t(x, dy) \in \mathcal{P}(E)$ is continuous in the weak topology on
$$\mathcal{P}(E) \text{ for all compact } K \subset E$$

(see Theorem 3.3 and Proposition 3.6 below). In fact, this is true for very general state spaces E (see Hypothesis 2.1 below). So, in such a very general case the transition semigroup of a Markov process with right continuous sample paths is uniquely determined by its strong derivative at zero with respect to the mixed topology $\tau_1^{\mathscr{M}}$ on $C_b(E)$ and can be reconstructed through an Euler formula (see Proposition 5.2 (f)).

We would like to mention here that for a special class of stochastic evolution equations on a Hilbert space, similar to those in Section 4.2 below (see the fundamental book [18] for the general theory) the strong continuity of the transition semigroups of their solutions at $t = 0$ in the mixed topology was first proved in [37] (and the reference therein, in particular [15]). The latter paper was a strong motivation for proving the much more general result above and for developing the corresponding general theory in the present paper. The necessity to relax the norm topology on $C_b(E)$ has been well known in the SPDE community and the problem was approached using many ad-hoc constructions, see for example [14, 15, 17, 59]. In the aforementioned works no underlying topology making the semigroups of interest strongly continuous was identified.

Let us summarize the single sections of this paper and at the same time present our further main results.

Section 2 contains our setup and necessary definitions, in particular, those concerning the mixed topology. We generalize the situation above by replacing $C_b(E)$ by $C_\kappa(E) := \frac{1}{\kappa} \cdot C_b(E)$ with corresponding weighted supremum norm $\| \cdot \|_\kappa$, where $\kappa \colon E \to (0, \infty)$ is a continuous function, and consider the mixed topology $\tau_\kappa^{\mathscr{M}}$ on $C_\kappa(E)$.

In Section 3 we introduce a general class of C_0-semigroups of operators $(P_t)_{t \geq 0}$ on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ (see Definition 3.1). In case these P_t are linear, we prove that the semigroup can be represented by a semigroup of measure kernels with certain properties and that any such gives rise to a (linear) C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ (see Theorem 3.3), which is another main result of this paper. The main underlying fact, why this works, is the well-known result that the topological dual of $(C_b(E), \tau_1^{\mathscr{M}})$ coincides with the set $\mathcal{M}_b(E)$ of all signed Radon measures on $(E, \mathscr{B}(E))$ (see Appendix A for references and a simple proof in Remark A.10 based on the Daniell-Stone Theorem).

Section 4 is devoted to examples on finite and infinite dimensional state spaces. We start with transition semigroups coming from a large class of SDEs on Hilbert spaces H (taking the role of E), including, e.g., the 2D-stochastic Navier-Stokes equations as well as stochastic (fast and slow diffusion) porous media equations (see Section 4.1). Here we consider both the norm topology on H and (in Section 4.2) also the bw-topology on H. Furthermore, we look at a class of SPDEs with Levy noise on Banach spaces E, more precisely SDEs of Ornstein-Uhlenbeck (O-U) type, but driven by Levy noise (see equation (4.18)). Their corresponding transition semigroups, called *generalized Mehler semigroups*, also turn out to be C_0-semigroups on $(C_b(E), \tau_1^{\mathscr{M}})$ both when E is considered with the norm topology (see Section 4.3) and, provided E is reflexive, also with the bw-topology (see Section 4.4). The

interesting feature of the bw-topology is that in this case $C_b(E)$ consists of all bounded sequentially weakly continuous functions on E.

In Section 5.1 we define the strong and weak *infinitesimal generator* $(L, D(L))$ of a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ and prove that they coincide (see Theorem 5.5). Furthermore, we show that the usual "invariance condition" for identifying cores for $(L, D(L))$ also holds in this case (see Proposition 5.3). We introduce the notions of *core operators and Markov core operators* for $(L, D(L))$ (see Definition 5.7). Subsequently, we prove that a sufficient condition for being a Markov core operator $(L_0, D(L_0))$ for $(L, D(L))$ is, that the Fokker-Planck-Kolmogorov equation for $(L_0, D(L_0))$ has a unique solution for all Dirac measures $\delta_x, x \in E$ (see Theorem 5.9). This is another main result of this paper, which is illustrated by a number of applications, where we identify the Kolmogorov operator of a large class of SDEs on $\mathbb{R}^d$ (see Section 5.2) or on a Hilbert space H (see Section 5.4) as a Markov core operator for the infinitesimal generator $(L, D(L))$ of the C_0-semigroup on $(C_\kappa(\mathbb{R}^d), \tau_\kappa^{\mathscr{M}})$ and $(C_\kappa(H), \tau_\kappa^{\mathscr{M}})$, respectively, given by the transition semigroup of the SDE's solutions. Furthermore, in Section 5.3 using results from [47], we identify the Kolmogorov operator of SDE (4.18), i.e., the SDE for the O-U-process with Levy noise on a Hilbert space E, which is a pseudo-differential operator (see equation (5.30)), as a core operator for the generator $(L, D(L))$ of the corresponding generalized Mehler semigroup on $(C_b(E), \tau_1^{\mathscr{M}})$. These results in Sections 5.1 - 5.3 constitute the fourth main contribution of this paper.

In Section 6 we consider the case where the C_0-semigroup $(P_t)_{t\geq0}$ on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ consists of convex operators. In this case we prove that $(P_t)_{t\geq0}$, gives rise to viscosity solutions (see Definition 6.1) to the Cauchy problem of its associated infinitesimal generator. Moreover, we show that every convex Markov C_0-semigroup on $(C_b(E), \tau_\kappa^{\mathscr{M}})$ gives rise to a notion of a nonlinear Markov process under a convex expectation. This provides an analytic counterpart to the recent investigations of G-expectations and nonlinear Markov processes, see [57]. The latter appear in the context of financial modeling in terms of a Brownian motion under volatility uncertainty. Generalizations to uncertainty in the generators of Levy processes and a class of Feller processes have been made in [53], [42], [20], [52]. In this context and, more generally, in Mathematical Finance, the so-called continuity from above on $C_b(E)$ of related risk measures plays an important role. The main results of this section are formulated in Theorems 6.2 and 6.4.

Section 7 contains examples from stochastic optimal control as applications of the result in Section 6, both on finite (Section 7.1) and infinite dimensional (Section 7.2) state spaces.

2. BASIC DEFINITIONS AND SETUP

In this section we recall basic definitions and some properties of the so called mixed topology on a space of continuous functions $\varphi \colon E \to \mathbb{R}$. A very general definition of this topology was introduced in [72] and, in the special case of the space of bounded continuous functions defined on a completely regular topological space E, it was studied in topological measure theory as one of the strict topologies, see [71]. In this paper we restrict our attention to a special class of completely regular topological spaces E, but many results presented in this section, when appropriately reformulated, hold for larger classes of spaces, or even for every completely regular Hausdorff topological space.

The following hypothesis about the space E is assumed to hold throughout the paper and will not be enunciated again.

Hypothesis 2.1. *The space E is a completely regular Hausdorff topological space, such that*

    (1) *compact subsets of E are metrizable,*
    (2) *the Borel σ-algebra $\mathscr{B}(E)$ is identical with the Baire σ-algebra $\mathrm{Ba}(E)$.*
    (3) *a function $\varphi : E \to \mathbb{R}$ is continuous if and only if φ is continuous on every compact subset of E.*

Remark 2.2.

    a) Topological spaces that satisfy condition (3) are known as k_f- or $k_\mathbb{R}$-spaces, see [38] or [71].
    b) Polish spaces satisfy all three conditions of Hypothesis 2.1.
    c) Let $E = F^\star$ be the dual of a separable Banach space F endowed with its weak* topology. Then, E is a Hausdorff topological vector space and thus completely regular, see [38, Theorem 2.9.2]. We say that a set $B \subset E$ is bw-closed if its intersection with every weak*-compact set is weak*-closed. The corresponding topology is completely regular, and is known as the bw-topology, [22, pages 427-428] or [51, Section 2.7]. Clearly, the bw-topology coincides with the weak*-topology on every weak*-compact set, and therefore weak*-compactness is equivalent to bw-compactness. As a consequence, any function $E \to \mathbb{R}$ that is continuous on all weak*-compacts of E endowed with the weak* topology is continuous on E endowed with the bw-topology. In fact, the bw-continuous functions are precisely the sequentially weak*-continuous functions. Thus, part (3) of Hypothesis 2.1 holds. Let us recall that a dual of an infinite-dimensional Banach space endowed with its weak* topology is never a k_f-space, see [54, Theorem 5.1] and [36, Corollary 1.14]. By [24, Theorem 2.3], $\mathscr{B}(E, \sigma(E,F)) = \mathrm{Ba}(E, \sigma(E,F))$, and since E is the countable union of weak*-compacts, we have $\mathscr{B}(E, \sigma(E,F)) = \mathscr{B}(E, bw)$. Hence, $\mathscr{B}(E, bw) = \mathrm{Ba}(E, bw)$ and thus (2) of Hypothesis 2.1 holds. Since balls in E equipped with the weak*-topology are metrizable, and weak*-compacts are norm-bounded, (1) of Hypothesis 2.1 also holds. In particular, every separable reflexive Banach space endowed with the bw-topology satisfies Hypothesis 2.1.

Throughout, we consider a continuous weight function $\kappa \colon E \to (0, \infty)$, and $C_\kappa(E)$ denotes the space of continuous functions $\varphi \colon E \to \mathbb{R}$ with

$$\|\varphi\|_\kappa = \sup_{x \in E} |\kappa(x)\varphi(x)| < \infty.$$

If $\kappa \equiv 1$, we use the notation $C_b(E)$ instead of $C_1(E)$.

On $C_\kappa(E)$, we consider various topologies. One of them is the norm topology $\tau_\kappa^{\mathscr{U}}$ w.r.t. $\|\cdot\|_\kappa$. For any compact set $C \subset E$, we define the seminorm

$$p_{\kappa,C}(\varphi) = \sup_{x \in C} |\kappa(x)\varphi(x)|, \quad \text{for } \varphi \in C_\kappa(E),$$

and we denote the locally convex topology on $C_\kappa(E)$ generated by the family of seminorms $\{p_{\kappa,C} \colon C \text{ compact}\}$ by $\tau_\kappa^{\mathscr{C}}$. Note that, by virtue of our assumptions on the weight function κ, the topology $\tau_\kappa^{\mathscr{C}}$ coincides with the topology $\tau_1^{\mathscr{C}}$ of uniform convergence on compact subsets of E, which is generated by the family of seminorms $p_C(\varphi) := \sup_{x \in C} |\varphi(x)|$, for $\varphi \in C_\kappa(E)$.

We continue with the definition of the mixed topology, which is fundamental for everything that follows. For an arbitrary sequence (C_n) of compact subsets of E and a sequence (a_n) of

positive numbers with $\lim_{n\to\infty} a_n = 0$, we define the seminorm

$$p_{\kappa,(C_n),(a_n)}(\varphi) := \sup_{n\in\mathbb{N}} \big(a_n p_{\kappa,C_n}(\varphi)\big) = \sup_{n\in\mathbb{N}} \sup_{x\in C_n} \big(a_n \kappa(x)|\varphi(x)|\big).$$

Definition 2.3. *The locally convex topology on $C_\kappa(E)$, defined by the family of seminorms*

$$\big\{p_{\kappa,(C_n),(a_n)}\colon C_n \subset E \text{ compact}, \, 0 < a_n \to 0\big\},$$

is called the mixed topology, and is denoted by $\tau_\kappa^{\mathcal{M}}$.

In the language of topological measure theory, $\tau_\kappa^{\mathcal{M}}$ belongs to the class of strict topologies, see [71]. By definition,

$$\tau_\kappa^{\mathcal{C}} \subset \tau_\kappa^{\mathcal{M}} \subset \tau_\kappa^{\mathcal{U}}.$$

For the reader's convenience, we collect some basic properties of the mixed topology in the Appendices A and B. For a more detailed discussion of mixed (or strict) topologies, we refer to [71] and [72].

We now introduce the dual objects of $C_\kappa(E)$. Let $M_b(E)$ denote the space of all signed Radon measures $\mu\colon \mathscr{B}(E) \to \mathbb{R}$ with $|\mu|(E) < \infty$, where $|\mu|$ stands for the total variation measure of μ. Recall that, under Hypothesis 2.1, every Baire measure is Borel, and that a Borel measure $\mu\colon \mathscr{B}(E) \to \mathbb{R}$ with $|\mu|(E) < \infty$ is Radon[1] if, for every Borel set B and every $\varepsilon > 0$, there exists a compact set $C_\varepsilon \subset B$ such that

$$|\mu|\,(B \setminus C_\varepsilon) < \varepsilon.$$

A family $\mathcal{F} \subset M_b(E)$ is said to be tight if, for every $\varepsilon > 0$, there exists a compact $C_\varepsilon \subset E$ such that

$$\sup_{\mu\in\mathcal{F}} |\mu|(E \setminus C_\varepsilon) < \varepsilon.$$

We denote the space of all Radon measures μ on $(E, \mathscr{B}(E))$ with

$$\int_E \frac{|\mu|(dx)}{\kappa(x)} < \infty$$

by $M_\kappa(E)$. That is, $M_\kappa(E) = \kappa \cdot M_b(E)$. Let $M_\kappa^+(E)$ be the subset of all nonnegative measures in $M_\kappa(E)$. If $\mu \in M_\kappa(E)$, then the mapping

$$C_\kappa(E) \ni \varphi \mapsto \int_E \varphi \, d\mu \quad \text{is norm-continuous.}$$

Throughout, we endow $M_\kappa(E)$ with the narrow topology, i.e., the weakest topology such that, for every $\varphi \in C_\kappa(E)$, the mapping

$$M_\kappa(E) \ni \mu \mapsto \int_E \varphi \, d\mu \quad \text{is continuous.}$$

By Theorem A.9, the space $M_\kappa(E)$ endowed with the narrow topology is the topological dual of $\big(C_\kappa(E), \tau_\kappa^{\mathcal{M}}\big)$.

In what follows, we consider (nonlinear) operators on $C_\kappa(E)$, i.e., $C_\kappa(E) \to C_\kappa(E)$. We say that an operator T on $C_\kappa(E)$ is norm-bounded if

$$\sup_{\varphi\in B} \|T\varphi\|_\kappa < \infty$$

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[1]In some papers in topological measure theory this is called a tight measure.

for all norm-bounded sets $B \subset C_\kappa(E)$, i.e., $\sup_{\varphi \in B} \|\varphi\|_\kappa < \infty$. An operator on $C_\kappa(E)$ is called $\tau_\kappa^{\mathcal{M}}$-continuous if it is continuous for the mixed topology $\tau_\kappa^{\mathcal{M}}$. In the Appendix B, we characterize norm-bounded linear operators T on $C_\kappa(E)$ that are $\tau_\kappa^{\mathcal{M}}$-continuous.

3. Strongly continuous semigroups on spaces of continuous functions with mixed topology

In this section, we introduce the notion of strongly continuous and locally equicontinuous semigroups on $\left(C_\kappa(E), \tau_\kappa^{\mathcal{M}} \right)$, which we will refer to as C_0-semigroups. The following definition is a straightforward generalisation of the definition of strongly continuous and equicontinuous semigroups of linear operators on locally convex spaces given in [73].

Definition 3.1. *A family of (possibly nonlinear) operators $P = (P_t)_{t \geq 0}$ on $C_\kappa(E)$ is called a semigroup on $C_\kappa(E)$ if*

(i) *$P_0 \varphi = \varphi$ for all $\varphi \in C_\kappa(E)$,*
(ii) *$P_s P_t \varphi = P_{s+t} \varphi$ for all $s, t \geq 0$ and $\varphi \in C_\kappa(E)$.*

The family P is called a C_0-semigroup on $\left(C_\kappa(E), \tau_\kappa^{\mathcal{M}} \right)$ if it additionally satisfies:

(iii) *The semigroup P is locally uniformly equicontinuous, i.e., for every $T \geq 0$, the family of operators $(P_t)_{0 \leq t \leq T}$ is $\tau_\kappa^{\mathcal{M}}$-uniformly equicontinuous. More precisely, for every $T \geq 0$, $\varepsilon > 0$, and every seminorm $p_{\kappa,(K_n),(a_n)}$, there exists a seminorm $p_{\kappa,(C_n),(b_n)}$ and $\delta > 0$ such that, for every $0 \leq t \leq T$ and $\varphi_1, \varphi_2 \in C_\kappa(E)$,*

$$p_{\kappa,(K_n),(a_n)} \left(P_t \varphi_1 - P_t \varphi_2 \right) < \varepsilon \quad \text{if} \quad p_{\kappa,(C_n),(b_n)}(\varphi_1 - \varphi_2) < \delta.$$

(iv) *The semigroup P is strongly $\tau_\kappa^{\mathcal{M}}$-right continuous, i.e., $P_t \varphi \to \varphi$ in $\tau_\kappa^{\mathcal{M}}$ as $t \to 0$ for every $\varphi \in C_\kappa(E)$. More precisely, for all $\varphi \in C_\kappa(E)$ and every seminorm $p_{\kappa,(K_n),(a_n)}$,*

$$\lim_{t \to 0} p_{\kappa,(K_n),(a_n)} \left(P_t \varphi - \varphi \right) = 0.$$

Remark 3.2.

(i) We note that (iii) and (iv) imply that P is strongly $\tau_\kappa^{\mathcal{M}}$-continuous. Indeed, let $T > 0$, $\varphi \in C_\kappa(E)$, $\epsilon > 0$, and $p_{\kappa,(K_n),(a_n)}$ be a seminorm. Then, by (iii), there exist a seminorm $p_{\kappa,(C_n),(b_n)}$ and $\delta > 0$ such that, for all $t \in [0, T]$ and $\varphi_1, \varphi_2 \in C_\kappa(E)$,

$$p_{\kappa,(K_n),(a_n)} \left(P_t \varphi_1 - P_t \varphi_2 \right) < \varepsilon \quad \text{if} \quad p_{\kappa,(C_n),(b_n)}(\varphi_1 - \varphi_2) < \delta.$$

By (iv),

$$p_{\kappa,(C_n),(b_n)} \left(P_{|s-t|} \varphi - \varphi \right) < \delta.$$

for all $s, t \in [0, T]$ with $|t - s|$ sufficiently small. Hence, if $|t - s|$ is sufficiently small and w.l.o.g. $t < s$, then

$$p_{\kappa,(K_n),(a_n)} \left(P_s \varphi - P_t \varphi \right) = p_{\kappa,(K_n),(a_n)} \left(P_t P_{|s-t|} \varphi - P_t \varphi \right) < \varepsilon.$$

As a consequence for every $\varphi \in C_\kappa(E)$ by the continuity of $P_t \varphi$ on E we easily obtain that for all compact $C \subset E$ the map

$$[0, \infty] \times C \ni (t, x) \mapsto P_t \varphi(x)$$

is continuous.

(ii) Now let us consider the linear case, i.e., each P_t of P is a linear operator. Then

$$(3.1) \qquad \sup_{t \leq T} \sup_{\|\varphi\|_\kappa \leq 1} \|P_t \varphi\|_\kappa < \infty.$$

Indeed, let $\varphi \in C_\kappa(E)$. Then by the uniform boundedness principle it suffices to show that

$$\sup_{t \leq T} \|P_t \varphi\|_\kappa < \infty.$$

If this is not the case, there exist $t_n \in [0, T]$, $n \in \mathbb{N}$, such that

$$(3.2) \qquad \|P_{t_n} \varphi\|_\kappa \geq n.$$

We may assume that $\lim_{n \to \infty} t_n = t \in [0, T]$. Hence by part (i) of this Remark

$$\tau_\kappa^{\mathscr{M}} - \lim_{n \to \infty} P_{t_n} \varphi = P_t \varphi,$$

consequently, by Proposition A.4 in the Appendix $\sup_{n \in \mathbb{N}} \|P_{t_n} \varphi\|_\kappa < \infty$, which contradicts (3.2).

By the semigroup property (3.1) is equivalent to: There exist $M \in [1, \infty)$ and $a \in \mathbb{R}$ such that

$$(3.3) \qquad \|P_t \varphi\|_\kappa \leq M e^{at} \|\varphi\|_\kappa \quad \text{for all } \varphi \in C_\kappa(E) \text{ and } t \geq 0.$$

(The equivalence of (3.1) and (3.3) is, of course, also true in the nonlinear case.)
Furthermore, if P consists of linear operators, then (iii) is equivalent to:
For every $T > 0$ and every seminorm $p_{\kappa,(C_n),(a_n)}$, there exist a seminorm $p_{\kappa,(K_n),(b_n)}$ and $C_T \in (0, \infty)$ such that

$$(3.4) \qquad p_{\kappa,(C_n),(a_n)}(P_t \varphi) \leq C_T p_{\kappa,(K_n),(b_n)}(\varphi) \quad \text{for all } \varphi \in C_\kappa(E) \text{ and } t \in [0, T].$$

We have the following characterization for C_0-semigroups on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$ consisting of linear operators.

Theorem 3.3. *Let $P = (P_t)_{t \geq 0}$ be a semigroup of linear operators on $C_\kappa(E)$. Then, the following conditions are equivalent.*
  (a) *The semigroup P is a C_0-semigroup on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$.*
  (b) *There exists a family of Borel measures $\{\mu_t(x, \cdot) : x \in E, t \geq 0\} \subset M_\kappa(E)$ such that:*
    (1) *The map $E \ni x \mapsto \mu_t(x, B)$ is measurable for every $B \in \mathcal{B}(E)$ and $t \geq 0$.*
    (2) *For every $t \geq 0, \mu_t(\cdot, dy)$ represents P_t, i.e.,*

$$(3.5) \qquad P_t \varphi(x) = \int_E \varphi(y) \mu_t(x, dy) \quad \text{for all } \varphi \in C_\kappa(E), x \in E.$$

   (3) *For every $T \geq 0$,*

$$\sup_{t \leq T} \sup_{x \in E} \left(\kappa(x) \int_E \frac{|\mu_t|\,(x, dy)}{\kappa(y)} \right) < \infty.$$

   (4) *For every $T \geq 0$ and every compact $C \subset E$, the family of measures*

$$\left\{ \frac{\kappa(x)\,|\mu_t|\,(x, dy)}{\kappa(y)} : x \in C,\, t \in [0, T] \right\}$$

   is tight.

(5) *For every $x \in E$ and any sequence $(x_n) \subset E$ with $\lim_{n\to\infty} x_n = x$ (in E), we have*

$$\lim_{(t,x_n)\to(0,x)} \mu_t(x_n, \cdot) = \delta_x$$

in $M_\kappa(E)$, where δ_x denotes the Dirac measure with barycenter x.

Proof. We start with the proof of the implication $(b) \Rightarrow (a)$. Assume that (b) is satisfied. We have to show (iii),(iv) in Definition 3.1. In order to show that (iii) is satisfied, let $T > 0$. For $n \in \mathbb{N}$ let $(a_n) \subset (0, \infty)$ and (C_n) be an increasing sequence of compact subsets of E. Let $b_n := 2^{1-n}$, $n \in \mathbb{N}$. By (4), for every $l \in \mathbb{N}$, there exists an increasing sequence $(K_{l,n})_{n\in\mathbb{N}}$ of compacts in E such that

$$(3.6) \qquad \sup_{t\in[0,T]} \sup_{x\in C_l} \left(\kappa(x) \int_{E\setminus K_{l,n}} \frac{|\mu_t|(x,dy)}{\kappa(y)} \right) \leq 2^{-2n-l} \quad \text{for all } n \in \mathbb{N}.$$

Define, for $n \in \mathbb{N}$,

$$K_n := \bigcap_{l=1}^{\infty} K_{l,n}.$$

Then, (K_n) is an increasing sequence of compacts, and, for all $n \in \mathbb{N}$,

(3.7)

$$\sup_{t\in[0,T]} \sup_{x\in C_l} \left(\kappa(x) \int_{E\setminus K_n} \frac{|\mu_t|(x,dy)}{\kappa(y)} \right) \leq \sup_{t\in[0,T]} \sup_{x\in C_l} \left(\sum_{l=1}^{\infty} \kappa(x) \int_{E\setminus K_{l,n}} \frac{|\mu_t|(x,dy)}{\kappa(y)} \right) \leq 2^{-2n}$$

by (3.6). Now, we are going to show (3.4). To that end, let $\varphi \in C_\kappa(E)$. By homogeneity, we may assume that

$$p_{\kappa,(K_n),(b_n)}(\varphi) = 1,$$

hence

$$(3.8) \qquad p_{\kappa,K_n}(\varphi) \leq 2^{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Setting $K_0 := \emptyset$, by (2), for all $t \in [0,T]$, we have

$$p_{\kappa,(C_l),(a_l)}(P_t\varphi)$$

$$\leq \sup_{l\in\mathbb{N}} a_l \sup_{x\in C_l} \left(\kappa(x) \int_E |\varphi|(y) \, |\mu_t|(x,dy) \right)$$

$$\leq \sup_{l\in\mathbb{N}} a_l \sup_{x\in C_l} \left(\sum_{n=1}^{\infty} p_{\kappa,K_n}(\varphi)\kappa(x) \int_{K_n\setminus K_{n-1}} \frac{|\mu_t|(x,dy)}{\kappa(y)} \right),$$

which, by (3.7) and (3.8), is dominated by

$$\sup_{l\in\mathbb{N}} a_l \left(\sum_{n=2}^{\infty} 2^{n-1} 2^{-2(n-1)} + \sup_{x\in E} \int_{K_1} |\mu_t|(x,dy) \right)$$

$$\leq \sup_{l\in\mathbb{N}} a_l \left(1 + \sup_{t\in[0,T]} \sup_{x\in E} \kappa(x) \int_E \frac{|\mu_t|(x,dy)}{\kappa(y)} \right) =: C_T,$$

where, by (3), this constant is finite. Hence, by the last part of Remark 3.2 (ii), Property (iii) follows.

We proceed to the proof of (iv). Since by Remark 3.2 (ii) we know that (3.1) holds, by Proposition A.4 we have to show that, for every compact $K \subset E$,

$$\lim_{t \to 0} p_{\kappa,K}(P_t \varphi - \varphi) = 0. \tag{3.9}$$

Suppose this does not hold. Then, we can find a compact $K \subset E$, $\varepsilon > 0$, $t_n \to 0$, and $(x_n) \subset K$ such that

$$|P_{t_n} \varphi(x_n) - \varphi(x_n)| \geq \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

Since K is compact and metrizable, there exists some $x \in K$ such that $x_{n_k} \to x$ for a subsequence (n_k). Since (3.10) also holds for this subsequence, we get a contradiction to condition (5).

It remains to establish the implication $(a) \Rightarrow (b)$. If (a) holds, then (3.1) holds by Remark 3.2 (ii). Hence by Theorem B.2, there exists a family of measures $\{\mu_t(x, \cdot) \colon t \geq 0,\ x \in E\}$ such that (1), (2), and (3) hold. If $C \subset E$ is compact, then so is $[0, T] \times C \subset \mathbb{R} \times E$, hence we can use the same arguments as in the proof of Theorem B.2 to prove Property (4). Property (5) is an immediate consequence of the strong continuity of P at zero. $\qquad\square$

Remark 3.4. As just proved above, the dependence of the constant C_T in (3.4) on the semigroup P_t for $t \in [0, T]$ is only via the quantity

$$\sup_{t \in [0,T]} \sup_{x \in E} \kappa(x) \int_E \frac{|\mu_t|(x, dy)}{\kappa(y)},$$

where $\mu_t(x, dy)$, $x \in E$, $t \geq 0$, are the representing measures for $(P_t)_{t \geq 0}$ in (3.5).

The following proposition renders a convenient sufficient condition to check conditions (4) and (5) in Theorem 3.3, if E is a so-called Prohorov space, whose definition we recall first (see [6, Definition 4.7.1(i)]).

Definition 3.5. *Let E be as above (i.e., as in Hypothesis 2.1). Then E is called a Prohorov space, if every compact subset of $M_b^+(E)$ (equipped with the narrow topology) is tight.*

Proposition 3.6. *Let E be Prohorov. Let $\mu_t(x, \cdot) \in M_\kappa^+(E)$, $t \geq 0$, and $x \in E$, such that $E \ni x \mapsto \mu_t(x, B)$ is $\mathcal{B}(E)$-measurable for all $B \in \mathcal{B}(E)$, $t \geq 0$, and $\mu_0(x, \cdot) = \delta_x$ for all $x \in E$. Suppose that (3) in Theorem 3.3 holds and that, for every $T \in (0, \infty)$ and every compact $C \subset E$, the map*

$$[0, T] \times C \ni (t, x) \longmapsto \int_E \varphi(y)\ \mu_t(x, dy)$$

is continuous for every $\varphi \in C_\kappa(E)$. Then (4) and (5) in Theorem 3.3 also hold.

Proof. Since the continuous image of a compact set is compact, by the assumptions, it follows that $\{\mu_t(x, \cdot) \colon x \in C,\ t \in [0, T]\}$ is a compact subset of $M_\kappa^+(E)$. Hence (4) holds, since E is assumed to be Prohorov. Condition (5) is fulfilled since $\{x_n \colon n \in \mathbb{N}\} \cup \{x\}$ is compact for every sequence $(x_n) \subset E$ with $x_n \to x \in E$. $\qquad\square$

Remark 3.7. If E is Polish, then E is Prohorov. Likewise, if E is as in Remark 2.2(3), and equipped with the bounded weak topology τ_{bw}, then [6, Proposition 4.7.6(i)] implies that (E, τ_{bw}) is Prohorov.

4. Examples for linear C_0-semigroups on $\big(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\big)$

We are now going to present large classes of examples for C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ given by transition semigroups of solutions to stochastic differential equations (SDEs) on infinite dimensional state spaces, hence including stochastic partial differential equations (SPDEs) as their main examples. The main tool to show that such transition semigroups are indeed C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ will be Proposition 3.6.

4.1. Transition semigroups of solutions to SDEs on Hilbert spaces of locally monotone type.

The first class of examples come from SDEs in Hilbert spaces of locally monotone type, introduced in [49]. Let us recall the necessary details from [49, Section 5.1].
Let $E := H$ be a separable Hilbert space with inner product $\langle\,,\,\rangle_H$ and H^* its dual. Let V be a reflexive Banach space, such that $V \subset H$ continuously and densely. Then for its dual space V^* it follows that $H^* \subset V^*$ continuously and densely. Identifying H and H^* via the Riesz isomorphism we have that

$$V \subset H \subset V^*$$

continuously and densely and if $_{V^*}\langle\,,\,\rangle_V$ denotes the dualization between V^* and V (i.e. $_{V^*}\langle z, v\rangle_V := z(v)$ for $z \in V^*, v \in V$), it follows that

$$_{V^*}\langle z, v\rangle_V = \langle z, v\rangle_H \quad \text{for all } z \in H, v \in V.$$

(V, H, V^*) is called a <u>Gelfand triple</u>. Note that since $H \subset V^*$ continuously and densely, also V^* is separable, hence so is V. Furthermore, $\mathcal{B}(V)$ is generated by V^* and $\mathcal{B}(H)$ by H^*. We also have by Kuratowski's theorem that $V \in \mathcal{B}(H)$, $H \in \mathcal{B}(V^*)$ and $\mathcal{B}(V) = \mathcal{B}(H) \cap V$, $\mathcal{B}(H) = \mathcal{B}(V^*) \cap H$.
Let $W(t)$, $t \in [0, \infty)$, be a cylindrical Wiener process in a separable Hilbert space U on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\mathcal{F}_t$, $t \in [0, \infty)$. We consider the following stochastic differential equation on H

(4.1) $$dX(t) = A(t, X(t))dt + B(t, X(t))dW(t),$$

where for some fixed time $T > 0$

$$A : [0, T] \times V \times \Omega \to V^*; \quad B : [0, T] \times V \times \Omega \to L_2(U, H)$$

are progressively measurable, where U is another separable Hilbert space and $L_2(U, H)$ denotes the set of all Hilbert-Schmidt operators from U to H.
The coefficients A and B are assumed to satisfy the following conditions:

There exist constants $\alpha \in\,]1, \infty[$, $\beta \in [0, \infty[$, $\theta \in\,]0, \infty[$, $C_0 \in \mathbb{R}$ and a nonnegative adapted process $f \in L^1([0, T] \times \Omega; \delta t \otimes \mathbb{P})$ such that the following conditions hold for all $u, v, w \in V$ and $(t, \omega) \in [0, T] \times \Omega$:

(H1) (Hemicontinuity) The map $\lambda \mapsto {}_{V^*}\langle A(t, u + \lambda v), w\rangle_V$ is continuous on $\mathbb{R}$.

(H2) (Local monotonicity)

$$2_{V^*}\langle A(t, u) - A(t, v), u - v\rangle_V + \|B(t, u) - B(t, v)\|_{L_2(U,H)}^2 \leq (f(t) + \rho(v)) \|u - v\|_H^2,$$

where $\rho : V \to [0, +\infty[$ is a measurable hemicontinuous function which is bounded on bounded sets in V.

$(H3)$ (Coercivity)
$$2_{V^*}\langle A(t,v), v\rangle_V + \|B(t,v)\|_{L_2(U,H)}^2 \leq C_0\|v\|_H^2 - \theta\|v\|_V^\alpha + f(t).$$

$(H4)$ (Growth)
$$\|A(t,v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (f(t) + C_0\|v\|_V^\alpha)(1 + \|v\|_H^\beta).$$

Definition 4.1. *A continuous H-valued $(\mathcal{F}_t)$-adapted process $(X(t))_{t\in[0,T]}$ is called a solution of (4.1), if for its $dt \otimes \mathbb{P}$-equivalent class $\hat{X}$ we have*
$$\hat{X} \in L^\alpha([0,T]\times\Omega, dt\otimes\mathbb{P}; V) \cap L^2([0,T]\times\Omega, dt\otimes\mathbb{P}; H)$$
with α in $(H3)$ and $\mathbb{P}$-a.s.
$$X(t) = X(0) + \int_0^t A(s, \bar{X}(s))ds + \int_0^t B(s, \bar{X}(s))dW(s), \ t \in [0,T],$$
where $\bar{X}$ is any V-valued progressively measurable $dt\otimes\mathbb{P}$-version of $\hat{X}$.

The main existence and uniqueness for (4.1) then reads as follows (see [49, Theorem 5.1.3]).

Theorem 4.2. *Suppose $(H1), (H2), (H3), (H4)$ hold for some $f \in L^{p/2}([0,T]\times\Omega; dt\otimes\mathbb{P})$ with some $p \geq \beta + 2$, and there exists a constant C such that*
$$\|B(t,v)\|_{L^2(U,H)}^2 \leq C(f(t) + \|v\|_H^2), \ t \in [0,T], v \in V;$$
$$\rho(v) \leq C(1 + \|v\|_V^\alpha)(1 + \|v\|_H^\beta), \ v \in V.$$
Then for every $X_0 \in L^p(\Omega, \mathcal{F}_0, P; H)$, (4.1) has a unique solution $(X(t))_{t\in[0,T]}$ such that $X(0) = X_0$. Furthermore, there exsists $C \in [0,\infty)$ such that

(4.2)
$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X(t)\|_H^p\right) \leq C\,\mathbb{E}\big(\|X_0\|_H^p + \int_0^T f^{\frac{p}{2}}(t)dt\big),$$

where $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$.
Moreover, if $A(t,\cdot)(\omega), B(t,\cdot)(\omega)$ are independent of $t \in [0,T]$ and $\omega \in \Omega$, then the laws $\mathbb{P} \circ X(\cdot, x)^{-1}$, $x \in H$, of the solutions $X(t,x)$, $t \in [0,\infty)$, of (4.1) started at $x \in H$, form a time-homogeneous Markov process.

As shown in [49, Section 5.1] the above framework and Theorem 4.2 apply to a large class of SPDEs including the stochastic heat equation (see [49, Remark 4.1.8]), the stochastic p-Laplace equation (see [49, Example 4.1.9]), the stochastic slow diffusion-porous media equation (see [49, Example 4.1.11]), the stochastic fast diffusion-porous media equation (see [60]), both with general diffusivity, the perturbed stochastic Burgers equation (see [49, Lemma 5.1.6 (1) and Example 5.1.8]) and the stochastic $2D$ Navier-Stokes equation (see [49, Example 5.1.10]).

For later use we need the following:

Lemma 4.3. *Consider the situation of Theorem 4.2 and let $X(t,x)$, $t \in [0,T]$, be the unique solution of (4.1) with $X(0,x) = x \in H$. Assume, in addition, that there exists $C_B \in (0,\infty)$ such that*

$$(4.3) \qquad \sup_{s \in [0,T]} \|B(s,x) - B(s,y)\|_{L_2(U,H)} \leq C_B \|x - y\|_H \quad \forall\, x, y \in V.$$

Then for all $x, y \in H$

$$\mathbb{E}\left[exp(-\int_0^T (f(s) + \rho(X(s,y)))ds) \sup_{s \in [0,T]} \|X(s,x) - X(s,y)\|_H^2\right]$$

$$\leq e^{\frac{9}{2}C_B^2 T} \|x - y\|_H^2.$$

In particular, if $x_n, y \in H$ such that $\lim_{n \to \infty} x_n = y$, then

$$\sup_{t \in [0,T]} \|X(t,x_n) - X(t,y)\|_H \xrightarrow[n \to \infty]{} 0 \quad in\ \mathbb{P}\text{-measure.}$$

Proof. Letting

$$F(t) := exp(-\int_0^t (f(s) + \rho(X(s,y)))ds)\ (> 0),\ t \in [0,T],$$

we have by Itô's formula (see e.g. [49, Theorem 4.2.5]) and (H2) that $\forall t \in [0,T]$

$$F(t)\|X(t,x) - X(t,y)\|_H^2 = \|x - y\|^2$$

$$+ 2\int_0^t F(s)\Big({}_{V^*}\langle A(s, X(s,x)) - A(s, X(s,y)), X(s,x) - X(s,y)\rangle_V$$

$$+ \|B(s, X(s,x)) - B(s, X(s,y))\|_{L_2(U,H)}^2 \Big)ds$$

$$- \int_0^t F(s)\big(f(s) + \rho(X(s,y))\big)\|X(s,x) - X(s,y)\|_H^2 ds$$

$$+ \int_0^t F(s)\langle X(s,x) - X(s,y), \big(B(s, X(s,x)) - B(s, X(s,y))\big)dW(s)\rangle_H.$$

Hence by (H2), the Burkholder-Davis-Gundy inequality with $p = 1$ and (4.3)

$$\mathbb{E}\left[\sup_{s \in [0,t]} \Big(F(s)\|X(s,x) - X(s,y)\|_H^2\Big)\right] \leq \|x - y\|_H^2$$

$$+ 3\mathbb{E}\left[\Big(\int_0^t F(s)^2 \|B(s, X(s,x)) - B(s, X(s,y))\|_{L_2(U,H)}^2 \|X(s,x) - X(s,y)\|_H^2\ ds\Big)^{\frac{1}{2}}\right]$$

$$\leq \|x - y\|_H^2$$

$$+ 3C_B^2 \mathbb{E}\left[\sup_{s \in [0,t]} \Big(F(s)^{\frac{1}{2}}\|X(s,x) - X(s,y)\|_H\Big)\Big(\int_0^t F(s)\|X(s,x) - X(s,y)\|_H^2\ ds\Big)^{\frac{1}{2}}\right]$$

$$\leq \|x - y\|_H^2 + \frac{1}{2}\mathbb{E}\left[\sup_{s \in [0,t]} \big(F(s)\|X(s,x) - X(s,y)\|_H^2\big)\right]$$

$$+ \frac{1}{2}9C_B^2 \int_0^t \mathbb{E}\left[\sup_{r \in [0,s]} \big(F(r)\|X(r,x) - X(r,y)\|_H^2\big)\right] ds.$$

Hence by Gronwall's lemma $\forall t \geq 0$

$$\mathbb{E}\left[exp\left(-\int_0^T (f(s) + \rho(X(s,y)))ds\right) \sup_{s \in [0,t]} \|X(s,x) - X(s,y)\|_H^2\right] \leq \|x - y\|_H^2 \; e^{\frac{9}{2}C_B^2 T}.$$

So, if $x_n \to y$ w.r.t. $\|\cdot\|_H$, then

$$\sup_{t \in [0,T]} \|X(t,x_n) - X(t,y)\|_H \underset{n \to \infty}{\longrightarrow} 0 \quad \text{in } \mathbb{P}\text{-measure.} \qquad \square$$

From now on in this section we assume that A and B above do not depend on $\omega \in \Omega$, $t \in [0, \infty)$, and that (H1)-(H4) hold with some constant $f \in [0, \infty)$ replacing the function f. Furthermore, we assume that (4.3) holds.

So, let us now consider the transition semigroup of the unique solution from Theorem 4.2, i.e. for $\varphi \in C_b(H)$, $x \in H$, $t \geq 0$,

$$(4.4) \qquad P_t\varphi(x) := \mathbb{E}[\varphi(X(t,x))] = \int_\Omega \varphi(X(t,x)(\omega))\mathbb{P}(d\omega)$$

$$= \int \varphi(y) \; \mu_t(x, dy),$$

where $X(t,x)$, $t \geq 0$, denotes the solution of (4.1) with initial condition $X(0,x) = x \in H$ and

$$\mu_t(x, dy) := (\mathbb{P} \circ X(t,x)^{-1})(dy).$$

<u>Claim 1.</u>
$(P_t)_{t \geq 0}$ is a Markov C_0-semigroup on $(C_b(H), \tau_1^{\mathscr{M}})$.
<u>Claim 2.</u>
Let $m \in [1, \infty)$ and

$$(4.5) \qquad \kappa(x) := (1 + \|x\|_H^m)^{-1}, \quad x \in H.$$

Then $(P_t)_{t \geq 0}$ is a Markov C_0-semigroup on $\left(C_\kappa(H), \tau_\kappa^{\mathscr{M}}\right)$.
Proof of Claim 1: Clearly, $(P_t)_{t \geq 0}$ and $\mu_t(x, dy)$, $t \geq 0$, $x \in H$, satisfy conditions (1), (2), (3) in Theorem 3.3 with $E := H$ (equipped with its norm topology) and $\kappa = 1$. To show that also (4) and (5) hold, by Proposition 3.6 we have to show that for every compact $C \subset H$ and every $\varphi \in C_b(H)$

$$(4.6) \qquad [0,T] \times C \ni (t,x) \longmapsto P_t\varphi(x) \text{ is continuous.}$$

So, let $t, t_n \in [0, T]$ and $x, x_n \in H$, $n \in \mathbb{N}$, such that

$$(t_n, x_n) \longrightarrow (t, x) \text{ in } [0, T] \times H \text{ as } n \to \infty.$$

Clearly, it then follows by Lemma 4.3 that

$$X(t_n, x_n) \longrightarrow X(t, x) \quad \text{in } \mathbb{P}\text{-measure,}$$

since $X(t_n, x) \to X(t, x)$ $\mathbb{P}$-a.s. Hence $\mu_{t_n}(x_n, \cdot) = \mathbb{P} \circ X(t_n, x_n)^{-1} \longrightarrow \mathbb{P} \circ X(t, x)^{-1} = \mu_t(x, \cdot)$ weakly as $n \to \infty$ and (4.6) follows. Therefore, (P_t) defined in (4.4) is a C_0-semigroup on $(C_b(E), \tau_1^{\mathscr{M}})$ by Theorem 3.3, and Claim 1 is proved. $\qquad \square$

Proof of Claim 2: Obviously, $(P_t)_{t\geq 0}$ satisfies (1),(2) and (for κ as in (4.5)) also (3) in Theorem 3.3, since by (4.2) (applied with $p = m$) for some $C_T \in (0, \infty)$ we have

$$(4.7) \qquad P_t\left(\frac{1}{\kappa}\right)(x) \leq C_T \frac{1}{\kappa(x)} \forall t \in [0, T], \, x \in H.$$

As above (4) and (5) in Theorem 3.3 follow from (4.6) above, however, to be proved for all $\varphi \in C_\kappa(H)$. So, let $\varphi \in C_\kappa(H)$ and $t_n \to t$ in $[0, T]$, $x_n \to x$ in H. Then

$$|P_{t_n}\varphi(x_n) - P_t\varphi(x)|$$
$$= \left|\mathbb{E}[\varphi(X(t_n, x_n)) - \varphi(X(t, x))]\right|$$
$$\leq \|\varphi\|_\kappa \mathbb{E}\left[\left|\|X(t_n, x_n)\|_H^m - \|X(t, x)\|_H^m\right|\right]$$
$$+ \mathbb{E}\left[\left|(\varphi\kappa)(X(t_n, x_n)) - (\varphi\kappa)(X(t, x))\right|(1 + \|X(t, x)\|_H^m)\right],$$

which converges to zero as $n \to \infty$, since we already know from the proof of Claim 1 that $X(t_n, x_n) \to X(t, x)$ in $\mathbb{P}$-measure and since $\|X(t_n, x_n)\|_H^m$, $n \in \mathbb{N}$, are uniformly integrable by (4.2), so that the generalized Lebesgue's dominated convergence theorem applies. Hence Theorem 3.3 implies Claim 2. $\qquad\square$

4.2. Transition semigroups of mild solutions to SDEs on Hilbert spaces with bounded weak topology.

Let H be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and norm $|\cdot|_H$. We will denote the space H endowed with the bounded weak topology by H_{bw}. In this section we will consider the following stochastic evolution equation in H:

$$(4.8) \qquad dX(t) = (AX(t) + F(X(t))) \, dt + G(X(t)) \, dW(t), \quad X(0) = x \in H \, .$$

We assume that
- W is a cylindrical Wiener process on a separable Hilbert space U,
- A generates a C_0-semigroup T_t, $t \geq 0$, on H,
- $F : H \to H$ satisfies the Lipschitz condition with a constant L:

$$|F(x) - F(y)|_H \leq L|x - y|_H, \quad x, y \in H,$$

- $G : E \to L(U, E)$ (:= all continuous linear operators from U to H) is strongly measurable and satisfies the conditions

$$\|T_t G(x)\|_{L_2(U,H)}^2 \leq k(t) \left(1 + |x|_H^2\right), \quad x \in H,$$

and

$$\|T_t (G(x) - G(y))\|_{L_2(U,H)}^2 \leq k(t)|x - y|_H^2, \quad x, y \in H,$$

where $k \in L_{loc}^1(0, \infty)$, $k \geq 0$. In the above we use the notation $\|B\|_{L_2(U,H)}$ for the Hilbert-Schmidt norm of an operator $B : U \to H$. Under the above assumptions equation (4.8) has a unique mild solution in H given by the formula

$$X(t, x) = T_t x + \int_0^t T_{t-s}F(X(s, x)) \, ds + \int_0^t T_{t-s}G(X(s, x)) \, dW(s), \quad t \in [0, T].$$

Moreover, by standard arguments we find, that for all $x \in H$, $T > 0$

$$(4.9) \qquad \sup_{t \leq T} \mathbb{E}\, |X(t, x)|_H^m \leq C_m(T)\, (1 + |x|_H^m)$$

and

$$(4.10) \qquad \sup_{x \in B_r} \mathbb{E} \, |X(t,x) - T_t x|_H^m \leq C_m(T,r) K_t^{m/2}, \quad t \in [0,T] \,,$$

where B_r denotes the open centered ball of radius r in H and

$$K_t = \int_0^t (1 + k(s)) \, ds, \quad t \leq T \,.$$

Let $S_t \varphi(x) := \mathbb{E}\varphi\left(X(t,x)\right)$ for $\varphi \in C_{\kappa_m}(H)$. Following the arguments from [37] we obtain for

$$(4.11) \qquad \kappa_m := (1 + |x|^m)^{-1}, \quad m \geq 2,$$

that

$$(4.12) \qquad \|S_t\|_{C_{\kappa_m}(H) \to C_{\kappa_m}(H)} \leq M_m e^{\gamma_m t}, \ t \geq 0.$$

By an easy modification of the proof in [37] one can prove that under the above assumptions the semigroup (S_t) is a C_0-semigroup in $\left(C_m(H), \tau_{\kappa_m}^{\mathcal{M}}\right)$ We will show that $(S(t))$ defines also a C_0-semigroup on the space $\left(C_m(H_{bw}), \tau_{\kappa_m}^{\mathcal{M}}\right)$. The next proposition extends the result in [50].

Proposition 4.4. *Assume that that the semigroup T_t, $t > 0$, is compact in H. Then the semigroup S_t, $t > 0$, defines a C_0-semigroup on $\left(C_{\kappa_m}(H_{bw}), \tau_{\kappa_m}^{\mathcal{M}}\right)$.*

Proof. By a result in [50] we have $S_t C_b(H_{bw}) \subset C_b(H_{bw})$ for any $t > 0$. Hence by (4.12) it easily follows that $S_t : C_{\kappa_m}(H_{bw}) \subset C_{\kappa_m}(H_{bw})$ and that

$$(4.13) \qquad \|S(t)\|_{C_{\kappa_m}(H_{bw}) \to C_{\kappa_m}(H_{bw})} \leq M_m e^{\gamma_m t}, \ t \geq 0.$$

This is the only part of the proof where compactness of T_t is required.

We will show that the semigroup (S_t) satisfies conditions (1) - (5) in part (b) of Theorem 3.3, where $\mu_t(x, V) = \mathbb{P}(X(t,x) \in V)$ for Borel sets $V \subset H$. We recall here that the Borel σ-algebras of H and H_{bw} coincide and clearly, the mapping

$$H_{bw} \ni x \to \mu_t(x, V)$$

is $\mathscr{B}(H_{bw})$-measurable for every $t \geq 0$ and $V \in B(H_{bw})$, hence condition (1) of Theorem 3.3 holds. By (4.13) condition (2) of Theorem 3.3 is satisfied as well. Invoking (4.9) we obtain for all $x \in H$, $T > 0$

$$\int_{H_{bw}} \frac{\mu_t(x, dy)}{\kappa_m(y)} = \mathbb{E}\left(1 + |X(t,x)|^m\right) \leq C_m(T)\left(1 + |x|_H^m\right), \quad t \in [0,T] \,,$$

and condition (3) of Theorem 3.3 follows. Since B_r is bw-compact for every $r > 0$ we can use (4.9) again to show that for every $T > 0$ and every $r > 0$ the family of measures

$$\left\{ \frac{\kappa_m(x)\mu_t(x, dy)}{\kappa_m(y)} : x \in B_r, \, t \leq T \right\}$$

is tight, which yields condition (4) of Theorem 3.3. It remains to prove that condition (5) is satisfied and it is enough to prove this condition for $m = 0$. Let $\varphi \in C_b(H_{bw})$. Let $t_n \to 0$ and $x_n \to x$ weakly with $\sup_{n \geq 1} |x_n|_H \leq r$ for a certain $r > 0$. For any $\varepsilon > 0$ and $T > 0$ we can choose $R \geq r$ such that

$$\sup_{x \in B_r} \sup_{t \leq T} |T_t x|_H < R \,,$$

and

$$\sup_{x \in B_r} \sup_{t \leq T} \mathbb{P}\left(\left|X(t,x)\right|_H > R\right) < \varepsilon \,.$$

Let $\{f_k; \, |f|_H = 1 \,, k \geq 1\}$ be a dense set in the sphere $\{f \in H; \, |f|_H = 1\}$. We recall that the metric

$$\rho(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\langle x - y, f_k \rangle|}{1 + |\langle x - y, f_k \rangle|}, \quad x, y \in B_R$$

defines a Polish topology identical with the weak topology on B_R. We have

$$S_{t_n} \varphi\left(x_n\right) = \mathbb{E}\varphi\left(X\left(t_n, x_n\right)\right) I_{B_R}\left(X\left(t_n, x_n\right)\right) + \mathbb{E}\varphi\left(X\left(t_n, x_n\right)\right) I_{B_R^c}\left(X\left(t_n, x_n\right)\right)$$
$$= \varphi(x)\mathbb{E}I_{B_R}\left(X\left(t_n, x_n\right)\right) + \delta\left(R, t_n, x_n\right)$$

where

$$\delta\left(R, t_n, x_n\right) = \mathbb{E}\left(\varphi\left(X\left(t_n, x_n\right)\right) - \varphi\left(x\right)\right) I_{B_R}\left(X\left(t_n, x_n\right)\right)$$
$$+ \mathbb{E}\varphi\left(X\left(t_n, x_n\right)\right) I_{B_R^c}\left(X\left(t_n, x_n\right)\right)$$
$$= \delta_1\left(R, t_n, x_n\right) + \delta_2\left(R, t_n, x_n\right)$$

hence

(4.14) $$\left|\delta\left(R, t_n, x_n\right)\right| \leq \left|\delta_1\left(R, t_n, x_n\right)\right| + \frac{C\|\varphi\|_\infty}{R} \,.$$

Let ω be the modulus of continuity of the function φ on B_R. Then

$$\left|\delta_1\left(R, t_n, x_n\right)\right| \leq \mathbb{E}\,\omega\left(\rho\left(X\left(t_n, x_n\right), x\right)\right) I_{B_R}\left(X\left(t_n, x_n\right)\right)$$

Setting $\psi(t, x) := T_t x$ we obtain

$$\rho\left(X\left(t_n, x_n\right), x\right) \leq \rho\left(X\left(t_n, x_n\right), \psi\left(t_n, x_n\right)\right) + \rho\left(\psi\left(t_n, x_n\right), x\right) \,.$$

For every $f \in H$ the function

$$[0, T] \times H_{bw} \ni (t, x) \rightarrow \langle \psi(t, x), f \rangle_H$$

is continuous, hence

$$\lim_{n \to \infty} \rho\left(\psi\left(t_n, x_n\right), x\right) = 0 \,.$$

Therefore, invoking (4.10) we find that for every $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}\left(\rho\left(X\left(t_n, x_n\right), x\right) > \varepsilon\right) = 0 \,,$$

hence $\left|\delta_1\left(R, t_n, x_n\right)\right| \to 0$ for $n \to \infty$. Finally, again by (4.9)

$$\limsup_n \left|S\left(t_n\right) \varphi\left(x_n\right) - \varphi(x)\right| \leq |\varphi(x)| \limsup_n \mathbb{E}I_{B_R^c}\left(X\left(t_n, x_n\right)\right) + \frac{C\|\varphi\|_\infty}{R}$$
$$\leq \frac{2C\|\varphi\|_\infty}{R} \,,$$

and condition (5) of Theorem 3.3 follows by taking $R \to \infty$. $\qquad\square$

4.3. Generalized Mehler semigroups on Banach spaces with norm topology.

Let E be a separable Banach space and let $\kappa = 1$. Let $(T_t)_{t\geq 0}$ be C_0-semigroup of operators on E. Furthermore, let μ_t, $t \in [0, \infty)$, be probability measures on $(E, \mathcal{B}(E))$ such that:

$$(4.15) \qquad [0, \infty) \ni t \longmapsto \mu_t \in \mathcal{M}_b(E) \text{ is narrowly (i.e., } \sigma(\mathcal{M}_b(E), C_b(E))\text{-) continuous.}$$

$$(4.16) \qquad \mu_{t+s} = \left(\mu_t \circ T_s^{-1}\right) * \mu_s \quad t, s \in [0, \infty).$$

Define for $t \in [0, \infty)$, $x \in E$

$$(4.17) \qquad P_t\varphi(x) := \int_E \varphi(T_t x + y)\, \mu_t(dy), \quad \varphi \in C_b(E).$$

Then $(P_t)_{t\geq 0}$ is (by (4.16)) a semigroup of linear operators on $C_b(E)$, called a "generalized Mehler semigroup". In this generality such semigroups have been first introduced in [10] and then further analyzed in [34] and many other papers (see e.g. the very recent work [2] and the references therein). They appear as transition semigroups of Ornstein-Uhlenbeck process with Levy noise, i.e. solutions to the following SDEs on E

$$(4.18) \qquad dX(t) = AX(t)dt + dY(t),$$

where A is the generator of (T_t) on E and $Y(t)$, $t \geq 0$, is the underlying Levy process corresponding to the Levy characteristics appearing in the Levy-Khintchine representation of the exponent of the Fourier transforms of μ_t, $t \geq 0$. We refer to [34] for details. Obviously, (P_t) has a representation as in (3.5) with

$$(4.19) \qquad \mu_t(x, dy) := (\delta_{T_t x} * \mu_t)(dy), \quad t \in [0, \infty), \ x \in E.$$

So, clearly conditions (1)-(3) in Theorem 3.3 hold. To show that (P_t) in (4.17) is a C_0-semigroup on $(C_b(E), \tau_1^{\mathcal{M}})$, it remains to prove that (4) and (5) hold, for which by Proposition 3.6 it suffices to show that for all $\varphi \in C_b(E)$ the map

$$[0, \infty) \times E \ni (t, x) \longmapsto \int_E \varphi(T_t x + y)\, \mu_t(dy)$$

is continuous. So, let $x_n, x \in E$, $t_n, t \in [0, \infty)$ such that $\lim_{n\to\infty} t_n = t$ and $\lim_{n\to\infty} x_n = x$ (w.r.t. the norm topology on E). Then we have to show that for all $\varphi \in C_b(E)$

$$\int_E \varphi\, d(\delta_{T_{t_n} x_n} * \mu_{t_n}) \longrightarrow \int_E \varphi\, d(\delta_{T_t x} * \mu_t) \quad \text{as } n \to \infty.$$

By the Portemanteau theorem we may assume that φ is Lipschitz with Lipschitz constant less or equal to one. Then we have

$$\left| \int_E \varphi(T_t x + y)\, \mu_t(dy) - \int_E \varphi(T_{t_n} x_n + y)\, \mu_{t_n}(dy) \right|$$

$$\leq \left| \int_E \varphi(T_t x + y)\, (\mu_t - \mu_{t_n})(dy) \right| + \|T_t x - T_{t_n} x_n\|_E,$$

which clearly converges to zero as $n \to \infty$ by (4.15) and because (T_t) is a C_0-semigroup on E.

4.4. **Generalized Mehler semigroups on Banach spaces with bounded weak topology.**

Let $\kappa = 1$ and E be a reflexive separable Banach space (in particular, E is as in Remark 2.2 (3)). Let us now consider (E, τ_{bw}), i.e. E equipped with the bounded weak topology (see Remark 2.2 (3)). Then, since E is separable, we have that $\mathscr{B}((E, \|\cdot\|_E)) = \mathscr{B}((E, \tau_{bw}))$. Let μ_t, $t \in [0, \infty)$, be as in (iii) above, satisfying (4.16), but instead of (4.15), we assume the weaker condition

$$(4.20) \qquad [0, \infty) \ni t \longrightarrow \mu_t \in \mathscr{M}_b((E, \tau_{bw}))$$

$$\text{is narrowly } \Big(\text{i.e., } \sigma\big(\mathscr{M}_b((E, \tau_{bw})), C_b((E, \tau_{bw}))\big)\Big) \text{ continuous.}$$

Let (P_t) be defined as in (4.17). We want to show that again by Theorem 3.3 and Proposition 3.6 (P_t) is a C_0-semigroup on $C_b\big(((E, \tau_{bw})), \tau_1^{\mathscr{M}}\big)$. We recall that $C_b((E, \tau_{bw}))$ are exactly the bounded sequentially weak*-continuous functions on E and that each τ_{bw}-compact $C \subset E$ is metrizable (see Remark 2.2 (3)). Obviously (P_t) is a semigroup of linear operators on $C_b((E, \tau_{bw}))$ satisfying conditions (1)-(3) in Theorem 3.3. It remains to prove (4) and (5), which again will follow by Proposition 3.6. So let $t_n \to t$ in $[0, T]$, $x_n \to x$ in (E, τ_{bw}) and $\varphi \in C_b((E, \tau_{bw}))$. We have to show that

$$(4.21) \qquad \lim_{n \to \infty} P_{t_n}\varphi(x_n) = P_t\varphi(x).$$

Let us recall the definition of the finitely based C_b^1-functions, i.e.

$$\mathscr{F}C_b^1 := \{f(l_1, \ldots, l_m) \mid m \in \mathbb{N}, \ f \in C_b^1(\mathbb{R}^m), \ l_1, \ldots, l_m \in E^*\}.$$

By Theorem A.8 in the Appendix and the Hahn-Banach theorem $\mathscr{F}C_b^1$ is dense in $C_b((E, \tau_{bw}), \tau_1^{\mathscr{M}})$. By (4.17) we have

$$(4.22) \qquad |P_t\varphi(x) - P_{t_n}\varphi(x_n)|$$

$$\leq \left| \int_E \varphi(T_t x + y) \ (\mu_t - \mu_{t_n})(dy) \right|$$

$$+ \int_E |\varphi(T_t x + y) - \varphi(T_{t_n} x_n + y)| \ \mu_{t_n}(dy).$$

Clearly, since its integrand is in $C_b((E, \tau_{bw}))$, the first integral on the r.h.s. of (4.22) converges to zero as $n \to \infty$ by assumption (4.20). To see that this also holds for the second, let $\varepsilon > 0$. Since (E, τ_{bw}) is a Skorohod space (see Remark 3.7), by (4.20) there exists a τ_{bw}-compact set $K_\varepsilon \subset E$ such that

$$(4.23) \qquad \sup_{t \in [0, T]} \mu_t(K_\varepsilon^c) < \varepsilon.$$

Since $T_{t_n} x_n \underset{n \to \infty}{\longrightarrow} T_t x$ weakly, there exists a τ_{bw}-compact set $C \subset E$ such that

$$\{T_{t_n} x_n \mid n \in \mathbb{N}\} \cup \{T_t x\} \subset C.$$

Furthermore, since $K_\varepsilon + C$ is τ_{bw}-compact, there exists $\psi = f(l_1, \ldots, l_m) \in \mathscr{F}C_b^1$ such that

$$(4.24) \qquad p_{1, K_\varepsilon + C}(\varphi - \psi) < \varepsilon.$$

Clearly, we may assume that $\|\psi\|_\infty \le \|\varphi\|_\infty$. Then we can estimate the second term on the r.h.s. of (4.22) by

$$\int_{K_\varepsilon} |\varphi - \psi|(T_t x + y)\, \mu_{t_n}(dy) + \int_{K_\varepsilon} |\varphi - \psi|(T_{t_n} x_n + y)\, \mu_{t_n}(dy)$$
$$+ 4\|\varphi\|_\infty\, \mu_{t_n}(K_\varepsilon^c) + \|Df\|_\infty\, |P_m(T_t x - T_{t_n} x_n)|_{\mathbb{R}^m}$$

where $P_m(z) = (l_1(z), \ldots, l_m(z))$, $z \in E$. Letting first $n \to \infty$ and then $\varepsilon \to 0$ by (4.23), (4.24) we obtain (4.21).

5. Strong and Weak infinitesimal generators

5.1. Definitions and (Markov) core operators.

Definition 5.1. Let $(P_t)_{t\ge0}$ be a C_0-semigroup on $\big(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\big)$. Then, we define its infinitesimal generator L by the formula
(5.1)
$$L\varphi := \tau_\kappa^{\mathscr{M}} - \lim_{t\to0} \frac{P_t\varphi - \varphi}{t} \quad for \quad \varphi \in D(L) := \left\{ \psi \in C_\kappa(E) : \tau_\kappa^{\mathscr{M}} - \lim_{t\to0} \frac{P_t\psi - \psi}{t} \text{ exists} \right\}.$$

In order to formulate the next result, we first recall that, if X is any sequentially complete locally convex linear space, then a continuous function $f : [0,T] \to X$ is Riemann integrable, and the function $F(t) = \int_0^t f(s)\, ds$ is differentiable with $\frac{dF}{dt} = f(t)$ for every $t \in (0,T)$ (see [30] for details). We also recall that, by Theorem A.6, the space $\big(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\big)$ is complete. In the next propostion we collect some known properties of C_0-semigroups of operators on $C_\kappa(E)$. Parts (b)-(e) of the Proposition were proved in a more general framework in [40], part (a) in [1].

Proposition 5.2. Let $P = (P_t)_{t\ge0}$ be a C_0-semigroup on $\big(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\big)$ consisting of linear operators with generator L. Then, the following holds:

(a) The $\tau_\kappa^{\mathscr{M}}$-closure of $D(L)$ is identical with $C_\kappa(E)$.
(b) The generator L is $\tau_\kappa^{\mathscr{M}}$-closed, that is for every net $(\varphi_\alpha) \subset D(L)$, such that $\varphi_\alpha \to \varphi$ and $L\varphi_\alpha \to \psi$ we have $\varphi \in D(L)$ and $L\varphi = \psi$.
(c) For every $\varphi \in D(L)$ we have $P_t\varphi \in D(L)$ and $LP_t\varphi = P_t L\varphi$. In particular, each $P_t : D(L) \to D(L)$ is continuous in the $\tau_\kappa^{\mathscr{M}}$-graph topology of L on $D(L)$.
(d) For every $\varphi \in D(L)$
$$P_t\varphi - \varphi = \int_0^t P_s L\varphi\, ds.$$

Moreover, for every $\varphi \in C_\kappa(E)$
$$\int_0^t P_s\varphi\, ds \in D(L), \quad and \quad P_t\varphi - \varphi = L \int_0^t P_s\varphi\, ds.$$

(e) For every $\lambda > \omega$ and $\varphi \in C_\kappa(E)$, the Riemann integral
$$J(\lambda)\varphi = \int_0^\infty e^{-\lambda t} P_t\varphi\, dt,$$

is convergent in the topology $\tau_\kappa^{\mathscr{M}}$ and $J(\lambda) = (\lambda - L)^{-1}$. In particular, P is the unique C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ of linear operators with generator L.

(f) *The Euler formula holds, i.e., for all $\varphi \in C_\kappa(E)$*

$$P_t\varphi = \tau_\kappa^{\mathscr{M}} - \lim_{n\to\infty} \left(\frac{n}{t}(\frac{n}{t} - L)^{-1}\right)^n \varphi.$$

Proof. According to the remarks preceding this theorem, we only have to prove (e). But by Remark 5.6 below this is an immediate consequence of Theorem 5.6 in [11] (attributed there to F. Kühnemund, see [43]) □

Proposition 5.3. *Let $\mathscr{D} \subset D(L)$ be a $\tau_\kappa^{\mathscr{M}}$-dense set in $C_\kappa(E)$ such that $P_t\mathscr{D} \subset \overline{\mathscr{D}}^L$ for every $t > 0$, where $\overline{\mathscr{D}}^L$ denotes the closure of $\mathscr{D}$ in the $\tau_\kappa^{\mathscr{M}}$-graph topology of L on $D(L)$. Then $\mathscr{D}$ is a $\tau_\kappa^{\mathscr{M}}$-core for L.*

Proof. Since by Proposition 5.2 (c) each P_t is continuous in the $\tau_\kappa^{\mathscr{M}}$-graph topology of $D(L)$, we have that $P_t(\overline{\mathscr{D}^L}) \subset \overline{\mathscr{D}}^L$, so we may assume that $\mathscr{D} = \overline{\mathscr{D}}^L$. Now let $\varphi \in D(L)$. We have to show that $\varphi \in \overline{\mathscr{D}}^L = \mathscr{D}$. There exists a net $(\varphi_\alpha) \subset \mathscr{D}$ such that

$$\tau_\kappa^{\mathscr{M}} - \lim_\alpha \varphi_\alpha = \varphi$$

We claim that for every α and $t \geq 0$

(5.2)
$$\int_0^t P_s\varphi_\alpha \, ds \in \mathscr{D}.$$

Indeed, this integral initially converges in $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$, but again by Proposition 5.2 (c) it also converges in $\overline{\mathscr{D}}^L = \mathscr{D}$ in the $\tau_\kappa^{\mathscr{M}}$-graph topology of L on $D(L)$. So (5.2) holds. Furthermore, by definition of the Riemann integral in $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ we have for all $t \geq 0$

$$\tau_\kappa^{\mathscr{M}} - \lim_\alpha \int_0^t P_s\varphi_\alpha \, ds = \int_0^t P_s\varphi \, ds$$

and since by Preposition 5.2 (d)

$$\tau_\kappa^{\mathscr{M}} - \lim_\alpha L \int_0^t P_s\varphi_\alpha \, ds = \tau_\kappa^{\mathscr{M}} - \lim_\alpha(P_t\varphi_\alpha - \varphi_\alpha) = P_t\varphi - \varphi,$$

we conclude that

(5.3)
$$\int_0^t P_s\varphi \, ds \in \overline{\mathscr{D}}^L = \mathscr{D}.$$

Finally, since $\varphi \in D(L)$, it follows that

$$\tau_\kappa^{\mathscr{M}} - \lim_{t\to 0} \frac{1}{t} \int_0^t P_s\varphi \, ds = \varphi$$

and

$$\tau_\kappa^{\mathscr{M}} - \lim_{t\to 0} L\left(\frac{1}{t} \int_0^t P_s\varphi \, ds\right) = \tau_\kappa^{\mathscr{M}} - \lim_{t\to 0} \frac{1}{t} \int_0^t P_s L\varphi \, ds = L\varphi.$$

Therefore, by (5.3), $\varphi \in \overline{\mathscr{D}}^L = \mathscr{D}$.

□

Definition 5.4. *Let $P = (P_t)_{t\geq 0}$ be a C_0-semigroup on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$. We say that $\varphi \in D\left(L_w\right) \subset C_\kappa(E)$ if and only if there exists some $f \in C_\kappa(E)$ such that $\frac{1}{t}(P_t\varphi - \varphi) \xrightarrow{t\to 0} f$ weakly in $\left(C_\kappa(E), \ \tau_\kappa^{\mathscr{M}}\right)$, i.e.,*

$$(5.4) \qquad \lim_{t\to 0} \int_E \frac{P_t\varphi(x) - \varphi(x)}{t} \ \nu(dx) = \int_E f(x) \ \nu(dx)$$

for each $\nu \in M_\kappa(E)$. In this case, we define the operator L_w by the formula

$$L_w\varphi = f.$$

We say that L_w is the weak generator of the C_0-semigroup P on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$ with domain $D\left(L_w\right)$.

Theorem 5.5. *Let $(P_t)_{t\geq 0}$ be a C_0-semigroup on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$ consisting of linear operators. Then,*

$$L = L_w.$$

Moreoever, $\varphi \in D(L)$ if and only if

$$(5.5) \qquad \sup_{t\leq 1} \left(\frac{1}{t} \|P_t\varphi - \varphi\|_\kappa\right) < \infty,$$

$$(5.6) \qquad f(x) := \lim_{t\to 0} \frac{P_t\varphi(x) - \varphi(x)}{t} \quad \text{exists for all } x \in E,$$

and $f \in C_\kappa(E)$. In this case, $f = L\varphi$.

Proof. We start with the proof that $L = L_w$, which is a modification of the proof of [56, p. 43, Corollary 1.2] given for C_0-semigroups in Banach spaces. If $\varphi \in D(L)$, then (5.4) follows by Theorem A.9. Hence, $L \subset L_w$. To show that $L_w \subset L$ choose $\varphi \in D\left(L_w\right)$. Then, by the semigroup property, $t \mapsto P_t\varphi$ is a weakly continuous differentiable curve in $\left(C_\kappa(E), \ \tau_\kappa^{\mathscr{M}}\right)$. Hence

$$\int_E \left(P_t\varphi(x) - \varphi(x)\right) \nu(dx) = \int_0^t \int_E P_s L_w\varphi(x) \ \nu(dx) \ ds = \int_E \left(\int_0^t P_s L_w\varphi \ ds\right)(x) \ \nu(dx),$$

where for the secound equality we used that a continuous linear functional on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$ interchanges with the $C_\kappa(E)$-valued Riemann integral. Taking $\nu := \delta_x$, for $x \in E$, we get, for all $x \in E$,

$$(5.7) \qquad P_t\varphi(x) - \varphi(x) = \int_0^t P_s L_w\varphi(x) \ ds = \left(\int_0^t P_s L_w \ \varphi \ ds\right)(x),$$

where the last integral is the Riemann integral in $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$ of $s \mapsto P_s L_w\varphi$, which is a continuous curve in $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$, since $L_w\varphi \in C_\kappa(E)$. It follows that

$$\tau_\kappa^{\mathscr{M}} - \lim_{t\downarrow 0} \frac{P_t\varphi - \varphi}{t} = L_w\varphi,$$

which shows that $\varphi \in D(L)$ and concludes the proof that $L = L_w$.

Assume that (5.5) and (5.6) hold. Then, one immediately sees that (5.4) is satisfied for every measure $\nu \in M_\kappa(E)$, hence $\varphi \in D(L_w) = D(L)$ with $f = L\varphi$ by the first part of the proof. Conversely, assume that $\varphi \in D(L)$. Then, (5.6) with $f = L\varphi$ is obvious and, by Proposition A.4, (5.5) holds. $\qquad\square$

Remark 5.6. It is very easy to check that in the linear case our C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ are special cases of the bi-continuous semigroups introduced in [43]. We also refer to [12], [27], [41] and [45] for further developments, and furthermore to [31], where only a sequential C_0-property is required. In particular, according to the main result in [43] there is a Hille-Yosida-type Theorem for characterizing their infinitesimal generators defined in Definition 5.1. Likewise we have a characterization for the latter through a Hille-Phillips-type Theorem by the very nice recent paper [13] (see Theorems 3.6 and 3.15 therein).

Next we want to discuss examples of infinitesimal generators for C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$, which are given by transition semigroups of solutions to S(P)DEs, and the relation to the Kolmogorov operators associated to the latter. In each case we shall proceed in two steps. First, we shall prove that the respective infinitesimal generator is an extension of the Kolmogorov operator associated to the latter. Second, we shall prove "strong uniqueness" or at least "Markov uniqueness" for the respective Kolmogorov operator, which thus uniquely determines the infinitesimal generator of the corresponding C_0-semigroup on $(C_\kappa, \tau_\kappa^{\mathscr{M}})$. We start with the following definitions.

Definition 5.7. *Let P_t, $t \geq 0$, be a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ with infinitesimal generator $(L, D(L))$ and let $(L_0, D(L_0))$ be a densely defined (i.e., $D(L_0)$ is dense in $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$) linear operator on $C_\kappa(E)$ such that $L_0 \subset L$ (i.e., $D(L_0) \subset D(L)$ and $L_0\varphi = L\varphi$ for all $\varphi \in D(L_0)$).*

(i) *The operator $(L_0, D(L_0))$ is called a core operator for $(L, D(L))$ if the closure of its graph $\Gamma(L_0) = \{(\varphi, L_0\varphi) \in C_\kappa(E) \times C_\kappa(E) \mid \varphi \in D(L_0)\}$ in $(C_\kappa(E), \tau_\kappa^{\mathscr{M}}) \times (C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ coincides with the graph $\Gamma(L)$.*

(ii) *Suppose that κ is bounded and that $(P_t)_{t\geq 0}$ is Markov, i.e. $C_\kappa(E) \ni \varphi \geq 0 \Rightarrow P_t\varphi \geq 0$, $t \geq 0$; and $P_t 1 = 1$, $t \geq 0$. The operator $(L_0, D(L_0))$ is called a Markov core operator for $(L, D(L))$ if $(L, D(L))$ is the only operator with $L_0 \subset L$, which is the infinitesimal generator of a Markov C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$.*

Remark 5.8. Suppose $(L_0, D(L_0))$ is a core operator for $(L, D(L))$. Then $(L, D(L))$ is the unique operator with $L_0 \subset L$, which is the infinitesimal generator of a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$. Indeed, if $(\tilde{L}, D(\tilde{L}))$ is another such operator, it follows that $L \subset \tilde{L}$, hence $1 - L \subset 1 - \tilde{L}$. But by Proposition 5.2 (e)

$$C_\kappa(E) = (1 - \tilde{L})(D(\tilde{L})) \supset (1 - L)(D(L)) = C_\kappa(E),$$

hence $D(\tilde{L}) = D(L)$, because $1 - \tilde{L}$ is injective (e.g., again by Proposition 5.2 (e)). So, $(\tilde{L}, D(\tilde{L})) = (L, D(L))$ and as a consequence, if κ is bounded and $(P_t)_{t\geq 0}$ is Markov, then $(L_0, D(L_0))$ is also a Markov core operator for $(L, D(L))$.

Theorem 5.9. *Let κ be bounded and $(P_t)_{t\geq 0}$ be a Markov C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ with infinitesimal generator $(L, D(L))$ and let $(L_0, D(L_0))$ be a densely defined linear operator on $C_\kappa(E)$ such that $L_0 \subset L$. Suppose that, for every $x \in E$, the Fokker-Planck-Kolmogorov equation*

$$(5.8) \qquad \int \varphi(y)\, \nu_t(dy) = \int \varphi(y)\, \delta_x(dy) + \int_0^t \int L_0\, \varphi(y)\, \nu_s(dy)\, ds, \quad t \geq 0, \varphi \in D(L_0),$$

(see [9]*) has a unique solution* $(\nu_t)_{t\geq 0} \in C([0,\infty), M_\kappa^+(E))$*, such that* $\nu_t(E) = 1$ *for all* $t \in [0,\infty)$ *and such that*

$$(5.9) \qquad \int_0^T \int_E \frac{1}{\kappa}\, d\nu_t\, dt < \infty, \quad T > 0.$$

Then $(L_0, D(L_0))$ *is a Markov core operator for* $(L, D(L))$*, on* $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$*.*

Proof. Let $(\tilde{L}, D(\tilde{L}))$ be the infinitesimal generator of a Markov C_0-semigroup $(\tilde{P}_t)_{t\geq 0}$ on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ such that $L_0 \subset \tilde{L}$. Let $\tilde{\mu}_t(x, \cdot)$, $x \in E$, $t \geq 0$, be its representing measures from Theorem 3.3. Clearly, $(\tilde{\mu}_t(x, \cdot))_{t\geq 0} \in C([0,\infty), M_\kappa^+(E))$, $\tilde{\mu}_t(x, E) = 1$ for all $t \in [0,\infty)$ and (5.9) holds with $\tilde{\mu}_t(x, \cdot)$ replacing ν_t for all $x \in E$ and by Theorem 5.5 (more precisely, (5.7)) it solves (5.8). Hence the assertion follows. □

Now let us start with an example on a finite-dimensional state space. In fact, the corresponding SDE on $\mathbb{R}^d$ and the assumptions on the coefficients are the standard ones. So, in this "generic case" our theory of C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ applies and thus identifies the corresponding Kolmogorov operator L_0 with domain $D(L_0) = C_b^2(\mathbb{R}^d)$ as a Markov core operator for the infinitesimal generator of the C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ given by the transition semigroup of the solutions to the SDE. To the best of our knowledge in this generality this is the first result confirming that the Kolmogorov operator determines the (truly) infinitesimal generator of the said transition semigroup of the Markov process given by the SDE's solution. This appears to have been an open problem for many years.

5.2. Applications to SDEs on $\mathbb{R}^d$.

Let $E := \mathbb{R}^d$ and $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with normal filtration $\mathscr{F}_t$, $t \geq 0$, and $(W_t)_{t\geq 0}$ be a (standard) $(\mathscr{F}_t)$-Wiener process on $\mathbb{R}^{d_1}$. Let $M(d \times d_1, \mathbb{R})$ denote the set of real $d \times d_1$-matrices equipped with the Hilbert-Schmidt norm $\|\cdot\|$ and let

$$\sigma\colon \mathbb{R}^d \to M(d \times d_1, \mathbb{R}),$$
$$b\colon \mathbb{R}^d \to \mathbb{R}^d,$$

be continuous maps satisfying the following standard assumptions. There exist $K \in L^1_{loc}([0,\infty))$ and $C \in [0,\infty)$ such that for all $R \geq 0$,

$$(5.10) \quad 2\langle x - y, b(x) - b(y)\rangle_H + \|\sigma(x) - \sigma(y)\|^2 \leq K(R)|x-y|^2, \quad x, y \in \mathbb{R}^d, |x|, |y| \leq R,$$

and

$$(5.11) \qquad 2\langle x, b(x)\rangle_H + \|\sigma(x)\|^2 \leq C(1 + |x|^2), \quad \text{for all } x \in \mathbb{R}^d.$$

Here $\langle\,,\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$ and $|\cdot|$ the corresponding norm. Then it is well-known (see e.g. [49, Section 3] and the references therein) that the SDE

$$(5.12) \qquad dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x \in \mathbb{R}^d,$$

has a unique strong solution $X(t,x)$, $t \geq 0$, such that for $p \geq 2$ there exists $C_{T,p} \in [0,\infty)$ such that

$$(5.13) \qquad \mathbb{E}[\sup_{t\in[0,T]} |X(t,x)|^p] \leq C_{T,p}(1 + |x|^p),$$

where $\mathbb{E}$ denotes expectation w.r.t.$\mathbb{P}$. Indeed, (5.13) is a direct consequence of (5.11) and Itô's formula. Let

$$(5.14) \qquad \kappa(x) := (1 + |x|^m)^{-1}$$

and for $\varphi \in C_\kappa(\mathbb{R}^d)$, $t \geq 0$, $x \in \mathbb{R}^d$,

$$(5.15) \qquad P_t\varphi(x) := \mathbb{E}_\mathbb{P}[\varphi(X(t,x))] = \int \varphi(y)\mu_t(x,dy),$$

where

$$\mu_t(x,dy) := \left(\mathbb{P} \circ X(t,x)^{-1}\right)(dy) \in M_\kappa(\mathbb{R}^d)$$

(cf. (4.4)). By [49, Proposition 3.2.1] exactly the same arguments, which prove Claim 2 in Section 4.1, imply that $(P_t)_{t\geq 0}$ is a Markov C_0-semigroup on $\left(C_\kappa(\mathbb{R}^d), \tau_\kappa^\mathcal{M}\right)$. Let $(L, D(L))$ be its infinitesimal generator and let us consider the Kolmogorov operator $(L_0, D(L_0))$ corresponding to (5.12), defined as

$$(5.16) \qquad L_0\varphi(x) := \frac{1}{2}\sum_{i,j=1}^d \left(\sigma(x)\sigma(x)^T\right)_{i,j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varphi(x) + \langle b(x), \nabla\varphi(x)\rangle_H, \quad x \in \mathbb{R}^d,$$

$$\varphi \in D(L_0) := C_b^2(\mathbb{R}^d).$$

By Theorem A.8 below $C_b^2(\mathbb{R}^d)$ is dense in $\left(C_\kappa(\mathbb{R}^d), \tau_\kappa^\mathcal{M}\right)$. To show that

$$(5.17) \qquad L_0 \subset L,$$

we need one more condition on b and σ, namely we additionally assume

$$(5.18) \qquad \sup_{x\in\mathbb{R}^d}\frac{|b(x)| + \|\sigma(x)\|}{1 + |x|^m} < \infty.$$

Now let us first show that (5.17). So, let $\varphi \in C_b^2(\mathbb{R}^d)$. Then by Itô's formula and (5.15) we have for all $x \in \mathbb{R}^d$

$$(5.19) \qquad P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))]$$

$$= \varphi(x) + \int_0^t \mathbb{E}[L_0\varphi(X(s,x))]\,ds$$

$$= \varphi(x) + \int_0^t\int_\mathbb{R}^d L_0\varphi(y)\,\mu_s(x,dy)\,ds.$$

Here we note that by (5.18) for some $c_0 \in (0, \infty)$

$$(5.20) \qquad |L_0\varphi(x)| \leq c_0(1 + |x|^m)\|\varphi\|_{C_b^2} \quad \forall x \in \mathbb{R}^d.$$

So, since $\mu_s(x,dy)$, $x \in \mathbb{R}^d$, $s \in [0,\infty)$, satisfy (3) in Theorem 3.3 by (5.13), the above use of Fubini's Theorem is justified and $L_0\varphi \in C_\kappa(\mathbb{R}^d)$. (5.19) implies that for all $x \in \mathbb{R}^d$

$$(5.21) \qquad \frac{1}{t}\left(P_t\varphi(x) - \varphi(x)\right) = \frac{1}{t}\int_0^t P_s(L_0\varphi)(x)\,ds,$$

So, by the fundamental theorem of calculus

$$(5.22) \qquad \lim_{t\to 0}\frac{1}{t}\left(P_t\varphi(x) - \varphi(x)\right) = L_0\varphi(x).$$

Since by (5.13), for $p = m$, we have

$$\sup_{t \in [0,T]} P_t \frac{1}{\kappa} \leq 2C_{T,p} \frac{1}{\kappa},$$

(5.20) and (5.21) imply

(5.23)
$$\sup_{0 \leq t \leq 1} \frac{1}{t} \|P_t \varphi - \varphi\|_\kappa < \infty,$$

and thus Theorem 5.5 implies (5.17).

We would like to stress at this point that at least if $\kappa = 1$, the proof of (5.17) is completely standard (as it is also in Section 5.4 below), as far as the part (5.22) is concerned, while part (5.23) is to be taken care of case by case.

To show that $(L_0, D(L_0))$ is a Markov core operator for $(L, D(L))$, we furthermore assume that

(5.24)   For every $K \subset \mathbb{R}^d$ compact there exists $c_K \in (0, \infty)$

such that for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$

$$\sum_{i,j=1}^d \left(\sigma(x)\sigma(x)^T\right)_{i,j} \xi_i \xi_j \geq c_K |\xi|^2, \quad x \in \mathbb{R}^d.$$

(5.25)   Each $\left(\sigma\sigma^T\right)_{i,j}$ is locally in $VMO(\mathbb{R}^d)$.

We recall that a $\mathscr{B}(\mathbb{R}^d)$-measurable function $g : \mathbb{R}^d \to \mathbb{R}$ belongs to the class $VMO(\mathbb{R}^d)$, if it is bounded and for

$$O(g, R) := \sup_{x \in \mathbb{R}^d} \sup_{r \leq R} |B_r(x)|^{-2} \iint_{y,z \in B_r(x)} |g(y) - g(z)| \, dydz, \quad R \in (0, \infty),$$

we have

$$\lim_{R \to 0} O(g, R) = 0,$$

where $B_r(x)$ denotes the ball in $\mathbb{R}^d$ of radius r, centered at $x \in \mathbb{R}^d$, and $|B_r(x)|$ its Lebesgue measure. g belongs locally to the class $VMO(\mathbb{R}^d)$ if $\zeta g \in VMO(\mathbb{R}^d)$ for every $\zeta \in C_0^\infty(\mathbb{R}^d)$.

Under the assumptions (5.10), (5.11), (5.18), (5.24), (5.25) on the continuous maps b and σ above, it now follows by Proposition 5.9 and Theorem 9.3.6 in [9] that the Kolmogorov operator $(L_0, D(L_0))$ in (5.16) corresponding to SDE (5.12) is a Markov core operator for $(L, D(L))$. In this case one also says that Markov uniqueness holds for $(L_0, D(L_0))$ on $\left(C_\kappa(\mathbb{R}^d), \tau_\kappa^{\mathscr{M}}\right)$.

Now let us give an example on an infinitely dimensional state space, where we even have that the Kolmogorov operator $(L_0, D(L_0))$ is a core-operator for $(L, D(L))$.

5.3. Applications to generalized Mehler semigroups (or OU-processes with Levy noise) on Hilbert spaces.

Let E be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|_E$ and let us come back to Section 4.3, i.e., $(P_t)_{t \geq 0}$ is the semigroup defined in (4.17), which as shown there, is a C_0-semigroup on $(C_b(E), \tau_1^{\mathscr{M}})$, (so $\kappa \equiv 1$). In order to calculate its infinitesimal generator $(L, D(L))$ explicitly on a core domain, we need some assumptions. Let $\lambda : E \to \mathbb{C}$ satisfy the following hypothesis:

$(H1)$ λ is negative definite and Sazonov continuous with $\lambda(0) = 0$.

We refer e.g. to [34, Section 2] for the corresponding definitions. Then, as is well-known (cf., e.g., [55, Theorem VI. 4.10]), λ posesses a unique Levy-Khintchin representation of the form

$$(5.26) \qquad \lambda(\xi) := -i\langle \xi, a \rangle + \frac{1}{2}\langle \xi, R\xi \rangle - \int_E \left(e^{i\langle \xi, x \rangle} - 1 - \frac{i\langle \xi, x \rangle}{1 + \|x\|_E^2} \right) M(dx),$$

where $a \in E$, $R : E \to E$ a symmetric trace class operator and M a Levy measure on $(E, \mathcal{B}(E))$, i.e. $M(\{0\}) = 0$ and $\int_E \|x\|_E^2 \wedge 1 \, M(dx) < \infty$. We note that each λ of the form (5.26) is automatically Sazonov continuous on E. Obviously, then there exists $D \in [0, \infty)$ such that

$$(5.27) \qquad |\lambda(\xi)| \leq D(1 + \|\xi\|_E^2), \ \xi \in E, \text{ and } \lambda(-\xi) = \overline{\lambda(\xi)}, \quad \xi \in E,$$

and the real part of λ is non-negative. Now we shall choose the measures in (4.17) in the following way. By [34, Section 2.1] the functions

$$E \ni \xi \mapsto \int_0^t \lambda(T_s^* \xi) \, ds, \quad t \geq 0,$$

are also negative definite, zero for $\xi = 0$ and Sazonov continuous, where T_s^* denotes the adjoint operator of T_s on E. Hence by the Minlos-Sazonov Theorem (see [68]) for each $t \geq 0$ there exists a unique probability measure μ_t on $(E, \mathcal{B}(E))$ with Fourier transform

$$(5.28) \qquad \hat{\mu}_t(\xi) := exp\left(-\int_0^t \lambda(T_s^* \xi) \, ds \right), \quad \xi \in E.$$

(5.28) implies that

$$\hat{\mu}_{t+s}(\xi) = \hat{\mu}_s(\xi)\hat{\mu}_t(T_s^* \xi), \ \xi \in E', \quad t, s \geq 0,$$

which in turn is equivalent to (4.16). Hence for such μ_t, $P_t, t \geq 0$ defined as in (4.17), form indeed a generalized Mehler semigroup, hence by Section 4.3 a Markov C_0-semigroup on $(C_\kappa(E), \tau_1^{\mathscr{M}})$. Now we want to identify its generator $(L, D(L))$ on a convenient and large enough domain which was suggested in [47]. Let us recall its definition. Let W_0 be the set of functions φ that have a representation of the form

$$(5.29) \qquad \varphi(x) = f\Big(\langle \xi_1, x \rangle, \ldots, \langle \xi_m, x \rangle \Big), \ x \in E,$$

for $m \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^m, \mathbb{C})$ (i.e., the Schwartz space of complex-valued functions, "rapidly decreasing" at infinity as well as their derivatives). Obviously, W_0 is closed under multiplication. With the notations above, let $f_0 : \mathbb{R}^m \to \mathbb{C}$ denote the inverse Fourier transform of f, i.e. the function f_0, such that for all $y \in \mathbb{R}^m$

$$f(y) = \int_{\mathbb{R}^m} e^{i\langle y, v \rangle} f_0(v) dv,$$

and let $\nu(dv) := f_0(v)dv$, where dv denotes Lebesgue measure on $\mathbb{R}^m$. Let $\Pi_m \colon \mathbb{R}^m \to E$ be defined by

$$\Pi_m(v_1, \ldots, v_m) := v_1\xi_1 + \cdots + v_m\xi_m,$$

and let $\nu := (\Pi_m)_*\nu_0$, i.e., the image measure of ν_0 under Π_m. Then a simple computation yields that $\varphi = \hat{\nu}$. Let W be the ($\mathbb{R}-$)vector space, generated by the $\mathbb{R}$-valued elements of W_0, i.e. those for which

$$f_0(-v) = \overline{f_0(v)}, \ v \in \mathbb{R}^m.$$

Let us now recall one of the main results in [47], for which we need to assume the following condition:

(H2) There exists an orthonormal basis $\{\xi_n | n \in \mathbb{N}\}$ of E, consisting of eigenvectors of the adjoint operator A^* of A on E.

From this it is easy to see that W is an algebra separating the points of E. Note also that by definition of the Fréchet derivative φ' of φ it follows that $\varphi'(x) \in D(A^*)$ for all $x \in E$.

Theorem 5.10. [47, Theorem 1.1] *For all real-valued* $\varphi = \hat{\nu} \in W_0$ *and any* $x \in E$ *define*

$$(5.30) \qquad L_0\varphi(x) := \int_{E'} \Big(i\langle A^*\xi, x\rangle - \lambda(\xi)\Big)e^{i\langle\xi,x\rangle}\nu(d\xi) \quad \textit{(Kolmogorov operator)}$$

and extend L_0 *by* $\mathbb{R}-linearity$ *to* $D(L_0) := W$. *Suppose that (H1), (H2) hold. Then*

(i) L_0 *maps* $D(L_0)$ *into* $C_b(E)$.
(ii) $P_t\varphi(x) - \varphi(x) = \int_0^t P_s L_0\varphi(x)\,ds$ *for all* $\varphi \in D(L_0), x \in E$ *and* $t \geq 0$.

From this result if follows easily that for $(L_0, D(L_0))$ we have $L_0 \subset L$. Indeed, by Theorem 5.10 and its consequence that $s \mapsto P_s L_0\varphi(x)$ is continuous on $[0, \infty)$ for all $\varphi \in D(L_0), x \in E$, we have that

$$\frac{d}{dt}_{\restriction t=0} P_t\varphi(x) = L_0\varphi(x)$$

and for all $t \in [0, 1]$

$$\frac{1}{t}\big|P_t\varphi(x) - \varphi(x)\big| \leq \sup_{0 \leq s \leq 1} \|P_s L_0\varphi\|_1$$

(where we recall that $\kappa \equiv 1$). Hence Theorem 5.5 implies that $D(L_0) \subset D(L)$ and $L_0\varphi = L\varphi$ for all $\varphi \in D(L_0)$, i.e.,

$$(5.31) \qquad\qquad\qquad L_0 \subset L.$$

Now we shall prove that $(L_0, D(L_0))$ is a core operator for $(L, D(L))$. For this, according to Proposition 5.3 it suffices to prove

$$(5.32) \qquad\qquad\qquad P_t(D(L_0)) \subset \overline{D(L_0)}^L, \quad t > 0.$$

Remark 5.11. If λ, restricted to $span\{\xi_1, \cdots, \xi_n\}$, is infinitely often differentiable for all $n \in \mathbb{N}$, then by [47, Theorem 1.3(i)]

$$P_t(D(L_0) \subset D(L_0), \quad t > 0.$$

Hence (5.32) holds. But this is in general not true for general λ as above.

So, to prove (5.32) let us fix $t > 0$ and $\varphi \in D(L_0)$. Because P_t is linear, we may assume that φ is of type (5.29) with f real-valued. It is easily seen that such a φ can be approximated in the $\tau_\kappa^{\mathscr{M}}$-graph topology of L on $D(L)$ by $\varphi_n, n \in \mathbb{N}$, of type (5.29) with the corresponding $f_n \in S(\mathbb{R}^m, \mathbb{C})$ being Fourier transforms of $f_{n,0} \in S(\mathbb{R}^m, \mathbb{C})$ with compact supports. Hence we may assume that φ is of the form (5.29) with compactly supported f_0. Consider the following approximation of λ (see (5.26)) for $\epsilon \in (0, 1)$

$$(5.33) \qquad \lambda_\varepsilon(\xi) := -i\langle \xi, a \rangle + \frac{1}{2}\langle \xi, R\xi \rangle - \int_E \left(e^{i\langle \xi, x \rangle} - 1 - \frac{i\langle \xi, x \rangle}{1 + \|x\|_E^2} \right) M_\varepsilon(dx),$$

where

$$M_\varepsilon(dx) := \mathbb{1}_{\{\varepsilon \le \|\cdot\|_E \le \frac{1}{\varepsilon}\}} M(dx).$$

Obviously, M_ϵ is again a Lévy measure on $(E, \mathcal{B}(E))$, and λ_ε satisfies (H1). Let $\mu_t^{(\varepsilon)}$, $t \ge 0$, be defined analogously to μ_t, $t \ge 0$, through (5.28) with λ_ε replacing λ, and $P_t^{(\varepsilon)}$, $t \ge 0$, correspondingly through (4.17) with $\mu_t^{(\varepsilon)}$ replacing μ_t. Let $(L^{(\varepsilon)}, D(L^\varepsilon))$ be the infinitesimal generator of the C_0−semigroup $(P_t^{(\varepsilon)})_{t\ge 0}$ on $(C_b(E), \tau_1^{\mathscr{M}})$. Since by [47, Proposition 3.3], each λ_ε fulfills the condition of Remark 5.11, the latter implies

$$(5.34) \qquad P_t^{(\varepsilon)}\varphi \in D(L_0), \varepsilon \in (0, 1).$$

Hence, if we can prove

$$(5.35) \qquad P_t^{(\varepsilon)}\varphi \xrightarrow{\varepsilon \to 0} P_t\varphi,$$

in the $\tau_1^{\mathscr{M}}$-graph topology of L on $D(L)$, we obtain that $P_t\varphi \in \overline{D(L_0)}^L$ and (5.32) is proved. (5.35) follows from the following two claims, under the additional condition (5.36) in Claim 1, which is, however, is always fulfilled, if λ is real-valued (see Lemma 5.12 below).

<u>Claim 1.</u>

Assume that

$$(5.36) \qquad \{\mu_t^{(\varepsilon)} | \varepsilon \in (0, 1)\}$$

is tight.

Let $g \in C_b(E)$. Then

$$\tau_1^{\mathscr{M}} - \lim_{\varepsilon \to 0} P_t^{(\varepsilon)} g = P_t g.$$

<u>Claim 2.</u>

$$\tau_1^{\mathscr{M}} - \lim_{\varepsilon \to 0} L_0 P_t^{(\varepsilon)}\varphi = LP_t\varphi.$$

Proof of Claim 1: By [47, Corollary 3.5] for $\psi \in D(L_0)$ we have

$$(5.37) \qquad \lim_{\varepsilon \to 0} \|P_t^{(\varepsilon)}\psi - P_t\psi\|_1 = 0.$$

Let $p_{1,(C_n),(a_n)}$ be any of the seminorms generating $\tau_1^{\mathscr{M}}$ on $C_b(E)$. Then there exists a seminorm $p_{1,(K_n),(b_n)}$ and $C \in (0, \infty)$ such that

$$p_{1,(C_n),(a_n)}(P_t^{(\varepsilon)}\psi) \le C \, p_{1,(K_n),(b_n)}, (\psi)$$

for all $\psi \in D(L_0)$ and all $\varepsilon \in [0,1)$, where we set $P_t^{(0)} := P_t$. The fact that the seminorm $p_{1,(C_n),(a_n)}$ can indeed be taken independent of $\varepsilon \in [0,1)$ is due to assumption (5.36). This can be seen as follows:

Consider the representing measures $\mu_t^{(\varepsilon)}(x, dy)$, $x \in E$, $t \geq 0$, of $P_t^{(\varepsilon)}$, $t \geq 0$, $\varepsilon \in [0,1)$, which are given by (see (4.19))

$$(5.38) \qquad \mu_t^{(\varepsilon)}(x, dy) := (\delta_{T_t x} * \mu_t^{(\varepsilon)})(dy).$$

But by (5.36) for every $\delta > 0$ there exists a compact $K_\delta \subset E$ such that

$$\mu_t^{(\varepsilon)}(K_\delta^c) < \delta \quad \text{ for all } \ \varepsilon \in [0,1).$$

Let $C \subset E$ be compact. Define

$$\tilde{K}_\delta := K_\delta + T_t C.$$

Then $\tilde{K}_\delta$ is a compact subset of E and

$$(K_\delta + T_t C - T_t x)^c \subset K_\delta^c \quad x \in C.$$

Hence, since $(K_\delta + T_t C)^c - T_t x \subset (K_\delta + T_t C - T_t x)^c$, we have for all $x \in C$

$$(\delta_{T_t x} * \mu_t^{(\varepsilon)})(\tilde{K}_\delta^c) = \mu_t^{(\varepsilon)}((K_\delta + T_t C)^c - T_t x) \leq \mu_t^{(\varepsilon)}(K_\delta^c) < \delta \quad \varepsilon \in [0,1).$$

Therefore, by (5.38), $\{\mu_t^{(\varepsilon)}(x, dy)|\varepsilon \in [0,1), x \in C\}$ is tight. Hence exactly the same arguments as in the proof of "$(4) \Rightarrow (iv)$" in the proof of Theorem 3.3, applied to $\{\mu_t^{(\varepsilon)}|\varepsilon \in [0,1), x \in C\}$ (with t fixed), and Remark 3.4 imply that $p_{1,(K_n),(b_n)}$ can be taken independent of $\varepsilon \in [0,1)$.

Hence for all $\psi \in D(L_0)$

$$(5.39) \qquad p_{1,(C_n),(a_n)}\Big(P_t g - P_t^{(\varepsilon)} g \Big) \leq 2C p_{1,(K_n),(b_n)}(g - \psi) + \sup_{n \in \mathbb{N}} a_n \| P_t^{(\varepsilon)} \psi - P_t \psi \|_1.$$

Since $D(L_0)$ is dense in $(C_b(E), \tau_1^{\mathscr{M}})$ by Theorem (A.8) below, (5.37) and (5.39) imply Claim 1. $\qquad \square$

Proof of Claim 2: By Proposition 5.2(c) and (5.31), (5.34) we have for all $\varepsilon \in (0,1)$

$$(5.40) \qquad L P_t \varphi - L P_t^{(\varepsilon)} \varphi = \big(P_t L_0 \varphi - P_t^{(\varepsilon)} L_0 \varphi \big) + P_t^{(\varepsilon)} \big(L_0 - L^{(\varepsilon)} \big) \varphi + \big(L^{(\varepsilon)} - L_0 \big) P_t^{(\varepsilon)} \varphi.$$

By Claim 1 we have that as $\varepsilon \to 0$ the first summand converges to zero in $(C_b(E), \tau_1^{\mathscr{M}})$. For the second summand we have

$$\| P_t^{(\varepsilon)} (L_0 - L^{(\varepsilon)}) \varphi) \|_1 \leq \| (L_0 - L^{(\varepsilon)}) \varphi \|_1.$$

Defining $L_0^{(\varepsilon)}$ with domain $D(L_0)$ as in (5.30) with λ_ε replacing λ and applying (5.31) with L, L_0 replaced by $L^{(\varepsilon)}, L_0^{(\varepsilon)}$ respectively, we obtain that for every $x \in E$

$$(5.41) \qquad L_0^{(\varepsilon)} \subset L^{(\varepsilon)}$$

and hence

$$(5.42) \qquad (L_0 - L^{(\varepsilon)})\varphi(x) = \int_E (\lambda_\varepsilon(\xi) - \lambda(\xi)) e^{i\langle \xi, x \rangle} \, \nu(d\xi)$$

$$= \int_{\mathbb{R}^m} \big(\lambda_\varepsilon(\Pi_m(v)) - \lambda(\Pi_m(v)) \big) e^{i\langle \Pi_m(v), x \rangle} f_0(v) dv.$$

By [47, Lemma 3.1], $\lambda_\varepsilon \to \lambda$ uniformly on bounded subsets of E as $\varepsilon \to 0$, so by (5.42) and because f_0 has compact support, we obtain

$$\lim_{\varepsilon \to 0} \|(L_0 - L^{(\varepsilon)})\varphi\|_1 = 0,$$

so by (3.1) and Remark 3.2 (ii) the second summand in the r.h.s. of (5.40) also converges to zero in $(C_b(E), \tau_1^{\mathscr{M}})$.

Now let us turn to the third summand. So, let $x \in E, \varepsilon \in (0,1)$. Then by an elementary calculation (see [47, (1.2)])

$$P_t^{(\varepsilon)}\varphi(x) = \int_E e^{i\langle x, T_t^*\xi\rangle}\, \widehat{\mu_t^{(\varepsilon)}}(\xi)\nu(d\xi) = \widehat{\nu_t^{(\varepsilon)}}(x),$$

where ν_t^ε is the image measure of $\widehat{\mu_t^{(\varepsilon)}}(\xi)\nu(d\xi)$ under $T_t^* : E \to E$. By (H2) we know that $A^*\xi_j = \alpha_j\xi_j, j \in \mathbb{N}$, for some $\alpha_j \in \mathbb{R}$. Hence by Euler's formula for $T_t^* = e^{A^*t}$ we have $T_t^*\xi_j = e^{t\alpha_j}\xi_j, j \in \mathbb{N}$. Define the diagonal $m \times m$-matrix $D_{t,m}$ by $(D_{t,m})_{i,j} = \delta_{i,j}e^{t\alpha_j}$. Then, obviously, $T_t^*\Pi_m(v) = \Pi_m\big(D_{t,m}(v)\big)$ and hence, since $\nu \equiv (\Pi_m)_*(f_0 dv)$,

(5.43) $$P_t^{(\varepsilon)}\varphi(x) = \int_{\mathbb{R}^m} exp\big(i\langle x, \Pi_m(v)\rangle\big)g_\varepsilon(v)dv,$$

where for $v \in \mathbb{R}^m$

$$g_\varepsilon(v) := \big(\prod_{j=1}^m e^{-t\alpha_j}\big)\widehat{\mu_t}^{(\varepsilon)}\big(\Pi_m\big(D_{t,m}^{-1}\big)(v)\big)f_0\big(D_{t,m}^{-1}(v)\big).$$

We note that

(5.44) $$|g_\varepsilon| \leq \prod_{j=1}^m e^{-t\alpha_j}\mathbb{1}_{supp(f_0 \circ D_{t,m}^{-1})}\sup_{v \in \mathbb{R}^m}|f_0(v)|$$

and that $g_\varepsilon \in S(\mathbb{R}^m, \mathbb{C})$ with $g_\varepsilon(-v) = \overline{g_\varepsilon(v)}, v \in \mathbb{R}^m$. Hence $P_t^{(\varepsilon)}\varphi \in D(L_0)$, and by (5.41) analogously to (5.42) we obtain

$$\big(L^{(\varepsilon)} - L_0\big)P_t^{(\varepsilon)}\varphi(x) = \big(L_0^{(\varepsilon)} - L_0\big)P_t^{(\varepsilon)}\varphi(x)$$

$$= \int_E \big(\lambda(\xi) - \lambda_\varepsilon(\xi)\big)e^{i\langle\xi,x\rangle}\nu_t^{(\varepsilon)}(d\xi)$$

$$= \int_{\mathbb{R}^m} \big(\lambda(\Pi_m(v))\big) - \lambda_\varepsilon(\Pi_m(v))e^{i\langle\Pi_m(v),x\rangle}g_\varepsilon(v)dv.$$

Hence by [47, Lemma 3.1], (5.44) and because $supp f_0$ is compact, we obtain

$$\lim_{\varepsilon \to 0} \|(L^{(\varepsilon)} - L_0)P_t^{(\varepsilon)}\varphi\|_1 = 0,$$

and Claim 2 is proved. $\qquad\square$

So, we have that $(L_0, D(L_0))$ is a core operator of the infinitesimal generator $(L, D(L))$ of our generalized Mehler semigroup $(P_t)_{t \geq 0}$ defined in (4.17), if (H1), (H2) and (5.36) hold.

We are not entirely sure whether (5.36) always holds, but it does, if λ is real-valued, according to the following:

Lemma 5.12. *Suppose $\lambda \colon E \to \mathbb{R}$. Then (5.36) holds.*

Proof. Let $\xi \in E$. Then by assumption, (5.26) and (5.33) for all $\varepsilon \in [0,1)$

$$\lambda_\varepsilon(\xi) = \frac{1}{2}\langle \xi, R\xi \rangle + \int_E (1 - cos\langle \xi, x \rangle) M_\varepsilon(dx)$$

and $\lambda_\varepsilon(\xi)$ is decreasing in ε. Then for all $\varepsilon \in [0,1)$ (recalling that $P_t^{(0)} := P_t, \mu_t^{(0)} := \mu_t$) we have

(5.45) $$\widehat{\mu_t^{(\varepsilon)}}(\xi) \geq \widehat{\mu_t}(\xi) \ \forall \xi \in E.$$

Since $\widehat{\mu_t}$ is Sazonov continuous, for $\delta > 0$ there exists a nonnegative definite symmetric trace class operator S_δ on E such that

$$1 - \widehat{\mu_t}(\xi) \leq \langle S_\delta \xi, \xi \rangle + \delta \ \forall \xi \in E.$$

(see [55, Chap. VI, Theorem 2.3]). Hence by (5.45)

$$1 - \widehat{\mu_t^{(\varepsilon)}}(\xi) \leq \langle S_\delta \xi, \xi \rangle + \delta \ \forall \xi \in E.$$

Hence the assertion follows again by [[55, Chap. VI, Theorem 2.3]. $\square$

In the above example the Kolmogorov operator (see (5.30)) was a pseudo differential operator on E with symbol λ only dependent on ξ (not on x), i.e., constant diffusion, and linear drift. Therefore, finally we give an example on infinite dimensional state space E, but where the Kolmogorov operator is a partial differential operator with non-constant second order (=diffusion) coefficients and nonlinear first order (=drift) coefficients.

5.4. **Applications to SDEs on Hilbert spaces of locally monotone type.**
Consider the situation of Section 4.1 (so $E := H :=$ a separable Hilbert space H which is the pivot space of a Gelfand-triple $V \subset H \subset V^*$ as defined there). Let the coefficients A and B be independent both of $\omega \in \Omega$ and $t \in [0,T]$ and that $U = H$. Assume that B satisfies (4.3) and we assume that A can be written as a sum of two operators C and F. More precisely, let $(C, D(C))$ be a self-adjoint operator on H such that $-C \geq \theta_0 \in (0, \infty)$. Define $V := D((-C)^{\frac{1}{2}})$, equipped with the graph norm of $(-C)^{\frac{1}{2}}$, and V^* to be its dual. Then it is easy to see that C extends uniquely to a continuous linear operator from V to V^*, again denoted by C such that for all $u, v \in V$

(5.46) $$_{V^*}\langle -Cu, v \rangle_V = \langle u, v \rangle_H.$$

Furthermore, let $F \colon H \to V^*$ be $\mathscr{B}(H)/\mathscr{B}(V^*)$-measurable such that F restricted to V satisfies (H1)-(H4) in Section 4.1 with $B \equiv 0$, f constant and A replaced by F for $\alpha = 2$, $\beta \in [0, \infty)$, and $\theta = 0$.
Define

(5.47) $$A(u) := Cu + F(u), \ u \in V.$$

Then it is easy to check that A satisfies (H1)-(H4) in Section 4.1 with $\theta = \theta_0$, and $\alpha = 2$, $\beta \in [0, \infty)$ and f constant. So, by Theorem 4.2, $(P_t)_{t \geq 0}$ defined in (4.4) is a Markov C_0-semigroup on $(C_\kappa(H), \tau_\kappa^{\mathscr{M}})$ with $\kappa := (1 + \| \cdot \|_H^m)^{-1}$, $m \in [1, \infty)$, due to Claim 2 in Section 4.1. Let $(L, D(L))$ be its infinitesimal generator.

Now fix an orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$ of H consisting of elements in $D(C)$ and define

$$(5.48) \qquad D(L_0) := \{f\left(\langle e_1, \cdot\rangle, \ldots, \langle e_N, \cdot\rangle \mid N \in \mathbb{N},\ f \in C_b^2(\mathbb{R}^N)\right)\}.$$

Define the Kolmogorov operator associated to SDE (4.1) with B as above and A as in (5.47) with domain $D(L_0)$ as follows:

$$(5.49) \qquad L\varphi(x) := \frac{1}{2}\sum_{i,j=1}^\infty \langle B(x)e_i, B(x)e_j\rangle \frac{\partial}{\partial e_i}\left(\frac{\partial \varphi}{\partial e_j}\right)(x)$$
$$+ \sum_{i=1}^\infty \left(\langle x, Ce_i\rangle + {}_{V^*}\langle F(x), e_i\rangle_V\right)\frac{\partial \varphi}{\partial e_i}(x), \quad x \in H,\ \varphi \in D(L_0).$$

Here $\frac{\partial}{\partial e_i}$ denotes partial derivative in directions e_i and we note that all sums in (5.49) are in fact finite sums, since $\varphi \in D(L_0)$.
Now let us prove that

$$(5.50) \qquad\qquad\qquad\qquad L_0 \subset L.$$

For this we need one more condition, i.e. we assume:

$$(5.51)$$
The eigenbasis of $(C, D(C))$ above can be chosen in such a way that $x \mapsto {}_{V^*}\langle F(x), e_i\rangle_V$ is continuous on H and
$$\sup_{x \in H} \frac{|{}_{V^*}\langle F(x), e_i\rangle_V|}{1 + |x|_H^m} < \infty \quad \text{for some } m \in [1, \infty) \text{ and all } i \in \mathbb{N}.$$

Remark 5.13.
(i) A typical example for $F : H \to V^*$ above is a demicontinuous function (i.e., $x \mapsto {}_{V^*}\langle F(x), u\rangle_V$ is continuous on H for all $u \in V$) with $F(0) \in H$, which is one sided Lipschitz and of at most polynomial growth.
(ii) A typical example for $(C, D(C))$ is the Laplace operator on an open bound domain $\mathcal{O} \subset \mathbb{R}^d$ with Dirichlet boundary conditions considered on $L^2(\mathcal{O})$.

Under condition (5.51) a straightforward application of Itô's formula for Itô-processes in $\mathbb{R}^N, N \in \mathbb{N}$, yields for all $\varphi \in D(L_0)$ and for the solution $X(t, x), t \geq 0, x \in E$, to (4.1) with A and B as above:

$$(5.52) \qquad\qquad P_t\varphi(x) = \mathbb{E}\big[\varphi(X(t, x))\big]$$

$$(5.53) \qquad\qquad\qquad\qquad = \varphi(x) + \int_0^t \mathbb{E}\big[L_0\,\varphi(X(s, x))\big]\,ds$$

$$(5.54) \qquad\qquad\qquad\qquad = \varphi(x) + \int_0^t \int_H L_0\,\varphi(y)\,\mu_s(x, dy)\,ds,$$

where

$$\mu_s(x, dy) := (\mathbb{P} \circ X(s, x)^{-1})(dy) \in M_\kappa(\mathbb{R}^d).$$

Now exactly the same arguments as in Section 5.2 prove that (5.50) holds.

To prove that $(L_0, D(L_0))$ is a Markov core operator for $(L, D(L))$ on $(C_\kappa(H), \tau_\kappa^{\mathcal{M}})$ we shall again use Proposition 5.9, i.e. we have to prove uniqueness for the corresponding Fokker-Planck-Kolmogorov equation, which is in general very difficult here, since our state space H is infinite dimensional, and more assumptions are needed. Though there are such results also when B depends on x (see [7]), for simplicity we shall assume that B is constant. More preasely, we additionally assume that:

$$(5.55) \qquad\qquad B(x) = B (\in L_2(H)$$

for all $x \in V$ with $B = B^*$, B non-negative definite with $ker B = \{0\}$, and that for the eigenvalues $\alpha_k \in (0, \infty), k \in \mathbb{N}$, of B. There exists $m \in [1, \infty)$ such that

$$(5.56) \qquad\qquad \sup_{x \in H} (1 + |x|^m)^{-1} \sum_{k=1}^{\infty} \alpha_k^{-1} |_{V^*}\langle F(x), e_k \rangle_V|^2 < \infty.$$

Of course, we may assume that both (5.51) and (5.56) hold with the same m (otherwise we take the maximum of the two). Then taking this m and $\kappa := (1 + \| \cdot \|_H^m)^{-1}$, it follows by [7, Remark 2.1 (iii) and Theorem 2.3] and by (5.50) that all assumtions in Theorem 5.9 are fulfilled. Hence $(L_0, D(L_0))$ is a Markov core operator for $(L, D(L))$ on $(C_\kappa(H), \tau_\kappa^{\mathcal{M}})$.

6. CONVEX C_0-SEMIGROUPS ON $(C_\kappa(E), \tau_\kappa^{\mathcal{M}})$

We now draw our attention to C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathcal{M}})$ consisting of convex increasing operators on $C_\kappa(E)$. We show that these lead to viscosity solutions to abstract differential equations that are given in terms of their generator. We start by introducing our notion of a viscosity solution for abstract differential equations of the form

$$(6.1) \qquad\qquad u'(t) = Lu(t), \quad \text{for all } t > 0.$$

In the following, an operator $T\colon C_\kappa(E) \to C_\kappa(E)$ is called $increasing$ if

$$T\varphi_1 \leq T\varphi_2 \quad \text{for all } \varphi_1, \varphi_2 \in C_\kappa(E) \text{ with } \varphi_1 \leq \varphi_2.$$

We say that an operator $T\colon C_\kappa(E) \to C_\kappa(E)$ is $convex$ if

$$T\big(\lambda\varphi_1 + (1 - \lambda)\varphi_2\big) \leq \lambda T\varphi_1 + (1 - \lambda)T\varphi_2$$

for all $\lambda \in [0, 1]$ and $\varphi_1, \varphi_2 \in C_\kappa(E)$.

Definition 6.1. *Let $L\colon D \to C_\kappa(E)$ be a nonlinear operator, defined on a nonempty set $D \subset C_\kappa(E)$. We say that $u\colon [0, \infty) \to C_\kappa(E)$ is a D-viscosity subsolution to the abstract differential equation (6.1) if u is continuous w.r.t. the mixed topology $\tau_\kappa^{\mathcal{M}}$ and, for every $t > 0$, $x \in M$, and every differentiable function $\psi\colon (0, \infty) \to C_\kappa(E)$ with $\psi(t) \in D$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$, and $\psi(s) \geq u(s)$ for all $s > 0$,*

$$\big(\psi'(t)\big)(x) \leq \big(L\psi(t)\big)(x).$$

Analogously, u is called a D-viscosity supersolution to (6.1) if $u\colon [0, \infty) \to C_\kappa(E)$ is continuous and, for every $t > 0$, $x \in M$, and every differentiable function $\psi\colon (0, \infty) \to C_\kappa(E)$ with $\psi(t) \in D$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$, and $\psi(s) \leq u(s)$ for all $s > 0$,

$$\big(\psi'(t)\big)(x) \geq \big(L\psi(t)\big)(x).$$

We say that u is a D-viscosity solution to (6.1) if u is a viscosity subsolution and a viscosity supersolution.

Note that the previous definition does, a priori, not require the class of test functions for a viscosity solution to be rich in any sense. Therefore, in order to obtain uniqueness in standard settings, one has to verify on a case by case basis that the operator L is defined on a sufficiently large set D in order to apply standard comparison methods. Concerning the existence of D-viscosity solutions, we have the following theorem.

Theorem 6.2. *Let P be a C_0-semigroup on $(C_\kappa, \tau_\kappa^{\mathcal{M}})$ consisting of convex increasing operators with generator L. Then, for every $\varphi \in C_\kappa(E)$, the function $u \colon [0, \infty) \to C_\kappa(E)$, $t \mapsto P_t\varphi$ is a $D(L)$-viscosity solution to the abstract initial value problem*

$$\begin{aligned} u'(t) &= Lu(t), \quad \text{for all } t > 0, \\ u(0) &= \varphi. \end{aligned}$$

Proof. Fix $t > 0$ and $x \in E$. We first show that u is a viscosity subsolution. To that end, let $\psi \colon (0, \infty) \to C_\kappa(E)$ be a differentiable function with with $\psi(t) \in D(L)$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ and $\psi(s) \geq u(s)$ for all $s > 0$. For $\lambda \in (0, 1)$, let $\psi_\lambda := \frac{\psi}{\lambda}$. Then, for $h \in (0, 1)$ with $h < t$, the semigroup property implies that

$$\begin{aligned} 0 = \frac{P_h P_{t-h}\varphi - P_t\varphi}{h} &= \frac{P_h u(t-h) - u(t)}{h} \leq \frac{P_h \psi(t-h) - u(t)}{h} \\ &\leq \frac{P_h \psi(t-h) - P_h\psi(t)}{h} + \frac{P_h\psi(t) - \psi(t)}{h} + \frac{\psi(t) - u(t)}{h} \\ &\leq \left(P_h\Big(\psi(t) + \tfrac{\psi(t-h)-\psi(t)}{h}\Big) - P_h\psi(t) \right) + \frac{P_h\psi(t) - \psi(t)}{h} + \frac{\psi(t) - u(t)}{h}, \end{aligned}$$

where, in the last inequality, we used the convexity of the map $v \mapsto P_h\big(\psi(t) + v\big) - P_h\psi(t)$. The strong continuity of the semigroup P and $\psi(t) \in D(L)$ imply that

$$P_h\Big(\psi(t) + \tfrac{\psi(t-h)-\psi(t)}{h}\Big) - P_h\psi(t) \to -\psi'(t) \quad \text{and} \quad \frac{P_h\psi(t) - \psi(t)}{h} \to L\psi(t)$$

as $h \downarrow 0$ in the mixed topology $\tau_\kappa^{\mathcal{M}}$. Using the equality $\big(u(t)\big)(x) = \big(\psi(t)\big)(x)$, it follows that

$$0 \leq -\big(\psi'(t)\big)(x) + \big(L\psi(t)\big)(x).$$

In order to show that u is a viscosity supersolution, let $\psi \colon (0, \infty) \to C_\kappa(E)$ differentiable with $\psi(t) \in D(L)$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$ and $\psi(s) \leq u(s)$ for all $s > 0$. Again, using the semigroup property, we find that, for all $h \in (0, 1)$ with $h < t$,

$$\begin{aligned} 0 = \frac{P_t\varphi - P_h P_{t-h}\varphi}{h} &= \frac{u(t) - P_h u(t-h)}{h} \leq \frac{u(t) - P_h\psi(t-h)}{h} \\ &= \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - P_h\psi(t)}{h} + \frac{P_h\psi(t) - P_h\psi(t-h)}{h} \\ &\leq \frac{u(t) - \psi(t)}{h} + \frac{\psi(t) - P_h\psi(t)}{h} + \left(P_h\Big(\psi(t-h) + \tfrac{\psi(t)-\psi(t-h)}{h}\Big) - P_h\psi(t-h) \right), \end{aligned}$$

where, in the last step, we used the convexity of the map $v \mapsto P_h\big(\psi(t-h) + v\big) - P_h\psi(t-h)$. Again, the strong continuity of the semigroup P and $\psi_t \in D(L)$ imply that

$$\frac{\psi(t) - P_h\psi(t)}{h} \to -L\psi(t) \quad \text{and} \quad P_h\Big(\psi(t-h) + \tfrac{\psi(t)-\psi(t-h)}{h}\Big) - P_h\psi(t-h) \to \psi'(t)$$

as $h \downarrow 0$ in the mixed topology $\tau_\kappa^{\mathcal{M}}$. Since $\big(u(t)\big)(x) = \big(\psi(t)\big)(x)$, we find that

$$0 \leq -\big(L\psi(t)\big)(x) + \big(\psi'(t)\big)(x),$$

and the proof is complete.   □

Finally, we derive a stochastic representation for P using convex expectations. For a measurable space $(\Omega, \mathcal{F})$, we denote the space of all bounded $\mathcal{F}$-measurable functions (random variables) $\Omega \to \mathbb{R}$ by $B_b(\Omega, \mathcal{F})$. For two bounded random variables $X, Y \in B_b(\Omega, \mathcal{F})$ we write $X \leq Y$ if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. For a constant $m \in \mathbb{R}$, we do not distinguish between m and the constant function taking that value.

Definition 6.3. *Let $(\Omega, \mathcal{F})$ be a measurable space. A functional $\mathcal{E} \colon B_b(\Omega, \mathcal{F}) \to \mathbb{R}$ is called a convex expectation if, for all $X, Y \in B_b(\Omega, \mathcal{F})$ and $\lambda \in [0, 1]$,*

(i) *$\mathcal{E}(X) \leq \mathcal{E}(Y)$ if $X \leq Y$,*
(ii) *$\mathcal{E}(m) = m$ for all constants $m \in \mathbb{R}$,*
(iii) *$\mathcal{E}(\lambda X + (1 - \lambda)Y) \leq \lambda \mathcal{E}(X) + (1 - \lambda)\mathcal{E}(Y)$.*

We say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a convex expectation space if there exists a set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ and a function $\alpha \colon \mathcal{P} \to [0, \infty)$ such that

$$\mathcal{E}(X) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(X) - \alpha(\mathbb{P}) \quad \text{for all } X \in B_b(\Omega, \mathcal{F}),$$

where $\mathbb{E}_{\mathbb{P}}(\cdot)$ denotes the expectation w.r.t. to the probability measure $\mathbb{P}$.

The following theorem is a consequence of [19, Theorem 5.6] and the fact that the $\tau_\kappa^{\mathscr{M}}$-continuity of P_t implies the so-called continuity from above or Daniell continuity of P_t, for $t \geq 0$.

Theorem 6.4. *Assume that E is a Polish space, $\kappa \equiv 1$, and P is a C_0-semigroup of increasing convex operators with $P_t m = m$ for all $t \geq 0$ and $m \in \mathbb{R}$. Then, there exists a quadruple $(\Omega, \mathcal{F}, (\mathcal{E}^x)_{x \in E}, (X(t))_{t \geq 0})$ such that*

(i) *$X(t) \colon \Omega \to E$ is $\mathcal{F}$-$\mathcal{B}$-measurable for all $t \geq 0$,*
(ii) *$(\Omega, \mathcal{F}, \mathcal{E}^x)$ is a convex expectation space with $\mathcal{E}^x(\varphi(X(0))) = \varphi(x)$ for all $x \in E$ and $\varphi \in C_b(E)$,*
(iii) *For all $0 \leq s < t$, $n \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_n \leq s$ and $\psi \in C_b(E^{n+1})$,*

$$(6.2) \qquad \mathcal{E}^x\big(\psi(X(t_1), \ldots, X(t_n), X(t))\big) = \mathcal{E}^x\big((P_{t-s}\psi(X(t_1), \ldots, X(t_n), \cdot))(X(s))\big).$$

In particular,

$$(6.3) \qquad\qquad\qquad (P_t\varphi)(x) = \mathcal{E}^x(\varphi(X(t))).$$

for all $t \geq 0$, $x \in E$, and $\varphi \in C_b(E)$.

Let E be a Polish space. The quadruple $(\Omega, \mathcal{F}, (\mathcal{E}^x)_{x \in E}, (X(t))_{t \geq 0})$ can be seen as a nonlinear version of a Markov process. As an illustration, we consider the case, where the semigroup P and thus $\mathcal{E}^x$ is linear for all $x \in E$, and choose $\psi(x, y) = \varphi(x)1_B(y)$, for $x, y \in E$, with $\varphi \in C_b(E)$ and $B \in \mathcal{B}^n$, where $\mathcal{B}^n$ denotes the product σ-algebra of the Borel σ-algebra $\mathcal{B}$. Then, $\mathcal{E}^x = \mathbb{E}_{\mathbb{P}^x}$ is the expectation w.r.t. a probability measure $\mathbb{P}^x$ on $(\Omega, \mathcal{F})$ for all $x \in E$. Using the continuity from above and Dynkin's lemma, Equation (6.2) reads as

$$\mathbb{E}_{\mathbb{P}^x}\big(\varphi(X(t))1_B(X(t_1), \ldots, X(t_n))\big) = \mathbb{E}_{\mathbb{P}^x}\big[(P_{t-s}\varphi)(X(s))1_B(X(t_1), \ldots, X(t_n))\big],$$

which is equivalent to the Markov property

$$(6.4) \qquad\qquad \mathbb{E}_{\mathbb{P}^x}\big(\varphi(X(t))|\mathcal{F}_s\big) = (P_{t-s}u)(X(s)) \quad \mathbb{P}^x\text{-a.s.},$$

where $\mathcal{F}_s := \sigma(\{X(u) \mid 0 \leq u \leq s\})$. On the other hand, if $\mathcal{E}^x = \mathbb{E}_{\mathbb{P}^x}$, the Markov property (6.4) implies Property (iii) from Theorem 6.4.

7. Examples: value functions of optimal control problems

7.1. **A finite-dimensional setting.** In this section, we show that value functions of a large class of optimal control problems are examples of nonlinear C_0-semigroups. We illustrate this by means of a simple controlled dynamics in $\mathbb{R}^d$, with $d \in \mathbb{N}$ where the contol acts on the drift of a diffusion process. However, with similar techniques also other classes of controlled diffusions fall into our setup. Throughout, let $W = (W(t))_{t \geq 0}$ be a Brownian Motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual assumptions and $\sigma > 0$. For $m \in \mathbb{N}$, we consider a fixed nonempty set $A \subset \mathbb{R}^m$ of controls with $0 \in A$ and define the the the set of admissible controls $\mathcal{A}$ as the set of all progressively measurable processes $\alpha \colon \Omega \times [0, T] \to A$ with

$$\mathbb{E}\left(\int_0^t |\alpha(s)| ds \right) < \infty.$$

For a fixed measurable function $b \colon \Omega \times \mathbb{R}^d \times A \to \mathbb{R}^d$, an admissible control $\alpha \in \mathcal{A}$, and an initial value $x \in \mathbb{R}^d$, we consider the controlled dynamics

(7.1) $$dX^\alpha(t, x) = b\big(X^\alpha(t, x), \alpha(t)\big)dt + \sigma dW(t), \quad \text{for } t \geq 0, \quad X^\alpha(0, x) = x.$$

We assume that the drift term b satisfies the following Lipschitz and growth conditions: there exists a constant $C \geq 0$ such that

$$b(x, 0) = 0, \quad \mathbb{P}\text{-a.s.}, \quad \text{for all } x \in \mathbb{R}^d,$$

$$|b(x_1, a) - b(x_2, a)| \leq C|x - y|, \quad \mathbb{P}\text{-a.s.}, \quad \text{for all } x_1, x_2 \in \mathbb{R}^d \text{ and } a \in A,$$

$$|b(x, a)| \leq C\big(1 + |x| + |a|\big), \quad \mathbb{P}\text{-a.s.}, \quad \text{for all } x \in \mathbb{R}^d \text{ and } a \in A.$$

Under these assumptions, by standard SDE theory, for each initial value $x \in \mathbb{R}^d$ and every admissible control $\alpha \in \mathcal{A}$, there exists a unique strong solution $(X^\alpha(t, x))_{t \geq 0}$ to the controlled SDE (7.1).

We consider the weight function $\kappa \equiv 1$ and a running cost function $g \colon A \to [0, \infty)$ with $g(0) = 0$ and

$$\overline{g}^*(y) := \sup_{a \in A} \big(|a|y - g(a)\big) < \infty$$

for all $y \geq 0$. For $\varphi \in C_b(\mathbb{R}^d)$, we consider the value function

$$V(t, x; \varphi) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left(\varphi\big(X\alpha(t, x)\big) - \int_0^t g\big(\alpha(s)\big)ds \right),$$

and we define $\big(P_t \varphi\big)(x) := V(t, x; \varphi)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. We first show that $P_t \colon C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is well-defined with $\|P_t \varphi\|_\infty \leq \|\varphi\|_\infty$ for all $\varphi \in C_b(\mathbb{R}^d)$. Using the Lipschitz condition of b together with Gronwall's lemma, we obtain the a priori estimate

$$\mathbb{E}\big(|X^\alpha(t, x_1) - X^\alpha(t, x_2)|\big) \leq e^{Ct}|x_1 - x_2|$$

for all $t \geq 0$, $x_1, x_2 \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}$. This shows that the value function V is continuous in the x-variable. Moreover, $\|V(t, \cdot, \varphi)\|_\infty \leq \|\varphi\|_\infty$ for all $\varphi \in C_b(\mathbb{R}^d)$, since $g(0) = 0$. Since the value function V satisfies the dynamic programming principle, cf. Pham [58] or Fabbri et al. [29], the family $P = (P_t)_{t \geq 0}$ is a semigroup.

Using the linear growth of b together with Gronwall's lemma,

(7.2) $$\mathbb{E}\big(|X^\alpha(t, x) - x|\big) + |x| \leq e^{Ct}\left(|x| + \sigma\sqrt{t} + Ct + \int_0^t C|\alpha(s)|ds \right)$$

for all $t \geq 0$, $x \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}$. Let $\varepsilon > 0$, $\varphi \in C_b(\mathbb{R}^d)$, and $t \geq 0$. Then, for every $r \geq 0$, there exists some $\delta > 0$ such that

$$|\varphi(y) - \varphi(x)| < \frac{\varepsilon}{2} \quad \text{for all } x, y \in \mathbb{R}^d \text{ with } |x| \leq r \text{ and } |x - y| < \delta.$$

Hence, for all $x \in \mathbb{R}^d$ with $|x| \leq r$, Equation 7.2 implies that

$$V(t, x; \varphi) - \varphi(x) \leq \frac{\varepsilon}{2} + 2\|\varphi\|_\infty \mathbb{E}\left(1_{\{|X^\alpha(t,x)-x|>\delta\}}\right) - \mathbb{E}\left(\int_0^t g(\alpha(s))ds\right)$$

$$\leq \frac{\varepsilon}{2} + \frac{2\|\varphi\|_\infty}{\delta}\mathbb{E}\left(|X^\alpha(t, x) - x|\right) - \mathbb{E}\left(\int_0^t g(\alpha(s))ds\right)$$

$$\leq \frac{\varepsilon}{2} + (e^{Ct} - 1)|x| + e^{Ct}(Ct + \sigma\sqrt{t}) + t\overline{g}^*\left(\frac{2\|\varphi\|_\infty}{\delta}Ce^{Ct}\right).$$

On the other hand, for all $x \in \mathbb{R}^d$ with $|x| \leq r$,

$$\varphi(x) - V(t, x; \varphi) \leq \frac{\varepsilon}{2} + 2\|\varphi\|_\infty \mathbb{E}(1_{\{|\sigma W(t)|>\delta\}}) \leq \frac{\varepsilon}{2} + \frac{2\|\varphi\|_\infty}{\delta}\sigma\sqrt{t}.$$

We thus see, that $P_t\varphi \to \varphi$ uniformly on compact sets.

Now, let $R \geq 0$, $\varepsilon > 0$, and $\varphi_1, \varphi_2 \in C_b(\mathbb{R}^d)$ with $\|\varphi_i\|_\infty \leq R$, for $i = 1, 2$, and

$$\sup_{|y|\leq r} |\varphi_1(y) - \varphi_2(y)| < \frac{\varepsilon}{3} \quad \text{for sufficiently large } r > 0.$$

We observe that, for $\varphi \in C_b(\mathbb{R}^d)$ with $\|\varphi\|_\infty \leq R$, $t \geq 0$, $x \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}$ with

$$(7.3) \qquad V(t, x; \varphi) \leq \frac{\varepsilon}{3} + \mathbb{E}\left(\varphi(X^\alpha(t, x)) - \int_0^t g(\alpha(s))ds\right),$$

it follows that

$$(7.4) \qquad \mathbb{E}\left(\int_0^t |\alpha(s)|ds\right) \leq t\overline{g}^*(1) + \mathbb{E}\left(\int_0^t g(\alpha(s))\right) \leq \frac{\varepsilon}{3} + t\overline{g}^*(1) + 2R$$

Let $T, c \geq 0$. Then, for $t \in [0, T]$, $x \in \mathbb{R}^d$ with $|x| \leq c$, and $\alpha \in \mathcal{A}$ satisfying Equation (7.3) for $\varphi = \varphi_1$.

$$V(t, x; \varphi_1) - V(t, x; \varphi_2) \leq \frac{2\varepsilon}{3} + 2R\mathbb{E}\left(1_{\{|X^\alpha(t,x)|>r\}}\right) \leq \frac{2\varepsilon}{3} + \frac{2R}{r}\mathbb{E}\left(|X^\alpha(t, x)|\right)$$

$$\leq \frac{2\varepsilon}{3} + \frac{2R}{r}e^{CT}\left(\frac{\varepsilon}{3} + 2R + c + \sigma\sqrt{T} + T(C + \overline{g}^*(1))\right),$$

where, in the last step, we used Equation (7.2) and Equation (7.4). Choosing $r > 0$ sufficiently large, a symmetry argument yields that

$$\sup_{|x|\leq c} |V(t, x; \varphi_1) - V(t, x; \varphi_2)| < \varepsilon \quad \text{for all } t \in [0, T].$$

With similar arguments together with Itô's formula, one finds that the generator L of P on $C_b^2(E)$ is given by

$$(L\varphi)(x) = \frac{\sigma^2}{2}\partial_{xx}\varphi(x) + \sup_{a\in A}\left(b(x, a)\partial_x\varphi(x) - g(a)\right).$$

By Theorem 6.2, we thus obtain that $(t,x) \mapsto V(t,x;\varphi) = (P_t\varphi)(x)$ is a $C_b^2(\mathbb{R}^d)$-viscosity solution to the HJB equation

$$\partial_t v(t,x) = \frac{\sigma^2}{2}\partial_{xx}\varphi(x) + \sup_{a \in A}\Big(b(x,a)\partial_x\varphi(x) - g(a)\Big), \quad v(0,x) = \varphi(x).$$

7.2. An infinite-dimensional example with linear growth.

In this section, we consider a similar setup as in the previous subsection in a separable Hilbert space H with orthonormal base $(e_k)_{k\in\mathbb{N}} \subset H$, endowed with the bw-topology. Throughout, let $W = (W(t))_{t\geq 0}$ be a Brownian Motion with trace class covariance operator $\Sigma\colon H \to H$ on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual assumptions. For $p \in (1,2]$, we define the the set of admissible controls $\mathcal{A}$ as the set of all progressively measurable processes $\alpha\colon \Omega \times [0,T] \to U$ with

$$\mathbb{E}\bigg(\int_0^t |\alpha_s|_H^p ds\bigg) < \infty.$$

For every admissible control $\alpha \in \mathcal{A}$ and every initial value $x \in H$, we consider the controlled dynamics

(7.5)
$$X^\alpha(t,x) = x + \int_0^t \alpha(s)ds + W(t) \quad \text{for all } t \geq 0.$$

We consider the weight function $\kappa(x) := (1+|x|_H)^{-1}$, for $x \in H$, and a running cost function $g\colon H \to [0,\infty)$ with $g(0) = 0$ and

$$\bar{g}_p^*(y) := \sup_{a\in H}\big(|a|_H^p y - g(a)\big) < \infty$$

for all $y \geq 0$. This implies that, for all $q \in [1,p]$ and $y \in\geq 0$,

$$\bar{g}_q^*(y) := \sup_{a\in H}\big(|a|_H^q y - g(a)\big) < \infty.$$

For $\varphi \in C_\kappa(H_{bw})$, we consider the value function

$$V(t,x;\varphi) := \sup_{\alpha\in\mathcal{A}}\mathbb{E}\bigg(\varphi\big(X^\alpha(t,x)\big) - \int_0^t g\big(\alpha(s)\big)ds\bigg),$$

and we define $(P_t\varphi)(x) := V(t,x;\varphi)$ for all $t \geq 0$ and $x \in H$. We first show that $P_t\colon C_\kappa(H_{bw}) \to C_\kappa(H_{bw})$ is well-defined. Let $\varphi \in C_b(H_{bw})$ such that, there exists a constant $L \geq 0$ and some $n \in \mathbb{N}$ with

(7.6)
$$|\varphi(x) - \varphi(y)| \leq L\sum_{i=1}^n |\langle x-y, e_i\rangle| \quad \text{for all } x,y \in H.$$

Then, for all $t \geq 0$ and $x,y \in H$

$$|V(t,x;\varphi) - V(t,y;\varphi)| \leq L\sum_{i=1}^n |\langle x-y, e_i\rangle|.$$

Moreover, for all $q \in [1,p]$,

$$\mathbb{E}\big(|X_t^{x,\alpha}|_H^p\big)^{1/q} \leq |x|_H + \|\Sigma\|_{\mathrm{tr}}\sqrt{t} + \mathbb{E}\bigg(\int_0^t |\alpha(s)|_H^q ds\bigg)^{1/q}$$

for all $t \geq 0$, $x \in H$, and $\alpha \in \mathcal{A}$. Using this estimate, we find that, for $t \geq 0$, $x \in H$, and $\varphi \in C_\kappa(H_{bw})$,

$$V(t, x; \varphi) \leq \|\varphi\|_\kappa \left(1 + \mathbb{E}\big(|X^\alpha(t, x)|_H \big) \right) - \mathbb{E}\left(\int_0^t g\big(\alpha(s)\big) \, ds \right)$$

$$\leq \|\varphi\|_\kappa \big(1 + |x|_H + \|\Sigma\|_{\mathrm{tr}}\sqrt{t} \big) + t\overline{g}_1^*\big(\|\varphi\|_\kappa\big).$$

Moreover, for all $t \geq 0$, $x \in H$, and $\varphi \in C_\kappa(H_{bw})$,

$$V(t, x; \varphi) \geq -\|\varphi\|_\kappa \big(1 + |x|_H + \|\Sigma\|_{\mathrm{tr}}\sqrt{t} \big),$$

which shows that

$$\|P_t\varphi\|_\kappa \leq \Big(\|\varphi\|_\kappa + \overline{g}^*\big(\|\varphi\|_\kappa\big) \Big) \big(1 + t + \|\Sigma\|_{\mathrm{tr}}\sqrt{t} \big).$$

Now, let $R \geq 0$, $\varepsilon > 0$, and $\varphi_1, \varphi_2 \in C_\kappa(H_{bw})$ with $\|\varphi_i\|_\kappa \leq R$, for $i = 1, 2$, and

$$\sup_{|y|_H \leq r} |\varphi_1(y) - \varphi_2(y)| < \frac{\varepsilon}{3} \quad \text{for sufficiently large } r \geq 1.$$

We observe that, for $\varphi \in C_\kappa(H_{bw})$ with $\|\varphi\|_\kappa \leq R$, $t \geq 0$, $x \in \mathbb{R}^d$, and $\alpha \in \mathcal{A}$ with

$$(7.7) \qquad V(t, x; \varphi) \leq \frac{\varepsilon}{3} + \mathbb{E}\left(\varphi\big(X^\alpha(t, x)\big) - \int_0^t g\big(\alpha(s)\big) ds \right),$$

it follows that

$$\mathbb{E}\left(\int_0^t g\big(\alpha(s)\big) \right) \leq \frac{\varepsilon}{3} + 4R\left(\big(1 + |x|_H + 2\|\Sigma\|_{\mathrm{tr}}\sqrt{t}\big)^p + \mathbb{E}\left(\int_0^t |\alpha(s)|^p ds \right) \right),$$

which implies that

$$(7.8) \qquad \mathbb{E}\left(\int_0^t |\alpha(s)|^p ds \right) \leq \frac{\varepsilon}{3} + 4R\big(1 + |x|_H + \|\Sigma\|_{\mathrm{tr}}\sqrt{t}\big)^p + t\overline{g}_p^*(1 + 4R)$$

Let $T, c \geq 0$. Then, for $t \in [0, T]$, $x \in H$ with $|x|_H \leq c$, and $\alpha \in \mathcal{A}$ satisfying Equation (7.7) for $\varphi = \varphi_1$.

$$V(t, x; \varphi_1) - V(t, x; \varphi_2) \leq \frac{2\varepsilon}{3} + 2R\mathbb{E}\left(\big(1 + |X^\alpha(t, x)|\big) 1_{\{|X^\alpha(t,x)|>r\}} \right)$$

$$\leq \frac{2\varepsilon}{3} + \frac{4R}{r^{p-1}} \mathbb{E}\big(|X^\alpha(t, x)|^p \big)$$

$$\leq \frac{2\varepsilon}{3} + \frac{16(1 + R)}{r^{p-1}} \Big(\frac{\varepsilon}{3} + \big(1 + c + \|\Sigma\|_{\mathrm{tr}}\sqrt{T}\big)^p + T\overline{g}^*(1 + 4R) \Big),$$

where, in the last step, we used Equation (7.8). Choosing $r > 0$ sufficiently large, a symmetry argument yields that

$$\sup_{|x|_H \leq c} \big| V(t, x; \varphi_1) - V(t, x; \varphi_2) \big| < \varepsilon \quad \text{for all } t \in [0, T].$$

Since the value function V satisfies the dynamic programming principle, cf. Fabbri et al. [29], the family $P = (P_t)_{t \geq 0}$ is a semigroup.

Let $\varepsilon > 0$, $\varphi \in C_b(H_{bw})$ with (7.6) with $L \geq 0$ and $n \in \mathbb{N}$, and $t \geq 0$. Then, for all $x \in H$,

$$V(t, x; \varphi) - \varphi(x) \leq L\|\Sigma\|_{\mathrm{tr}}\sqrt{t} + \mathbb{E}\left(\int_0^t L|\alpha(s)| - g\big(\alpha(s) ds\big) \right)$$

$$\leq L\|\Sigma\|_{\mathrm{tr}}\sqrt{t} + t\overline{g}^*(L) \to 0 \quad \text{as } t \to 0.$$

Moreover, for all $x \in H$,

$$\varphi(x) - V(t, x; \varphi) \leq L\|\Sigma\|_{\mathrm{tr}}\sqrt{t}.$$

In particular, since the bounded finitely based Lipschitz functions are dense in $C_\kappa(H_{bw})$, it follows that $P_t\varphi \to \varphi$ uniformly on compacts for all $\varphi \in C_\kappa(H_{bw})$.

By Itô's formula and Theorem 6.2, we obtain that $(t, x) \mapsto V(t, x; \varphi) = (P_t\varphi)(x)$ is a $C_b^2(H_{bw})$-viscosity solution to the HJB equation

$$\partial_t v(t, x) = \frac{\sigma^2}{2}\partial_{xx}\varphi(x) + \sup_{a \in A}\left(\langle a, \partial_x\varphi(x)\rangle - g(a)\right), \quad v(0, x) = \varphi(x).$$

APPENDIX A. SOME FACTS ON THE MIXED TOPOLOGY

In this section we collect some general properties of the mixed topology that was introduced in Section 2 in a special case suitable for our purposes.

We recall the definition of the mixed topology $\tau^{\mathscr{M}}$. We closely follow [72], where a more general situation is studied.

Let X be a linear space, endowed with two topologies τ_1 and τ_2, with corresponding bases $\mathscr{U}(\tau_1)$ and $\mathscr{U}(\tau_2)$ of neighbourhoods of zero. We assume that (X, τ^1) and (X, τ^2) are Hausdorff topological vector spaces with $\tau^1 \subset \tau^2$. For a sequence $\gamma = (U_n^1) \subset \mathscr{U}(\tau^1)$, and any $U^2 \in \mathscr{U}(\tau^2)$, we define a set

$$U\left(\gamma, U^2\right) = \bigcup_{n=1}^{\infty}\sum_{k=1}^{n}\left(U_k^1 \cap kU^2\right).$$

Then, the family

$$\left\{U\left(\gamma, U^2\right) : \gamma = (U_n^1) \subset \mathscr{U}\left(\tau^1\right), U^2 \in \mathscr{U}\left(\tau^2\right)\right\}$$

defines a basis of neighbourhoods of zero for a locally convex topology denoted $\tau^{\mathscr{M}} = \tau^{\mathscr{M}}\left(\tau^1, \tau^2\right)$ making $(X, \tau^{\mathscr{M}})$ a Hausdorff topological vector space. The topology $\tau^{\mathscr{M}}$ is known as the mixed topology In the present paper, we use this definition only in the case, where $X = C_\kappa(E)$ with E a completely regular topological Hausdorff space, $\tau_1 = \tau_\kappa^{\mathscr{C}}$ and $\tau_2 = \tau_\kappa^{\mathscr{U}}$, cf. Section 2 for the notations. We list some basic properties of the mixed topology in this case.

Lemma A.1. ([72], Theorem 3.1.1) The mixed topology $\tau_\kappa^{\mathscr{M}} := \tau\left(\tau_\kappa^{\mathscr{C}}, \tau_\kappa^{\mathscr{U}}\right)$ is identical with the topology $\tau_\kappa^{\mathscr{M}}$ defined in Section 2 via the family of seminorms $p_{\kappa, (C_n), (a_n)}$.

Proof. For $\kappa \equiv 1$, this lemma has been proved in [72]. The case of arbitrary κ is an easy modification of the proof in [72]. $\square$

Let us recall that a subset of a locally convex space is bounded if it is absorbed by every neighbourhood of zero.

Proposition A.2.
  (a) ([72, Section 2.2]) The topology $\tau_\kappa^{\mathscr{M}}$ is the strongest locally convex topology on $C_\kappa(E)$ that coincides with $\tau_\kappa^{\mathscr{C}}$ on bounded sets of $\tau_\kappa^{\mathscr{U}}$.
  (b) ([72, Corollary on p. 56]) A set $B \subset C_\kappa(E)$ is bounded in the topology $\tau_\kappa^{\mathscr{M}}$ if and only if it is bounded in the topology $\tau_\kappa^{\mathscr{U}}$.

(c) *([72, Corollary 2.2.4]) The topology $\tau_\kappa^{\mathscr{M}}$ can be defined as the weakest topology τ on $C_\kappa(E)$ such that for every locally convex space F and every linear operator $T \colon C_\kappa(E) \to F$, T is τ-continuous if and only if T is $\tau_\kappa^{\mathscr{C}}$-continuous on $\tau_\kappa^{\mathscr{U}}$-bounded sets.*

Another equivalent definition of the mixed topology $\tau_\kappa^{\mathscr{M}}$ is given by the following construction. Let $\mathscr{W}_0(E)$ denote the family of weights consisting of all bounded functions $w \colon E \to [0, \infty)$ such that, for every $\varepsilon > 0$, the set $\{x \in E \colon \kappa(x)w(x) \geq \varepsilon\}$ is compact. For every weight $w \in \mathscr{W}_0(E)$ we define the seminorm

$$p_{\kappa,w}(\varphi) = \sup_{x \in E} |w(x)\kappa(x)\varphi(x)| \quad \text{for all } \varphi \in C_\kappa(E).$$

Proposition A.3. *([71, Theorem 10.6]) The locally convex topology on $C_\kappa(E)$ defined by the family of seminorms $\{p_{\kappa,w} \colon w \in \mathscr{W}_0(E)\}$ is identical with the topology $\tau_\kappa^{\mathscr{M}}$.*

The following result is a special case of Theorem 10.6 in [72]. Since in our case the proof is simple, we include it fot the reader's convenience.

Proposition A.4. *A sequence $(\varphi_n) \subset C_\kappa(E)$ is $\tau_\kappa^{\mathscr{M}}$-convergent to $\varphi \in C_\kappa(E)$ if and only if*

$$\sup_{n \geq 1} \|\varphi_n\|_\kappa < \infty \quad \text{and} \quad \lim_{n \to \infty} \varphi_n = \varphi \quad \text{in the topology } \tau_\kappa^{\mathscr{C}}.$$

Proof. We note that if

(A.1) $$\tau_\kappa^{\mathscr{M}} - \lim_{n \to \infty} \varphi_n = 0,$$

and $\sup_n \|\varphi\|_\kappa = \infty$, then there exist $x_n \in E$, $n \in \mathbb{N}$, such that $\kappa(x_n)\varphi_n(x_n) \geq n$. Choosing $C_n := \{x_n\}$, $a_n = \frac{1}{n}$, we have $p_{\kappa,(C_n),(a_n)}(\varphi_n) \geq 1$ for all $n \in \mathbb{N}$, contradicting (A.1). Hence the assertion follows from Lemma A.1 and Proposition A.2 (a). $\square$

Lemma A.5. *For each weight κ, the mapping*

$$\mathcal{I}_\kappa : C_b(E) \to C_\kappa(E), \quad \mathcal{I}_\kappa \varphi = \kappa^{-1}\varphi,$$

is a linear homeomorphism of $\big(C_b(E), \tau_1^{(i)}\big)$ onto $\big(C_\kappa(E), \tau_\kappa^{(i)}\big)$ for all three topologies $\tau_1^{\mathscr{U}}, \tau_1^{\mathscr{C}}, \tau_1^{\mathscr{M}}$ and $\tau_\kappa^{\mathscr{U}}, \tau_\kappa^{\mathscr{C}}, \tau_\kappa^{\mathscr{M}}$, respectively.

Proof. The proof is obvious for $\tau_\kappa^{\mathscr{C}}$ and $\tau_\kappa^{\mathscr{U}}$. By Proposition A.2(c), it is enough to check continuity of $\mathcal{I}_\kappa$ and $\mathcal{I}_\kappa^{-1}$ on balls of $C_b(E)$ and $C_\kappa(E)$ respectively. But this follows from Proposition A.2(a). $\square$

Theorem A.6. *([65, Theorem 7.1]) If Hypothesis 2.1 holds, then the space $\big(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\big)$ is complete.*

Theorem A.7. *A set $B \subset C_\kappa(E)$ is relatively $\tau_\kappa^{\mathscr{M}}$-compact if and only if the following two conditions hold:*

(1) $$\sup_{\varphi \in B} \|\varphi\|_\kappa < \infty,$$

(2) *B is equicontinuous on every compact subset of E.*

Proof. Assume that (1) and (2) hold. By (1) and Proposition A.2(a), it is enough to prove that B is relatively $\tau_\kappa^{\mathscr{C}}$-compact but this follows immediately from (1), (2), and an appropriate version of Ascoli's theorem, see, e.g., [26, Theorem 8.2.11]. The converse statement is obvious. $\square$

Theorem A.8. *([33, Theorem 11]) Suppose that $\mathscr{A} \subset C_\kappa(E)$ is an algebra that separates the points of E, and that, for each $x \in E$, there exists $a \in \mathscr{A}$ with $a(x) \neq 0$. Then $\mathscr{A}$ is $\tau_\kappa^{\mathscr{M}}$-dense in $C_\kappa(E)$.*

Proof. For $\kappa = 1$, the theorem was proved in [33]. For general κ, it follows from Lemma A.5. $\qquad\square$

Theorem A.9. *The space $M_\kappa(E)$ is the topological dual space of $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$. Moreover, if $M \subset M_\kappa(E)$ such that the set of measures $\left\{\kappa^{-1}\mu; \, \mu \in M\right\}$ is tight and bounded in variation norm, then M considered as a set of functions on $C_\kappa(E)$ is $\tau_\kappa^{\mathscr{M}}$-equicontinuous.*

Proof. If $\kappa = 1$ then by [46] (see also [33]) the dual space of $\left(C_1(E), \tau_1^{\mathscr{M}}\right)$ is identified as the space of Baire measures of finite variation on E for any completely regular space E. In this paper Borel and Baire σ-algebras on E coincide, hence the first claim follows. For the proof of the second part of the assertion see [16, p. 136, Proposition 3.6] . For arbitrary κ the proof follows from Lemma A.5. $\qquad\square$

Remark A.10. We note that if, in addition, to Hypothesis 2.1 we assume that our underlying space E is Radon, i.e. every finite measure on $(E, \mathscr{B}(E))$ is tight, which is the case in all examples in this paper, then the first assertion of Theorem A.9 is a trivial consequence of the Daniell-Stone Theorem (see e.g. [21]). Indeed, again by Lemma A.5 we may assume that $\kappa = 1$. Obviously, each $\mu \in M_b(E)$ is in the topological dual $\left(C_b(E), \tau_1^{\mathscr{M}}\right)'$ of $\left(C_b(E), \tau_1^{\mathscr{M}}\right)$. To prove the converse we first note that it is well-known that every element ℓ of the latter can be written as a difference $\ell = \ell^+ - \ell^-$, with $\ell^+, \ell^- \in \left(C_b(E), \tau_1^{\mathscr{M}}\right)'$ and both are nonnegative on nonnegative elements in $C_b(E)$ (see e.g. [44]). Hence we may assume that ℓ itself has this property. Since $C_b(E)$ is a Stone vector lattice, which by assumption (2) in Hypothesis 2.1 generates $\mathscr{B}(E)$, we only have to show the Daniell continuity, because then ℓ is represented by a unique finite nonnegative measure μ, which, since E is a Radon space, is in $M_b(E)$. But if $\varphi_n \in C_b(E)$, $\varphi_n \geq 0$, $n \in \mathbb{N}$, such that $\varphi_n \downarrow 0$ pointwise on E, then by Proposition A.4 and Dini's Theorem, we conclude that

$$\tau_1^{\mathscr{M}} - \lim_{n \to \infty} f_n = 0 \, ,$$

hence $\lim\limits_{n \to \infty} \ell(f_n) = 0$, and Daniell continuity holds.

Theorem A.11. *([33, Corollary on p. 119]) Let $M \subset M_\kappa(E)$ such that $\kappa^{-1}M$ is tight and bounded in variation norm. Then, M is narrowly relatively compact in $M_\kappa(E)$.*

APPENDIX B. CONTINUOUS OPERATORS FOR THE MIXED TOPOLOGY

The aim of this section is to characterise norm-bounded linear operators $T: C_\kappa(E) \to C_\kappa(E)$ that are $\tau_\kappa^{\mathscr{M}}$-continuous, i.e., continuous in the topology $\tau_\kappa^{\mathscr{M}}$.

Proposition B.1. *([16, p. 8, Corollary 1.7]) Let F be a locally convex space, and let $\mathcal{T}$ be an arbitrary family of linear mappings $T: C_\kappa(E) \to F$. The family $\mathcal{T}$ is $\tau_\kappa^{\mathscr{M}}$-equicontinuous if and only if the family*

$$\mathcal{T}|_B := \left\{T|_B: T \in \mathcal{T}\right\}$$

is $\tau_\kappa^{\mathscr{C}}$-equicontinuous for every norm-bounded set $B \subset C_\kappa(E)$.

Proof. The result follows immediately from Lemma A.1 and [16, Corollary 1.7]. $\qquad\square$

Theorem B.2. *Let* $T\colon C_\kappa(E) \to C_\kappa(E)$ *be a norm-bounded linear operator. Then, the following conditions are equivalent.*

(i) *T is $\tau_\kappa^{\mathscr{M}}$-continuous.*

(ii) *There exists a family $\{\mu(x,\,\cdot\,)\colon x \in E\} \subset M_\kappa(E)$ such that*

　(a) *for every $\varphi \in C_\kappa(E)$,*

$$T\varphi(x) = \int_E \varphi(y)\mu(x,dy),$$

　(b) *the mapping $E \ni x \mapsto \mu(x,B)$ is measurable for every Borel set $B \subset E$,*

　(c)

(B.1)
$$\sup_{x \in E} \left(\kappa(x) \int_E \frac{|\mu|(x,dy)}{\kappa(y)} \right) < \infty,$$

　and for every $\varepsilon > 0$ and every compact set $K_1 \subset E$, there exists another compact $K_2 \subset E$ such that

(B.2)
$$\sup_{x \in K_1} \left(\kappa(x) \int_{E \setminus K_2} \frac{|\mu|(x,dy)}{\kappa(y)} \right) < \varepsilon$$

Proof. We first prove the theorem for the case $\kappa \equiv 1$.

(i) $\Rightarrow$ (ii): Assume (i). We start by showing (a). By Proposition B.1, for every $x \in E$, the functional $l_x(\varphi) = T\varphi(x)$ is continuous in the topology $\tau_1^{\mathscr{M}}$. Therefore, by Theorem A.9, there exists a measure $\mu(x,\,\cdot\,) \in M_1(E)$ such that

$$l_x(\varphi) = T\varphi(x) = \int_E \varphi(y)\mu(x,dy),$$

which proves (a).

In order to prove (b), let $U \subset E$ be open. Let $\mathbb{R}^\infty$ denote the Polish space of infinite sequences of real numbers. Since the Baire σ-algebra $\mathrm{Ba}(E)$ is identical with the Borel σ-algebra $\mathscr{B}(E)$, by [5, Lemma 6.3.3], there exists an open set $V \subset \mathbb{R}^\infty$ and a continuous function $f\colon E \to \mathbb{R}^\infty$ such that $U = f^{-1}(V)$. Without loss of generality, we may assume that the measures $\mu(x,\,\cdot\,)$ are non-negative. Since $\mathbb{R}^\infty$ is a Polish space, there exists a sequence $(\varphi_n) \subset C_b(\mathbb{R}^\infty)$ such that $0 \leq \varphi_n \leq 1$ and $\lim_{n\to\infty} \varphi_n(z) = I_V(z)$. By the dominated convergence theorem,

$$\mu(x,U) = \int_{\mathbb{R}^\infty} I_U(y)\mu(x,dy) = \int_{\mathbb{R}^\infty} I_V(f(y))\mu(x,dy)$$

$$= \lim_{n\to\infty} \int_{f^{-1}(V)} \varphi_n(f(y))\mu(x,dy)$$

$$= \lim_{n\to\infty} T\left(\varphi_n \circ f\right)(x) \quad \text{for all } x \in E.$$

Hence, the function $x \to \mu(x,U)$ is Borel measurable as a pointwise limit of continuous functions. Finally, the measurability of the function $\mu(\,\cdot\,,B)$ for any Borel set $B \subset E$ follows from Dynkin's lemma.

Next, we prove (c). Invoking the lattice properties of $C_b(E)$ and $M_b(E)$, we have

$$\sup_{0 \leq \varphi \leq 1} |l_x(\varphi)| = |\mu|(x,\,\cdot\,).$$

Therefore,

$$\sup_{x \in E} \int_E |\mu|(x, dy) = \sup_{0 \leq \varphi \leq 1} \sup_{x \in E} |T\varphi(x)|,$$

which shows that (B.1) holds.

Let $K_1 \subset E$ be compact. Since T is $\tau_1^{\mathcal{M}}$-continuous, for every $\varepsilon > 0$ there exists a $\tau_1^{\mathcal{M}}$-neighbourhood of zero such that, for $\varphi \in U$, we have $p_{K_1}(T\varphi) < \varepsilon$. For $x \in E$, we have $T\varphi(x) = l_x(\varphi)$ and

$$p_{K_1}(T\varphi) = \sup_{x \in K_1} |l_x(\varphi)|.$$

Therefore, the family $\{\mu(x, \cdot) : x \in K_1\}$ is equicontinuous on U. Now, (B.2) follows from [65, Theorem 5.1].

(ii) $\Rightarrow$ (i): If (ii) holds, then

$$\sup_{x \in E} |T\varphi(x)| < \infty.$$

By Proposition A.2, the operator T is $\tau_1^{\mathcal{M}}$-continuous if and only if it is $\tau_1^{\mathscr{C}}$-continuous on every ball $B_r = \{\varphi \in C_b(E) : \|\varphi\|_\infty \leq r\}$ with $r \geq 0$. For $\varepsilon > 0$ and a compact $C \subset E$, let $U := \{\varphi \in B_r : p_C(\varphi) < \varepsilon\}$. Let the compact $K \subset E$ be chosen in such a way that

$$\sup_{x \in C} |\mu|(x, E \setminus K) < \frac{\varepsilon}{2}(1 + r)^{-1}.$$

Then,

$$\sup_{x \in K} \left(\int_E |\varphi(y)| \, |\mu|(x, dy) \right) \leq \int_K |\varphi(y)| \, |\mu|(x, dy) + \frac{\varepsilon}{2}.$$

Let

$$U_1 = \left\{ \varphi \in B_r : \sup_{y \in K} |\varphi(y)| < \frac{\varepsilon}{2M} \right\},$$

where $M := 1 + \sup_{x \in E} |\mu|(x, E)$. Then, for every $\varphi \in U_1$,

$$\sup_{x \in K} \int_E |\varphi(y)| |\mu|(x, dy) < \frac{\varepsilon}{2M} M + \frac{1}{2}\varepsilon = \varepsilon.$$

This shows that $T\varphi \in U$, for every $\varphi \in U_1$, and concludes the proof for $\kappa \equiv 1$.

For general κ, observe that, if (ii) holds, then

$$\mathcal{I}_\kappa^{-1} T \mathcal{I}_\kappa \varphi(x) = \kappa(x) \int_E \varphi(y) \frac{\mu(x, dy)}{\kappa(y)}$$

is $\tau_1^{\mathcal{M}}$-$\tau_1^{\mathcal{M}}$-continuous on $C_b(E)$ by the first part of the proof. Hence, by Lemma A.5, T is $\tau_\kappa^{\mathcal{M}}$-$\tau_\kappa^{\mathcal{M}}$-continuous on $C_\kappa(E)$. The converse implication follows by a similar argument. $\square$

Corollary B.3. *Assume that a linear operator $T : C_\kappa(E) \to C_\kappa(E)$ is $\tau_\kappa^{\mathcal{M}}$-$\tau_\kappa^{\mathcal{M}}$-continuous. Then T is positive if and only its representing measures $\mu(x, \cdot)$ are a non-negative for every $x \in E$.*

It follows from Theorem B.2 that every norm-bounded $\tau_\kappa^{\mathcal{M}}$-$\tau_\kappa^{\mathcal{M}}$-continuous linear operator on $C_\kappa(E)$ can be extended to a linear operator from $\kappa^{-1}B_b(E)$ to $\kappa^{-1}B_b(E)$, where $B_b(E)$ refers to the space of all bounded Borel measurable functions.

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