3-MANIFOLDS WITH NILPOTENT EMBEDDINGS IN S^4

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Abstract. We consider embeddings of 3-manifolds M in S^4 such that the two complementary regions X and Y each have nilpotent fundamental group. If $\beta = \beta_1(M)$ is odd then these groups are abelian and $\beta \leq 3$. In general $\pi_1(X)$ and $\pi_1(Y)$ have 3-generator presentations, and $\beta \leq 6$. We determine all such nilpotent groups which are torsion-free and have Hirsch length ≤ 5.

This is a continuation of the papers [5, 7], in which we considered the complementary regions of a closed hypersurface in S^4. Let M be a closed orientable 3-manifold and $j : M \to S^4 = X \cup_M Y$ be a locally flat embedding. Then $\chi(X) + \chi(Y) = 2$ and we may assume that $\chi(X) \leq \chi(Y)$. In [5, §7] we considered the possibilities for $\chi(X)$, $\pi_X = \pi_1(X)$ and $\pi_Y = \pi_1(Y)$, and showed that if π_X is abelian then $\beta = \beta_1(M; \mathbb{Q}) \leq 4$ or $\beta = 6$, while in [7] we attempted to apply 4-dimensional surgery to classify embeddings such that both π_X and π_Y are abelian. Here we shall cast our net a little wider. In order to use 4-dimensional surgery arguments we must restrict the possible groups π_X and π_Y. Under our present understanding of the Disc Embedding Theorem, these groups should be in the class G of groups generated from groups with subexponential growth by increasing unions and extensions [2]. This class includes all elementary amenable groups and is included in the class of *restrained* groups, those which have no non-cyclic free subgroups. We shall also assume that the embedding j is *bi-epic*, i.e., that each of the homomorphisms $j_{X*} : \pi = \pi_1(M) \to \pi_X$ and $j_{Y*} : \pi \to \pi_Y$ is an epimorphism. This is so if π_X and π_Y are nilpotent, since $H_1(j_X)$ and $H_1(j_Y)$ are always epimorphisms.

Our main results are, firstly, that if j is bi-epic and π_X and π_Y are restrained then $0 \leq \chi(X) \leq \chi(Y)$, so $\chi(X)$ and $\chi(Y)$ are determined by β, and if β is even then $\chi(X) = \chi(Y) = 1$ and so $\beta_2(\pi_X) \leq \beta_1(\pi_X)$ and $\beta_2(\pi_Y) \leq \beta_1(\pi_Y)$. Secondly, if π_X and π_Y are nilpotent then either $\beta = 1$ or 3 and π_X and π_Y are free abelian groups, or $\beta = 0, 2, 4$

1991 *Mathematics Subject Classification.* 57N13.
Key words and phrases. embedding, homologically balanced, nilpotent, 3-manifold, restrained, surgery.

or 6. If we assume further that π_X and π_Y are torsion-free nilpotent groups then the evidence suggests that, with one exception, they are free abelian of rank ≤ 3 or one of the $\mathbb{N}il^3$-groups Γ_q. The exception has Hirsch length 4. We show also that all nilpotent groups N of Hirsch length 5 have $\beta_2(N) > \beta_1(N)$, and so do not arise in this context. However, it is not yet known whether this is always so for torsion-free nilpotent groups of Hirsch length ≥ 6.

We shall say that an embedding has a group-theoretic property (e.g., abelian, nilpotent, ...) if the groups π_X and π_Y have this property.

1. RESTRAINED EMBEDDINGS

We begin with an observation that can be construed as a minimality condition.

Lemma 1. *Let $J = j_{K,\gamma}$ be an embedding obtained from $j : M \to S^4$ by a proper 2-knot surgery using the 2-knot K and the loop $\gamma \in \pi_{X(j)}$. Then J is not bi-epic, and $\pi_{X(J)}$ is not restrained, unless $\pi_{X(J)}$ is itself a restrained 2-knot group, in which case $\beta = \beta_1(M) = 1$ or 2.*

Proof. Let $C \cong \mathbb{Z}/q\mathbb{Z}$ be the subgroup of $\pi_{X(j)}$ generated by γ, and let t be a meridian for the knot group πK. Then

$$\pi_{X(J)} \cong \pi_{X(j)} *_C \pi K/\langle\langle t^q \rangle\rangle.$$

Since the 2-knot surgery is proper, $\langle\langle t^q \rangle\rangle$ is a proper normal subgroup of πK. Since the image of $\pi_1(M)$ lies in $\pi_{X(j)}$, the embedding J cannot be bi-epic. Moreover $\pi_{X(J)}$ can only be restrained if $\pi_{X(j)} \cong \mathbb{Z}$, in which case $\pi_{X(J)} \cong \pi K$ and $\chi(X(j)) = 0$ or 1, and so $\beta = 1$ or 2. □

Let G be a group. Then G' and ζG shall denote the commutator subgroup and centre of G, respectively.

If G is finitely generated and restrained then $def(G) \leq 1$. If $def(G) = 1$ then G is an ascending HNN extension, and so $\beta_1^{(2)}(G) = 0$. Hence $g.d.G \leq 2$, by [3, Theorem 2.5]. The argument is homological, and so it suffices that the augmentation ideal in $\mathbb{Z}[G]$ have a presentation of deficiency 1 as a $\mathbb{Z}[G]$-module. A finitely generated group G is *balanced* if it has deficiency ≥ 0, and is *homologically balanced* if the augmentation ideal in $\mathbb{Z}[G]$ has a square presentation matrix (i.e., has a presentation of deficiency 0 as a $\mathbb{Z}[G]$-module). We then have $\beta_2(G; R) \leq \beta_1(G; R)$, for all simple coefficients R.

Let $BS(1, m)$ be the Baumslag-Solitar group with presentation $\langle t, a \mid tat^{-1} = a^m \rangle$, for $m \in \mathbb{Z} \setminus \{0\}$. Then $BS(1, 1) \cong \mathbb{Z}^2$, while $BS(1, -1) \cong \pi_1(Kb)$ is the Klein bottle group.

If π_X and π_Y are each restrained then $\chi(X), \chi(Y) \geq 0$. Hence $(\chi(X), \chi(Y))$ is determined by the parity of $\beta_1(M)$, since $0 \leq \chi(X) \leq \chi(Y) \leq 2$ and $\chi(X) \equiv \chi(Y)$ *mod* (2).

Theorem 2. *Let $j : M \to S^4$ be a bi-epic embedding such that π_X and π_Y are restrained.*

If $\beta = \beta_1(M; \mathbb{Q})$ is odd then $\chi(X) = 0$ and $\chi(Y) = 2$, and X is aspherical. If, moreover, π_X is almost coherent or elementary amenable then $\pi_X \cong \mathbb{Z}$ or $BS(1, m)$, for some $m \neq 0$, and $\beta = 1$ or 3.

If β is even then $\chi(X) = \chi(Y) = 1$, and π_X and π_Y are homologically balanced.

Proof. Since j is bi-epic, $c.d.X \leq 2$ and $c.d.Y \leq 2$, by [5, Theorem 5.1]. Hence if $\chi(X) = 0$ and π_X is restrained then X is aspherical, by [3, Theorem 2.5]. If, moreover, π_X is elementary amenable or almost coherent then $\pi_X \cong \mathbb{Z}$ or $BS(1, m)$ for some $m \neq 0$, by [3, Corollary 2.6.1]. Hence $\beta = \beta_1(X) + \beta_2(X) = 1$, if $\pi_X \ncong BS(1, 1) = \mathbb{Z}^2$, and $\beta = 3$ if $\pi_X \cong \mathbb{Z}^2$.

Since $c.d.X \leq 2$ and X is homotopy equivalent to a finite complex the cellular chain complex $C_*(X; \mathbb{Z}[\pi_X])$ is chain homotopy equivalent to a finite free complex of length 2. If β is even and $\chi(X) = 1$ it follows that π_X is homologically balanced. Similarly for π_Y. ☐

There are examples of each type. (See below). There is also a partial converse. If $\chi(X) = 0$ and $\pi_X \cong BS(1, m)$ for some $m \neq 0$ then X is aspherical and j_{X*} is an epimorphism, by [5, Theorem 5.1].

If G is restrained and homologically balanced is there a bound on the minimal number of generators of G? on $\beta_1(G)$?

2. NILPOTENT EMBEDDINGS

Nilpotent embeddings are always bi-epic, since homomorphisms to a nilpotent group which induces epimorphisms on abelianization are epimorphisms.

Lemma 3. *Let $j : M \to S^4$ be an embedding such that π_X and π_Y are nilpotent.*

If $\beta = \beta_1(M; \mathbb{Q})$ is odd and π_X and π_Y are nilpotent then either $X \simeq S^1$ and $Y \simeq S^2$ or $X \simeq T$ and $Y \simeq S^1 \vee S^2$.

If β is even then $\beta = 0, 2, 4$ or 6, and π_X and π_Y are each 3-generated.

Proof. Suppose first that β is odd. Then X is aspherical, since $\chi(X) = 0$ and π_X is nilpotent. Hence $\pi_X \cong \mathbb{Z}$ or $\mathbb{Z}^2$, since $c.d.X \leq 2$. Since π_Y is nilpotent and $H_1(Y; \mathbb{Z}) \cong H^2(X; \mathbb{Z}) = 0$ or $\mathbb{Z}$, $\pi_Y = 1$ or $\mathbb{Z}$. The further details in this case are given in [7, Theorem 14].

We may assume that β is even and $\beta > 4$. Since $\chi(X) = 1$ and $H_i(X; R) = 0$ for $i > 2$ and any simple coefficients R, we have $\beta_2(X) = \beta_1(X)$, and so $\beta_2(\pi_X) \leq \beta_1(\pi_X)$. Since π_X is finitely generated and nilpotent, there is a prime p such that π_X can be generated by $d = \beta_1(\pi_X; \mathbb{F}_p)$ elements. Let $\widehat{\pi_X}$ be the pro-p completion of π_X. Since π_X is nilpotent, it is p-good, and so $\beta_i(\widehat{\pi_X}; \mathbb{F}_p) = \beta_i(\pi_X; \mathbb{F}_p)$, for all i. The group $\widehat{\pi_X}$ is a pro-p analytic group, and so has a minimal presentation with $d = \beta_1(\widehat{\pi_X}; \mathbb{F}_p)$ generators and $r = \beta_2(\widehat{\pi_X}; \mathbb{F}_p)$ relators. Since $\beta > 2$, $\widehat{\pi_X} \not\cong \widehat{\mathbb{Z}}_p$, and so $r > \frac{d^2}{4}$, by [9, Theorem 2.7]. (Similarly for π_Y.) Therefore $d \leq 3$ and $\beta \leq 2d \leq 6$. □

If π_X is nilpotent and $\beta_1(X) = 0$ then π_X is finite, while if $\beta_1(X) = 1$ then $\pi_X \cong F \rtimes \mathbb{Z}$, where F is finite. Thus if π_X and π_Y are torsion-free nilpotent and $\beta \leq 3$ then π_X and π_Y are abelian. See [7, Theorems 10, 11 and 16] for more on such embeddings.

It is reasonable to restrict consideration further to torsion-free nilpotent groups, as such groups satisfy the Novikov conjecture, and the surgery obstructions are maniable.

If G is torsion-free nilpotent of Hirsch length h then $c.d.G = h$. The first non-abelian examples are the $\mathbb{N}il^3$-groups Γ_q, with presentations $\langle x, y, z \mid [x, y] = z^q,\ [x, z] = [y, z] = 1\rangle$. Some of the argument of [7, Theorem 18] for the group $\mathbb{Z}^3$ extends to the groups Γ_q. Since $c.d.X \leq 2$, $\chi(X) = 1$ and $c.d.\Gamma_q = 3$ we see that $\pi_2(X)$ is a projective $\mathbb{Z}[\Gamma_q]$-module of rank 1. It is stably free since $\widetilde{K}_0([\mathbb{Z}[G]) = 0$ for torsion-free poly-Z groups G. However, Artamanov has shown that if G is a nonabelian poly-Z group then there are infinitely many isomorphism classes of projective $\mathbb{Z}[G]$-modules P such that $P \oplus \mathbb{Z}[G] \cong \mathbb{Z}[G]^2$ [1]. We do not know which can be realized as $\pi_2(X)$, for an embedding j with $\pi_X \cong \Gamma_q$. (This contrasts strongly with the situation when $\pi_X \cong \mathbb{Z}^3$, when $\pi_2(X)$ is free of rank 1.)

We know of only one other balanced, torsion-free nilpotent group.

Lemma 4. *There is just one torsion-free nilpotent group of Hirsch length* 4 *which has a balanced presentation.*

Proof. Let N be a torsion-free nilpotent group of Hirsch length 4 with a balanced presentation. Since N is an orientable PD_4-group, $\beta_2(N; R) = 2(\beta_1(N; R) - 1)$, for R any field. Hence $N/N' \cong \mathbb{Z}^2$, so $N' \cong \mathbb{Z}^2$ also. If $\zeta N \cong \mathbb{Z}^2$ then $\beta_1(N) = 3$, so we must have $\zeta N \cong \mathbb{Z}$. Moreover $N/\zeta N$ is torsion-free. It is then not hard to show that $N \cong \mathbb{Z}^3 \rtimes_A \mathbb{Z}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}).$$

which has the 2-generator balanced presentation $\langle t, u \mid [t,[t,[t,u]]] = [u,[t,u]] = 1\rangle$. (The generator u corresponds to the column vector $(1,0,0)^{tr}$.) □

This group is a quotient of the relatively free group $F(2)/F(2)_{[4]}$. No relatively free nilpotent group with Hirsch length ≥ 4 is balanced. For if $G = F(m)/F(m)_{[k]}$ then $H_2(G;\mathbb{Z}) \cong F(m)_{[k]}/F(m)_{[k+1]}$, by the five-term exact sequence of low degree for G as a quotient of $F(m)$. Hence $H_2(G;\mathbb{Z})$ has rank $> m$ unless $\beta = 1$ or $\beta = 2$ and $k \leq 3$ or $\beta = 3$ and $k = 2$, by the Witt formulae [11, Theorems 5.11 and 5.12]. Thus the only such groups with $\beta_2(G;\mathbb{Q}) \leq \beta_1(G;\mathbb{Q})$ are $G \cong \mathbb{Z}^k$ with $k \leq 3$ or $G \cong \Gamma = \Gamma_1 = F(2)/F(2)_{[3]}$. In each case $h(G) \leq 3$.

Theorem 5. *Let N be a finitely generated nilpotent group of Hirsch length $h(N) = 5$. Then $\beta_2(N;\mathbb{Q}) > \beta_1(N;\mathbb{Q})$, and so N is not balanced.*

Proof. If G is any finitely generated group then the kernel of the homomorphism $\psi_G : \wedge^2 H^1(G;\mathbb{Q}) \to H^2(G;\mathbb{Q})$ induced by cup product is isomorphic to $Hom(G^\tau/[G,G^\tau],\mathbb{Q})$, where G^τ is the preimage in G of the torsion subgroup of G/G' [4]. If G is solvable then $G^\tau/[G,G^\tau]$ has rank $\leq h(G^\tau) = h(G) - \beta_1(G;\mathbb{Q})$. Hence $\beta_2(G;\mathbb{Q}) - \beta_1(G;\mathbb{Q}) \geq \binom{\beta_1(G;\mathbb{Q})}{2} - h(G)$. Thus we may assume that $\beta = \beta_1(N;\mathbb{Q}) \leq 3$. Since N is nilpotent and $h(N) > 1$ we must then have $\beta = 2$ or 3. The quotient of N by its maximal finite normal subgroup is torsion free [12, Proposition 5.2.7], and has the same rational Betti numbers as N, so we may also assume that N is torsion-free.

The intersection $N' \cap \zeta N$ is nontrivial [12, Proposition 5.2.1], and so we may choose a maximal infinite cyclic subgroup $A \leq N' \cap \zeta N$. Let $\overline{N} = N/A$. Then $h(\overline{N}) = 4$ and $\overline{N}$ is also torsion-free, since the preimage of any finite subgroup in N is torsion-free and virtually $\mathbb{Z}$. Hence $\overline{N}$ is an orientable PD_4-group. Moreover, $\beta_1(\overline{N}) = \beta < 4$, since $A \leq N'$, and so $\overline{N}^\tau/[\overline{N},\overline{N}^\tau] \neq 1$.

Let $e \in H^2(\overline{N};\mathbb{Z})$ classify the extension

$$0 \to A \to N \to \overline{N} \to 1.$$

There is an associated "Gysin" exact sequence [10, Example 5C]:

$$0 \to \mathbb{Q}e \to H^2(\overline{N};\mathbb{Q}) \to H^2(N;\mathbb{Q}) \to H^1(\overline{N};\mathbb{Q}) \xrightarrow{\cup e} H^3(\overline{N};\mathbb{Q}) \to \dots$$

Suppose first that $\beta = 2$. Then $\beta_2(\overline{N}) = 2$ also, since $\chi(\overline{N}) = 0$. Hence $\psi_{\overline{N}} = 0$, since $\overline{N}^\tau/[\overline{N},\overline{N}^\tau] \neq 1$ and $\binom{\beta}{2} = 1$. Since the cup product of $H^3(\overline{N};\mathbb{Q})$ with $H^1(\overline{N};\mathbb{Q})$ is a non-singular pairing, it follows that $\alpha \cup e = 0$ for all $\alpha \in H^1(\overline{N};\mathbb{Q})$. Hence $\beta_2(N;\mathbb{Q}) = 2 - 1 + 2 = 3 > \beta$.

When $\beta = 3$ we must look a little more closely at the consequences of Poincaré duality. We note first that $\beta_2(\overline{N};\mathbb{Q}) = 2(\beta - 1) = 4$. Since cup-product of odd-dimensional cohomology classes is skew-symmetric, for any $e \in H^2(\overline{N};\mathbb{Q})$ the homomorphism $-\cup e$ from $H^1(\overline{N};\mathbb{Q})$ to $H^3(\overline{N};\mathbb{Q})$ has a skew-symmetric matrix, if these cohomology groups are given bases which are Kronecker dual with respect to the cup-product pairing into $H^4(\overline{N};\mathbb{Q}) \cong \mathbb{Q}$. Since $\beta = 3$ is odd, $\det(-\cup e) = 0$, and so we see that $\beta_2(N) \geq \beta_2(\overline{N}) - 1 + 1 = 4 > \beta$ again. □

If we consider more general solvable groups, we can find many metabelian groups of Hirsch length 5 with balanced presentations. One such group is a a Cappell-Shaneson 3-knot, with commutator subgroup $\mathbb{Z}^4$ and presentation

$$\langle t, x \mid t^4xt^{-4} = t^2x^2t^{-1}x^{-1}t^{-1}x^{-1},\ xt^2xt^{-2} = t^2xt^{-2}x\rangle.$$

If G is torsion-free nilpotent and $h(G) \geq 6$ is $def(G) < 0$? In particular, is this so if $G/G' \cong \mathbb{Z}^3$?

3. Examples

Pairs of groups with balanced presentations and isomorphic abelianizations can be realized by embeddings of 3-manifolds [8].

Our examples are based on 3- and 4-component links $L = L_a \cup L_u \cup L_v$ or $L_a \cup L_b \cup L_u \cup L_v$, where $L_a \cup L_b$ and $L_u \cup L_v$ are trivial links. The 3-manifold M obtained by 0-framed surgery on L embeds in S^4, and the complementary regions have Kirby-calculus presentations in which one of these sublinks is 0-framed and the other dotted (the roles being exchanged for the two regions). The components L_a and L_b represent words A and B in $F(u,v)$ and the components L_u and L_v represent words U and V in $F(a,b)$, and we may arrange that π_X and π_Y have presentations $\langle u, v \mid A, B\rangle$ and $\langle a, b \mid U, V\rangle$, respectively. (See the Figure.) The embeddings constructed in this way are always bi-epic, since π_X and π_Y are generated by the images of meridians of L. We shall use the tabulation of links in [13].

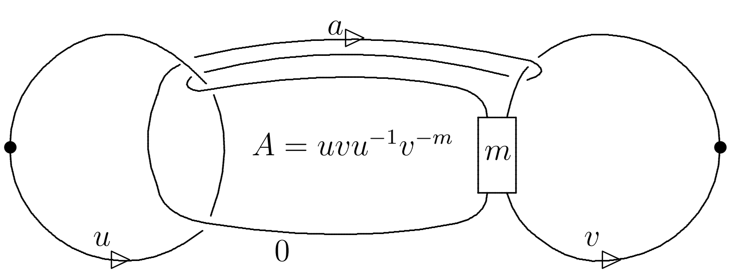

Figure 1

For example, consider the 3-component link in Figure 1, in which the strands in the box have m full twists, $A = uvu^{-1}v^{-m}$, $B = U = 1$ and $V = a^{m-1}$.

When $m = 0$ the link is the split union of an unknot and the Hopf link 2_1^2, $M \cong S^2 \times S^1$, $X \cong D^3 \times S^1$ and $Y \cong S^2 \times D^2$. When $m = 1$ the link is the Borromean rings (6_2^3), and X is a regular neighbourhood of the unknotted embedding of the torus T in S^4. When $m = -1$, the link is 8_9^3, and X is a regular neighbourhood of the unknotted embeding of the Klein bottle Kb in S^4 with normal Euler number 0. In general, X is aspherical, $\pi_X \cong BS(1, m)$ and $\pi_Y \cong \mathbb{Z}/(m-1)\mathbb{Z}$. (Note however that the boundary of a regular neighbourhood of the Fox 2-knot with group $BS(1, 2)$ gives an embedding of $S^2 \times S^1$ with $\pi_X \cong BS(1, 2)$ and $\chi(X) = 0$, but this embedding is not bi-epic and X is not aspherical.)

We may also construct embeddings such that $\pi_X \cong BS(1, m)$ and $\chi(X) = 1$, while $\pi_Y \cong BS(1, m)$ or $\mathbb{Z} \oplus \mathbb{Z}/(m-1)\mathbb{Z}$. These require 4-component links.

This is also the case if $\pi_X \cong \Gamma_q$, for then $\beta_2(X) = \beta_1(X) = 2$. If π_Y is abelian then $\pi_Y \cong \mathbb{Z}^2$ [5, Theorem 7.1], and so $q = 1$.

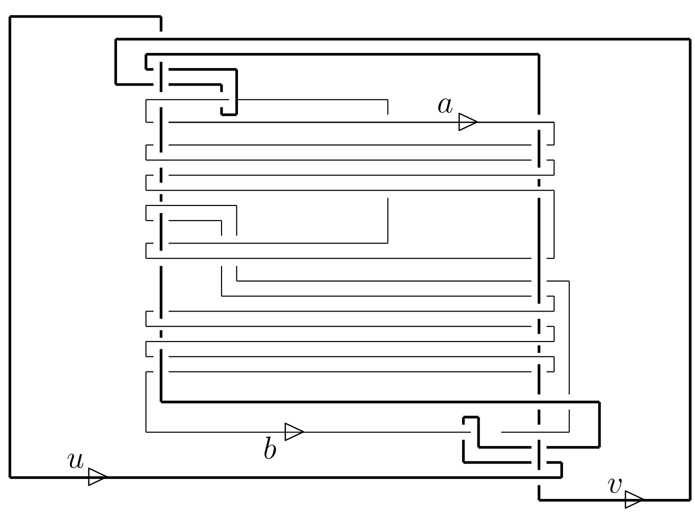

Figure 2

It is easy to find a 4-component link $L = L_a \cup L_b \cup L_u \cup L_v$ with each 2-component sublink trivial, and such that L_a and L_b represent (the conjugacy classes of) $A = [u, [u, v]]$ and $B = [v, [u, v]]$ in $F(u, v)$, respectively, while L_u and L_v have image 1 in $F(a, b)$. Arrange the link diagram so that L_u is on the left, L_v on the right, L_a at the top and L_b at the bottom. We may pass one bight of L_a which loops around L_u

under a similar bight of L_b, so that U now represents $[a,b]$ in $F(a,b)$. Finally we use claspers to modify L_u and L_v so that they represent $[b,v]$ in $F(b,v)$ and $[a,u]$ in $F(a,u)$. We obtain the link of Figure 2.

This link may be partitioned into two trivial links in three distinct ways, giving three embeddings of the 3-manifold obtained by 0-framed surgery on L. If the two sublinks are $L_a \cup L_b$ and $L_u \cup L_v$ then

$$A = vu^{-1}v^{-1}u^{-1}vuv^{-1}u, \; B = vuv^{-1}u^{-1}v^{-1}uvu^{-1},$$
$$U = b^{-1}aba^{-1} \quad \text{and} \quad V = 1.$$

Hence $\pi_X \cong \Gamma_1$ and $\pi_Y \cong \mathbb{Z}^2$.

Each of the other partitions determine abelian embeddings, with $\pi_X \cong \pi_Y \cong \mathbb{Z}^2$ and $\chi(X) = \chi(Y) = 1$.

With a little more effort, instead of passing just one bight of L_a under L_b (as above), we may interlace the loops of L_a and L_b around each of L_u and L_v so that u and V represent $[a,[a,b]]$ and $[b[a,b]]$, respectively, and so that each 2-component sublink of L is still trivial. If we then use claspers again we may arrange that u represents $[a,v]$ and v represents $[b,u]$, so that we obtain a 3-manifold which has one embedding with $\pi_X \cong \pi_Y \cong \Gamma_1$ and another with $\pi_X \cong \pi_Y \cong \mathbb{Z}^2$. Can we refine this construction so that the third embedding has $\pi_X \cong \Gamma_1$ and $\pi_Y \cong \mathbb{Z}^2$?

Acknowledgment. This work was begun at the BIRS conference on "Unifying Knot Theory in Dimension 4", 4-8 November 2019.

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