RESCALING NONLINEAR NOISE FOR 1D STOCHASTIC PARABOLIC EQUATIONS

BEN GOLDYS
SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, AUSTRALIA
AND MISHA NEKLYUDOV$^\sharp$
DEPARTAMENTO DE MATEMATICA, UFAM, BRASIL

ABSTRACT. We show regularisation effect of nonlinear gradient noise to the solution of 1D stochastic parabolic equation. We demonstrate convergence to a martingale (independent upon space variable) when we rescale noise at the extremum points of the process.

CONTENTS

1. INTRODUCTION

Regularisation of partial differential equations by noise has been an object of intense study for a number of years, see book of Flandoli [3], paper of Flandoli, Gubinelli and Priola [5] and, more recently, a review of the literature in Gess, Souganidis [9]. For instance, It was shown in Flandoli, Gubinelli and Priola [5] that the equation

$$du + b(x)\nabla u\, dt = \partial_x u \circ d\beta_t$$

can be well posed even if the corresponding deterministic equation is not. Their proof was based on linearity and homogenuity of the noise. An example of nonlinear equation, where the noise does not improve regularity can be found in Flandoli [3]. The effect of regularization by nonlinear stochastic perturbations in the setting of stochastic conservation laws has been recently considered in Gess, Souganidis [9, 10] and Gassiat, Gess [8]. The purpose of our work is to study regularisation by noise in the parabolic setting. Our estimates (Theorem 4.1 and Proposition 6.1) show that nonlinear gradient noise, when appropriately scaled, leads to flattening out of the system (compare with Example 6.1 in Section 6 for the linear case).

The process we consider can be described as an Ornstein-Uhlenbeck process with the noise that is "rescaled" at stationary points of the solution. Informally, it can be described as a limit,

Date: July 16, 2019.
2010 *Mathematics Subject Classification.* 60H15, 37K99.
Key words and phrases. regularization by noise, regularity.
$^\sharp$Corresponding author, e-mail: misha.neklyudov@gmail.com.

when $\epsilon \to 0$, of solutions to the stochastic PDE of the form

$$d\psi^\epsilon = A\psi^\epsilon\,dt + g\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\circ dW_t^Q, \quad x\in\mathbb{S}^1,\, t\geq 0, \tag{1.1}$$
$$\psi(0)=\psi_0\in L^2(\mathbb{S}^1).$$

where $\mathbb{S}^1$ stands for the unit circle, $\left(W_t^Q\right)$ is an $L^2\left(\mathbb{S}^1\right)$-valued Wiener process with the trace-class covariance operator Q, stochastic integral is understood in Stratonovich sense, A is a dissipative operator, and g is a bounded function with derivative g' that is concentrated near zero (Precise definitions are given later). A typical example of g is $g(z)=\frac{|z|}{\sqrt{1+z^2}}, z\in\mathbb{R}$. Intuition behind this example is that we are "switching off" the noise at the critical points of g and, as $\epsilon\to 0$, the limit of equation (1.1) can be informally written as

$$d\psi = A\psi\,dt + \mathrm{id}_{\{\psi_x\neq 0\}}\circ dW_t^Q, \quad x\in\mathbb{S}^1,\ t\geq 0\,, \tag{1.2}$$

since $g\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\to \mathrm{id}_{\{\psi_x\neq 0\}}$ pointwise. Our result shows that the limit $\epsilon\to 0$ of ψ^ϵ is actually space independent function. In particular, we cannot define meaningful solution of equation (1.2) in this way.

The motivation of the setup comes from micromagnetics. It is well known [1] that the theory of stochastic Landau-Lifshitz-Gilbert equation

$$d\mathbf{u} = (\mathbf{u}\times\triangle\mathbf{u} - \alpha\mathbf{u}\times(\mathbf{u}\times\triangle\mathbf{u}))\,dt + \nu\mathbf{u}\times\circ dW(t,x), \quad t\geq 0, x\in\mathbb{S}^1, \mathbf{u}\in\mathbb{S}^2 \tag{1.3}$$

$\alpha,\nu>0$ (where stochastic integral is understood in Stratonovich sense) does not cover physically important case of W being $\mathbb{R}^3$-valued, cylindrical Wiener process. It is expected that the following toy model can give insight into this difficulty:

$$d\mathbf{u} = \alpha(\triangle\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u})\,dt + \nu\mathbf{u}^\perp\circ d\eta(t,x), \alpha,\nu>0, t\geq 0, x\in\mathbb{S}^1, \mathbf{u}\in\mathbb{S}^1, \tag{1.4}$$

where $\mathbf{u}=(u^1,u^2)$ takes values in the circle instead of sphere, $\mathbf{u}^\perp=(-u^2,u^1)$, stochastic integral is understood in Stratonovich sense and $d\eta$ is 1D white in time and colored in space noise. Then putting $\mathbf{u}=e^{\imath\phi}$ and using the Itô formula we find that

$$d\phi = \alpha\triangle\phi\,dt + \nu d\eta.$$

Note that now ϕ is an Ornstein-Uhlenbeck process that is well defined even if $d\eta$ is the space-time white noise and, in this case, ϕ has enough regularity to define $e^{\imath\phi}$. Furthermore, ϕ has a unique Gaussian invariant measure, which can be transformed into the invariant measure of $\mathbf{u}$. Parameters α and ν are connected with macroscopic temperature T of the system through fluctuation-dissipation relation

$$\frac{2\alpha}{\nu^2} = \frac{1}{k_B T}$$

Now rescaling of ϕ at the extremum points can be interpreted as "cooling off" (for the function $g=\frac{|z|}{\sqrt{1+z^2}}$) the system[1] at extremal points. Our result states that such "cooling off" (or "heating up") at the extremal points leads to flattening out of the system i.e. we deduce that ψ^ϵ weakly converges to a martingale ψ independent of the space variable. That seems to be of interest because we change the system only locally while the result is global.

2. Definitions

We identify $\mathbb{S}^1$ with the interval $[0,2\pi)$. Let $H=L^2(\mathbb{S}^1,\mathbb{R})$ with scalar product $(\cdot,\cdot)$. Then the system

$$e_1=\frac{1}{\sqrt{2\pi}},\quad e_{2k+1}=\frac{1}{\sqrt{\pi}}\cos kx,\quad e_{2k}=\frac{1}{\sqrt{\pi}}\sin kx,\quad k\geq 1\,,$$

[1] for different g it could also be "heating up"

is an orthonormal basis in H. Let

$$H_l := \mathrm{lin}\{e_1, \ldots, e_l\}, \quad l \in \mathbb{N},$$

and let $\pi_l : H \to H_l$ denote the orthogonal projection onto H_l. For $n \in \mathbb{N} \cup \{0\}$ and $p \geq 1$ w denote by $W^{n,p}(\mathbb{S}^1)$ the Sobolev space of all functions $f \in L^p(\mathbb{S}^1)$ such that their weak derivatives up to order n have finite L^p norm. For $p = 2$ we will use a simplified notation $W^n(\mathbb{S}^1) = W^{n,2}(\mathbb{S}^1)$. For a Hilbert space X we denote by $W^{\alpha,p}([0,T], X), \alpha \in (0,1), p > 1$, the Sobolev space of functions $u \in L^p([0,T], X)$ such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|_X^p}{|t-s|^{1+\alpha p}} \, dt \, ds < \infty$$

endowed with the norm

$$||u||^p_{W^{\alpha,p}([0,T],X)} = \int_0^T |u(t)|_X^p \, dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|_X^p}{|t-s|^{1+\alpha p}} \, dt \, ds$$

We will assume that the H-valued Wiener process W_t^Q is defined by the series

$$W_t^Q(x) = \sum_{i=1}^{\infty} q_i \beta_t^i e_i(x) ,$$

where $\{\beta_t^i\}_{i=1}^{\infty}$ is a sequence of independent real-valued Brownian motions and

$$\sum_{l=1}^{\infty} q_l^2 < \infty, \quad \sum_{l=1}^{\infty} l^2 q_{2l}^2 < \infty , \tag{2.1}$$
$$q_{2k}^2 = q_{2k+1}^2, \quad k \in \mathbb{N} .$$

Assumption 1. *$A : W^2(\mathbb{S}^1) \to H$ is a linear operator such that for certain $\alpha > 0$ and $\beta \in \mathbb{R}$*

$$(-Af, f)_{W^1(\mathbb{S}^1)} \geq \alpha |f|^2_{W^2(\mathbb{S}^1)} + \beta |f|^2_{W^1(\mathbb{S}^1)}, \quad . \tag{2.2}$$

From now on and until Section 6 we will assume that

Assumption 2.

$$g, \, g' \in C_b(\mathbb{R}) ,$$

where $C_b(\mathbb{R})$ stands for the space of bounded continuous functions defined on $\mathbb{R}$.

Equation (1.1) can be reformulated as an Itô equation as follows:

$$\begin{cases} d\psi^\epsilon = \left(A\psi^\epsilon + \frac{1}{2\epsilon^2} M |g'|^2 \left(\frac{\psi_x^\epsilon}{\epsilon} \right) \psi_{xx}^\epsilon \right) dt + g\left(\frac{\psi_x^\epsilon}{\epsilon} \right) dW_t^Q, \\ \\ \psi^\epsilon(0) = \psi_0, \end{cases} \tag{2.3}$$

where $M = \frac{1}{2\pi} \sum_{l=1}^{\infty} q_l^2$ (derivation of the Stratonovich correction is given in appendix).

In order to define a weak solution to equation (2.3) we need some preparations. First, we define a function

$$G(x) = \int_0^x |g'|^2(y) \, dy, \quad x \in \mathbb{R}. \tag{2.4}$$

Then the following simple lemma follows from integration by parts.

Lemma 2.1. *For any $\phi, \psi \in C^2(\mathbb{S}^1)$ we have*

$$\frac{1}{\epsilon^2}\int\limits_{\mathbb{S}^1} \phi\psi_{xx}\left|g'\right|^2\left(\frac{\psi_x}{\epsilon}\right)\,dx = -\int\limits_{\mathbb{S}^1}\frac{\phi_x}{\epsilon}G\left(\frac{\psi_x}{\epsilon}\right)\,dx\,, \tag{2.5}$$

where G is defined by (2.4).

Next, we note that in view Assumption 1 we have a Gelfand triple

$$W^2\left(\mathbb{S}^1\right) \subset W^1\left(\mathbb{S}^1\right) \subset \left(W^2\left(\mathbb{S}^1\right)\right)^\star = H\,,$$

with continuous and dense imbeddings and A is coercive in $W^1\left(\mathbb{S}^1\right)$. Therefore, the operator $A^\star : W^2\left(\mathbb{S}^1\right) \to H$ is bounded. In particular $C^2\left(S^1\right)$ is dense in the domain of $A^\star$. Now, we are ready to define a weak solution to (2.3).

Definition 2.1. *Let $\epsilon > 0$ be fixed. We say that there exists a weak martingale solution of equation (2.3) if there exist a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and a progressively measurable, $W^1\left(\mathbb{S}^1\right)$-valued process ψ^ϵ, such that for every $T > 0$*

$$\psi^\epsilon(\cdot) \in C\left([0,T], H\right) \cap L^2\left(0,T; W^1\left(\mathbb{S}^1\right)\right), \quad \mathbb{P}-a.s.$$

and for every $t > 0$ and any $\phi \in C^2(\mathbb{S}^1)$

$$\begin{aligned}(\psi^\epsilon(t), \phi) = (\psi_0, \phi) &+ \int\limits_0^t (\psi^\epsilon, A^*\phi)\,ds - \frac{M}{2}\int\limits_0^t\left(\frac{\phi_x}{\epsilon}, G\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\right)\,ds \\ &+ \int\limits_0^t\left(g\left(\frac{\psi_x^\epsilon}{\epsilon}\right), \phi\right)dW_s^Q, \quad \mathbb{P}-a.s.,\end{aligned}$$

We say that ψ^ϵ is a strong martingale solution if it is a weak martingale solution, such that for every $T > 0$

$$\psi^\epsilon(\cdot) \in C\left([0,T], W^1\left(\mathbb{S}^1\right)\right) \cap L^2\left(0,T; W^2\left(\mathbb{S}^1\right)\right)$$

and for every $t > 0$

$$\begin{aligned}\psi^\epsilon(t) = \psi_0 &+ \int\limits_0^t\left(A\psi^\epsilon + \frac{M}{2\epsilon^2}\left|g'\right|^2\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\psi_{xx}^\epsilon\right)\,ds \\ &+ \int\limits_0^t g\left(\frac{\psi_x^\epsilon}{\epsilon}\right)dW_s^Q, \quad \mathbb{P}-a.s.\end{aligned} \tag{2.6}$$

We will denote

$$A^\epsilon : W^2(\mathbb{S}^1) \to H, \quad A^\epsilon f := Af + \frac{M}{2\epsilon^2}\left|g'\right|^2\left(\frac{f_x}{\epsilon}\right)f_{xx}$$

$$\sigma^\epsilon : W^1(\mathbb{S}^1) \to L^\infty(\mathbb{S}^1), \quad \sigma^\epsilon(f) := g\left(\frac{f_x}{\epsilon}\right).$$

We define the Galerkin approximation of equation (1.1) as follows

$$\begin{cases} d\psi^{m,\epsilon} = \pi_m(A^\epsilon(\pi_m\psi^{m,\epsilon}))\,dt + \pi_m\left(\sigma^\epsilon\left(\pi_m\psi^{m,\epsilon}\right)\right)\pi_m dW_s^Q, \\ \psi^{m,\epsilon}(0) = \pi_m\psi_0. \end{cases} \tag{2.7}$$

For every $m \geq 1$ equation (2.7) can be considered as an SDE in $\mathbb{R}^m$ with continuous coefficients and therefore has a local solution.

3. A Priori Estimates

In the following proposition we will deduce energy estimates uniform in ϵ and m to conclude the existence of a global solution to equation (2.7).

Proposition 3.1. *For every $\epsilon > 0$, $t > 0$ and any $m = 1, 2, \ldots$*

$$\mathbb{E}\,|\psi^{m,\epsilon}(t)|_H^2 - 2\mathbb{E}\int_0^t (A\psi^{m,\epsilon}, \psi^{m,\epsilon})_H\,ds + M\mathbb{E}\int_0^t\int_{\mathbb{S}^1} \frac{\psi_x^{m,\epsilon}}{\epsilon} G\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) dx\,ds \leq \mathbb{E}|\psi_0^{m,\epsilon}|_H^2 + M\mathbb{E}\int_0^t\int_{\mathbb{S}^1} |g|^2\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) dx\,ds. \tag{3.1}$$

Moreover, we have following estimate from below

$$\mathbb{E}|\psi^{m,\epsilon}|_H^2(t) - 2\mathbb{E}\int_0^t (A\psi^{m,\epsilon}, \psi^{m,\epsilon})_H\,ds + M\mathbb{E}\int_0^t\int_{\mathbb{S}^1} \frac{\psi_x^{m,\epsilon}}{\epsilon} G\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) dx\,ds \geq \mathbb{E}|\psi_0^{m,\epsilon}|_H^2. \tag{3.2}$$

Furthermore,

$$\mathbb{E}|\psi_x^{m,\epsilon}|_H^2(t) - 2\mathbb{E}\int_0^t ((A\psi^{m,\epsilon})_x, \psi_x^{m,\epsilon})_H\,ds \leq \mathbb{E}|\psi_{0x}^{m,\epsilon}|_H^2 + M_2\mathbb{E}\int_0^t\int_{\mathbb{S}^1} |g|^2\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) dx\,ds, \tag{3.3}$$

where $M_2 = \frac{1}{\pi}\sum_{l=1}^{\infty} l^2 q_{2l}^2$.

Proof.

- Since $\psi^{m,\epsilon}(t) \in H_m$ for every $t \geq 0$ and $m \geq 1$, we can apply the Itô formula to deduce that

(3.4)

$$|\psi^{m,\epsilon}|_H^2(t) = |\psi^{m,\epsilon}|_H^2(0) + 2\int_0^t (A\psi^{m,\epsilon}, \psi^{m,\epsilon})_H\,ds + \frac{M}{\epsilon^2}\int_0^t\int_{\mathbb{S}^1} \psi^{m,\epsilon}\psi_{xx}^{m,\epsilon}|g'|^2\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) dx\,ds$$
$$+ 2\sum_{i=1}^{\infty}\int_0^t\int_{\mathbb{S}^1} \psi^{m,\epsilon} g\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) q_i e_i(y)\,dy d\beta^i(s) + \sum_{i=1}^{\infty}\int_0^t\int_{\mathbb{S}^1} q_i^2 |\pi_m\left(g\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) e_i\right)|^2\,dy\,ds$$

Combining identity (3.4) and Lemma 2.1 we get

$$|\psi^{m,\epsilon}|_H^2(t) - 2\int_0^t (A\psi^{m,\epsilon},\psi^{m,\epsilon})_H\,ds + M\int_0^t\int_{\mathbb{S}^1} \frac{\psi_x^{m,\epsilon}}{\epsilon} G\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right)dx\,ds \tag{3.5}$$

$$= |\psi^{m,\epsilon}|_H^2(0) + 2\sum_{i=1}^{\infty} q_i \int_0^t\int_{\mathbb{S}^1} \psi^{m,\epsilon} g(\frac{\psi_x^{m,\epsilon}}{\epsilon}) e_i(y)\,dyd\beta^i(s)$$

$$+\sum_{i=1}^{\infty}\int_0^t\int_{\mathbb{S}^1} q_i^2 |\pi_m\left(g(\frac{\psi_x^{m,\epsilon}}{\epsilon})e_i\right)|^2\,dy\,ds\,.$$

We note that

$$M_{m,\epsilon}(t) := 2\sum_{i=1}^{\infty} q_i \int_0^t\int_{\mathbb{S}^1} \psi^{m,\epsilon} g\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) e_i(y)\,dyd\beta^i(s),\quad t\ge 0,$$

is a local martingale. Define stopping time

$$\tau_{m,\epsilon}(k) := \inf\{t\ge 0;\ |\psi^{m,\epsilon}(t)|_H^2 \ge k\}\,.$$

Then $N_{m,\epsilon}(t) := M_{m,\epsilon}(t\wedge\tau_{m,\epsilon}(k)), t\ge 0$ is a martingale. We will show that for every $m\ge 1$ and $\epsilon>0$

$$\lim_{k\to\infty}\tau_{m,\epsilon}(k) = \infty,\quad \mathbb{P}-a.s.$$

Putting $t := l\wedge\tau_{m,\epsilon}(k)$ in identity (3.5) and taking supremum over all $l\le r$ we obtain

$$\sup_{l\le r}|\psi^{m,\epsilon}|_H^2(l\wedge\tau_{m,\epsilon}(k)) - 2\int_0^{r\wedge\tau_{m,\epsilon}(k)} (A\psi^{m,\epsilon},\psi^{m,\epsilon})_H\,ds \tag{3.6}$$

$$+M\int_0^{r\wedge\tau_{m,\epsilon}(k)}\int_{\mathbb{S}^1}\frac{\psi_x^{m,\epsilon}}{\epsilon}G(\frac{\psi_x^{m,\epsilon}}{\epsilon})\,dx\,ds \le |\psi^{m,\epsilon}|_H^2(0) \tag{3.7}$$

$$+2\sup_{l\le r}\Big|\sum_{i=1}^{\infty} q_i\int_0^{l\wedge\tau_{m,\epsilon}(k)}\int_{\mathbb{S}^1}\psi^{m,\epsilon}g(\frac{\psi_x^{m,\epsilon}}{\epsilon})e_i(y)\,dyd\beta^i(s)\Big| \tag{3.8}$$

$$+\sum_{i=1}^{\infty} q_i^2\int_0^{r\wedge\tau_{m,\epsilon}(k)}\int_{\mathbb{S}^1}|\pi_m\left(g(\frac{\psi_x^{m,\epsilon}}{\epsilon})e_i\right)|^2\,dy\,ds$$

Consequently, taking expectation of inequality (3.6), applying the Burkholder-Davis-Gundy inequality and the Gronwall inequality we find that there exists $C>0$ such that

$$\sup_k \mathbb{E}\sup_{l\le r}|\psi^{m,\epsilon}|_H^2(l\wedge\tau_{m,\epsilon}(k)) \le C\,.$$

Therefore,

$$\mathbb{P}(\tau_{m,\epsilon}(k)\le t) = \mathbb{P}(\sup_{l\le t}|\psi^{m,\epsilon}|_H^2(l)\ge k) \le \frac{1}{k}\mathbb{E}\sup_{l\le t}|\psi^{m,\epsilon}|_H^2(l\wedge\tau_{m,\epsilon}(k)) \le \frac{C}{k}\to 0$$

and a.s. convergence $\tau_{m,\epsilon}(k)\to\infty$ for $k\to\infty$ follows (by taking subsequence over k).

Now we have from identity (3.5)

$$\mathbb{E}|\psi^{m,\epsilon}|_H^2(t\wedge\tau_{m,\epsilon}(k)) - 2\mathbb{E}\int_0^{t\wedge\tau_{m,\epsilon}(k)} (A\psi^{m,\epsilon},\psi^{m,\epsilon})_H\,ds$$

$$+ M\mathbb{E}\int_0^{t\wedge\tau_{m,\epsilon}(k)}\int_{\mathbb{S}^1}\frac{\psi_x^{m,\epsilon}}{\epsilon}G(\frac{\psi_x^{m,\epsilon}}{\epsilon})\,dx\,ds \tag{3.9}$$

$$= \mathbb{E}|\psi^{m,\epsilon}|_H^2(0) + \sum_{i=1}^{\infty} q_i^2\mathbb{E}\int_0^{t\wedge\tau_{m,\epsilon}(k)}\int_{\mathbb{S}^1}|\pi_m\left(g\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right)e_i\right)|^2\,dy\,ds.$$

Now we pass with $k\to\infty$ in (3.9) and notice that projection $|\pi_m|_{\mathcal{L}(H,H)}\le 1$:

$$\mathbb{E}|\psi^{m,\epsilon}|_H^2(t) - 2\mathbb{E}\int_0^t (A\psi^{m,\epsilon},\psi^{m,\epsilon})_H\,ds + M\mathbb{E}\int_0^t\int_{\mathbb{S}^1}\frac{\psi_x^{m,\epsilon}}{\epsilon}G\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right)\,dx\,ds \tag{3.10}$$

$$\le \mathbb{E}|\psi^{m,\epsilon}|_H^2(0) + M\mathbb{E}\int_0^t\int_{\mathbb{S}^1}|g|^2(\frac{\psi_x^{m,\epsilon}}{\epsilon})\,dy\,ds,$$

and the result follows.

- From identity (3.9) follows that

$$\mathbb{E}|\psi^{m,\epsilon}|_H^2(t\wedge\tau_{m,\epsilon}(k)) - 2\mathbb{E}\int_0^{t\wedge\tau_{m,\epsilon}(k)} (A\psi^{m,\epsilon},\psi^{m,\epsilon})_H\,ds + M\mathbb{E}\int_0^{t\wedge\tau_{m,\epsilon}(k)}\int_{\mathbb{S}^1}\frac{\psi_x^{m,\epsilon}}{\epsilon}G\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right)\,dx\,ds \tag{3.11}$$

$$\ge \mathbb{E}|\psi^{m,\epsilon}|_H^2(0).$$

Passing with $k\to\infty$ we obtain estimate (3.2).

- We apply the Itô formula to deduce that

$$|\psi_x^{m,\epsilon}|_H^2(t) - 2\int_0^t ((A\psi^{m,\epsilon})_x,\psi_x^{m,\epsilon})_H\,ds + \frac{M}{\epsilon^2}\int_0^t\int_{\mathbb{S}^1}|g'|^2(\frac{\psi_x^{m,\epsilon}}{\epsilon})|\psi_{xx}^{m,\epsilon}|^2\,dx\,ds \tag{3.12}$$

$$+ 2\sum_{i=1}^{\infty} q_i\int_0^t\int_{\mathbb{S}^1}\psi_{xx}^{m,\epsilon}g(\frac{\psi_x^{m,\epsilon}}{\epsilon})e_i\,dx\,d\beta^i(s)$$

$$= |\psi_x^{m,\epsilon}|_H^2(0) + \sum_{i=1}^{\infty} q_i^2\int_0^t\int_{\mathbb{S}^1}\left|\pi_m\left[\frac{\psi_{xx}^{m,\epsilon}}{\epsilon}g'(\frac{\psi_x^{m,\epsilon}}{\epsilon})e_i + g(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x\right]\right|^2\,dx\,ds$$

The last term in (3.12) can be rewritten as follows

$$\sum_{i=1}^{\infty} q_i^2\int_0^t\int_{\mathbb{S}^1}\left|\pi_m\left[\frac{\psi_{xx}^{m,\epsilon}}{\epsilon}g'(\frac{\psi_x^{m,\epsilon}}{\epsilon})e_i\right]\right|^2\,dx\,ds + \sum_{i=1}^{\infty} q_i^2\int_0^t\int_{\mathbb{S}^1}\left|\pi_m\left[g(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x\right]\right|^2\,dx\,ds \tag{3.13}$$

$$+ 2\sum_{i=1}^{\infty} q_i^2\int_0^t\int_{\mathbb{S}^1}\pi_m\left[\frac{\psi_{xx}^{m,\epsilon}}{\epsilon}g'(\frac{\psi_x^{m,\epsilon}}{\epsilon})e_i\right]\pi_m\left[g(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x\right]\,dx\,ds$$

$$= \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \left|\pi_m \left[\frac{\psi_{xx}^{m,\epsilon}}{\epsilon} g'(\frac{\psi_x^{m,\epsilon}}{\epsilon}) e_i\right]\right|^2 dx\, ds + \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \left|\pi_m \left[g(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x\right]\right|^2 dx\, ds$$
$$- 2\sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} (\mathrm{id} - \pi_m)\left[\frac{\psi_{xx}^{m,\epsilon}}{\epsilon} g'(\frac{\psi_x^{m,\epsilon}}{\epsilon}) e_i\right] (\mathrm{id} - \pi_m)\left[g(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x\right] dx\, ds$$

$$\leq \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \left|\pi_m \left[\frac{\psi_{xx}^{m,\epsilon}}{\epsilon} g'(\frac{\psi_x^{m,\epsilon}}{\epsilon}) e_i\right]\right|^2 dx\, ds + \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \left|\pi_m \left[g(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x\right]\right|^2 dx\, ds$$
$$+ \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \left|((\mathrm{id} - \pi_m)\left[\frac{\psi_{xx}^{m,\epsilon}}{\epsilon} g'(\frac{\psi_x^{m,\epsilon}}{\epsilon}) e_i\right]\right|^2 dx\, ds + \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \left|(\mathrm{id} - \pi_m)\left[g(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x\right]\right|^2 dx\, ds$$
$$= \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \frac{(\psi_{xx}^{m,\epsilon})^2}{\epsilon^2} |g'|^2(\frac{\psi_x^{m,\epsilon}}{\epsilon}) e_i^2\, dx\, ds + \sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} |g|^2(\frac{\psi_x^{m,\epsilon}}{\epsilon})(e_i)_x^2\, dx\, ds$$

where the first equality follows from the fact that

$$2\sum_{i=1}^{\infty} q_i^2 \int_0^t \int_{\mathbb{S}^1} \psi_{xx}^{m,\epsilon} g'\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) e_i g\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right)(e_i)_x\, dx\, ds = 0,$$

because

$$\sum_{i=1}^{\infty} q_i^2 e_i (e_i)_x = \frac{1}{2}\left(\sum_{i=1}^{\infty} q_i^2 |e_i|^2\right)_x = 0\,,$$

and thge second inequality is a consequence of the Cauchy-Schwartz inequality.

Combining formula (3.12) with inequality (3.13) we can deduce that

$$|\psi_x^{m,\epsilon}|_H^2(t) - 2\int_0^t ((A\psi^{m,\epsilon})_x, \psi_x^{m\epsilon})_H\, ds$$
$$+ 2\sum_{i=1}^{\infty} q^i \int_0^t \int_{\mathbb{S}^1} \psi_{xx}^{m,\epsilon} g\left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) e_i\, dx\, d\beta^i(s)$$
$$\leq |\psi_x^{m,\epsilon}|_H^2(0) + M_2 \int_0^t \int_{\mathbb{S}^1} |g|^2 \left(\frac{\psi_x^{m,\epsilon}}{\epsilon}\right) dx\, ds, \tag{3.14}$$

where $M_2 = \frac{1}{\pi}\sum\limits_{l=1}^{\infty} l^2 q_{2l}^2$. Conclusion of the proof follows in the same fashion as in part 1 (i.e. considering appropriate stopping time to stop local martingale in formula (3.14), taking expectation and the limit).

□

Corollary 3.1. *Assume that there exists constant $C > 0$ such that $g \in L^\infty(\mathbb{R})$, $0 \neq g' \in L^2 \cap L^\infty(\mathbb{R})$ satisfies*

$$\left(\int_{-\infty}^{-z} + \int_z^{\infty}\right) |g'|^2(y)\, dy \leq \frac{C}{z}, \quad z > 0, \tag{3.15}$$

and

$$\kappa = \min\left\{ \int_0^\infty |g'|^2(y)\,dy, \int_{-\infty}^0 |g'|^2(y)\,dy \right\} > 0\,. \tag{3.16}$$

Then there exists $C(t,\alpha,\beta,|g|_{L^\infty},\psi_0) > 0$ *independent of* m *and* ϵ *such that*

$$\int_0^t \mathbb{E}|\psi_x^{m,\epsilon}|_{L^1}\,ds \le C\frac{\epsilon}{\kappa}, \tag{3.17}$$

Proof. By boundedness of g, dissipativity of A (2.2) and a priori estimates (3.1), (3.3) we have that

$$\mathbb{E}\int_0^t \int_{\mathbb{S}^1} \frac{\psi_x^{m,\epsilon}}{\epsilon} G(\frac{\psi_x^{m,\epsilon}}{\epsilon})\,dx\,ds \le C(t,\alpha,\beta,|g|_{L^\infty},\psi_0). \tag{3.18}$$

Hence we have that

$$\mathbb{E}\int_0^t \int_{\{\psi_x^{m,\epsilon}\ge 0\}} \frac{\psi_x^{m,\epsilon}}{\epsilon} G(\frac{\psi_x^{m,\epsilon}}{\epsilon})\,dx\,ds + \mathbb{E}\int_0^t \int_{\{\psi_x^{m,\epsilon}< 0\}} \frac{\psi_x^{m,\epsilon}}{\epsilon} G(\frac{\psi_x^{m,\epsilon}}{\epsilon})\,dx\,ds \le C(t,\alpha,\beta,|g|_{L^\infty},\psi_0). \tag{3.19}$$

Consequently, condition (3.15) together with the estimate (3.19) gives us that decomposing $\int\limits_0^\infty = \int\limits_0^{\psi_x^{m,\epsilon}/\epsilon} + \int\limits_{\psi_x^{m,\epsilon}/\epsilon}^\infty$:

$$\begin{aligned}\mathbb{E}\int_0^t \int_{\{\psi_x^{m,\epsilon}\ge 0\}} \frac{\psi_x^{m,\epsilon}}{\epsilon} \int_0^\infty |g'|^2(y)\,dy\,dx\,ds \\ + \mathbb{E}\int_0^t \int_{\{\psi_x^{m,\epsilon}< 0\}} -\frac{\psi_x^{m,\epsilon}}{\epsilon} \int_{-\infty}^0 |g'|^2(y)\,dy\,dx\,ds \le C(t,\alpha,\beta,|g|_{L^\infty},\psi_0),\end{aligned}$$

and the result follows. □

The a priori estimates of Proposition 3.1 are uniform w.r.t. both parameter ϵ and dimension m of the approximation space H_m. The next a priori estimate will give us bound on fractional time derivative of the solution. The estimate is not uniform w.r.t. ϵ.

Lemma 3.1. *For any* $\epsilon > 0$, $T > 0$, $\alpha \in \left(0,\frac{1}{2}\right)$ *there exists* $C(\epsilon,T,\alpha)$ *such that*

$$\mathbb{E}|\psi^{m,\epsilon}|^2_{W^{\alpha,2}([0,T],H)} \le C(\epsilon,T,\alpha). \tag{3.20}$$

Proof. By definition of $\psi^{m,\epsilon}$ given in (2.7) $\psi^{m,\epsilon}$ has the representation

$$\psi^{m,\epsilon}(t) = \psi^{m,\epsilon}(0) + \int_0^t \pi_m(A^\epsilon(\pi_m\psi^{m,\epsilon}))\,ds + \int_0^t \pi_m(\sigma^\epsilon(\pi_m\psi^{m,\epsilon})\pi_m dW_s^Q).$$

Now for any fixed $\epsilon > 0$ the drift term is bounded in $L^2(\Omega, W^{1,2}([0,T],H))$ by a priori estimate (3.3). Furthermore, diffusion term is bounded in $L^2(\Omega, W^{\alpha,2}([0,T],H))$ for any $\alpha \in \left(0,\frac{1}{2}\right)$ by Lemma 2.1, of [4]. □

Now we are ready to take m to infinity in Galerkin approximation (2.7) and show the existence of strong solution of equation (1.1) for any $\epsilon > 0$.

4. Main result

Proposition 4.1. *Assume that* $g \in C_b(\mathbb{R})$, $g' \in L^2(\mathbb{R}) \cap C_b(\mathbb{R})$ *satisfies conditions* (3.15) *and* (3.16). *If* $\psi_0 \in W^{1,2}(\mathbb{S}^1)$ *then there exists a strong martingale solution* ψ^ϵ *of the system* (2.3) *and a constant* $C(t, \alpha, \beta, M, |g|_{L^\infty}, \psi_0) > 0$ *such that*

$$\int_0^t \mathbb{E}|\psi_x^\epsilon|_{L^1}\, ds \le C\frac{\epsilon}{\kappa}, \tag{4.1}$$

where κ *is defined by* (3.16). *In particular,*

$$\limsup_{\epsilon \to 0} \int_0^t \mathbb{E}|\psi_x^\epsilon|_{L^1}\, ds = 0.$$

Theorem 4.1. *Assume that conditions of Proposition 4.1 are satisfied and* $A^*(1) = 0$ *i.e.*

$$\int_{\mathbb{S}^1} A\phi\, dx = 0, \ \forall \phi \in C^\infty(\mathbb{S}^1). \tag{4.2}$$

Then there exists a martingale $\psi \in L^2(\Omega, C([0,T], \mathbb{R}))$ *and a sequence* $\{\epsilon_l\}_{l=1}^\infty, \epsilon_l \searrow 0$, *such that for any* $\phi \in C^\infty([0,T] \times \mathbb{S}^1)$ *we have*

$$\int_0^t \int_{\mathbb{S}^1} \psi^{\epsilon_l}(s, x, \cdot)\phi(s, x)\, dx\, ds \xrightarrow{l \to \infty} \int_0^t \psi(s, \cdot) \int_{\mathbb{S}^1} \phi(s, x)\, dx\, ds \quad \text{in law.}$$

Furthermore,

$$\mathbb{E}\psi(t) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \psi_0(x)\, dx.$$

Remark 4.1. It remains an open problem to find the quadratic variation of ψ.

Remark 4.2. Assumption (4.2) in Theorem 4.1 is made for simplicity. Otherwise, we would obtain, for $\epsilon \to 0$, a martingale with additional drift term. The structure of the drift would depend on the exact form of the operator A.

5. Proofs of Proposition 4.1 and Theorem 4.1

Proof of Proposition 4.1. Let $\{\psi^{m,\epsilon}\}_{m\in\mathbb{N}, \epsilon>0}$ be the Galerkin approximation introduced in (2.7). According to Proposition 3.1 and Lemma 3.1 we have the following a priori estimate

$$\sup_{m\in\mathbb{N}} \left[\mathbb{E}|\psi^{m,\epsilon}|^2_{W^{\alpha,2}([0,T],H)} + \mathbb{E}|\psi^{m,\epsilon}|^2_{L^2([0,T],W^{2,2}(\mathbb{S}^1))\cap C([0,T],W^{1,2}(\mathbb{S}^1))}\right] < \infty, \quad \alpha \in \left(0, \frac{1}{2}\right).$$

If $0 < \alpha < \frac{1}{2}$ then by Theorem 2.1 in [4]

$$L^2([0,T], W^{2,2}(\mathbb{S}^1)) \cap W^{\alpha,2}([0,T], H) \subset L^2([0,T], W^{1,2}(\mathbb{S}^1)),$$

with compact imbedding, hence the family of probability laws $\mathcal{L}(\psi^{m,\epsilon})$ is tight in $L^2([0,T], W^{1,2}(\mathbb{S}^1))$. Hence, there exists a subsequence $\psi^{m,\epsilon}$ (still denoted as $\psi^{m,\epsilon}$), such that $\mathcal{L}(\psi^{m,\epsilon})$ weakly converges in $L^2([0,T], W^{1,2}(\mathbb{S}^1))$ (for fixed $\epsilon > 0$). By the Skorokhod embedding theorem (cf. [7], p.9) there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ and random variables $\widetilde{\psi}^\epsilon$, $\widetilde{\psi}^{m,\epsilon}$, $m \in \mathbb{N}$, defined on it and taking values in $L^2([0,T], W^{1,2}(\mathbb{S}^1))$, such that

$$\widetilde{\psi}^{m,\epsilon} \to \widetilde{\psi}^\epsilon \quad \text{in} \quad L^2([0,T], W^{1,2}(\mathbb{S}^1)) \quad \mathbb{P} - a.s.$$

and the probability laws of $\widetilde{\psi}^{m,\epsilon}$ and $\psi^{m,\epsilon}$ on $L^2([0,T], W^{1,2}(\mathbb{S}^1))$ are the same. Therefore, $\widetilde{\psi}^{m,\epsilon}$ satisfy the same a priori estimate as $\psi^{m,\epsilon}$. Consequently,

$$\widetilde{\psi}^\epsilon \in L^2([0,T], W^{2,2}(\mathbb{S}^1)) \cap C([0,T], W^{1,2}(\mathbb{S}^1)), \quad \mathbb{P}\text{-a.s.}, \tag{5.1}$$

and $\widetilde{\psi}^{m,\epsilon} \to \widetilde{\psi}^{\epsilon}$ in $L^2(\Omega\times[0,T],W^{2,2}(\mathbb{S}^1))$ weakly. Define

$$M^{m,\epsilon}(t) := \widetilde{\psi}^{m,\epsilon}(t) - \pi_m\widetilde{\psi}^{m,\epsilon}(0) - \int\limits_0^t \pi_m(A^{\epsilon}(\pi_m\widetilde{\psi}^{m,\epsilon}))\,dt, t\geq 0.$$

Then $\{M^{m,\epsilon}\}_{t\geq 0}$ is a a square integrable martingale with respect to the filtration $(\mathcal{G}^{m,\epsilon})_t = \sigma(\{\widetilde{\psi}^{m,\epsilon}(s), s\leq t\})$ with quadratic variation

$$\langle M^{m,\epsilon}\rangle_t = \sum_{i=1}^{\infty} q_i^2 \int\limits_0^t |\pi_m(\sigma(\widetilde{\psi}^{m,\epsilon})e_i)|^2\,ds.$$

Indeed, since the laws $\mathcal{L}(\widetilde{\psi}^{m,\epsilon})$ and $\mathcal{L}(\psi^{m,\epsilon})$ are the same we have that for all $0\leq s\leq t$, $\lambda\in C_b(L^2([0,T),W^{1,2}(\mathbb{S}^1)))$ and $\phi,\gamma\in C^{\infty}(\mathbb{S}^1)$

$$\mathbb{E}[(M^{m,\epsilon}(t)-M^{m,\epsilon}(s),\phi)\lambda(\widetilde{\psi}^{m,\epsilon}|_{[0,s]})] = 0, \tag{5.2}$$

and

$$\begin{aligned}\mathbb{E}\lambda(\widetilde{\psi}^{m,\epsilon}|_{[0,s]})[(M^{m,\epsilon}(t),\phi)(M^{m,\epsilon}(t),\gamma) - (M^{m,\epsilon}(s),\phi)(M^{m,\epsilon}(s),\gamma)\\ - \sum_{i=1}^{\infty} q_i^2 \int\limits_s^t (\pi_m(\sigma(\widetilde{\psi}^{m,\epsilon})e_i)\phi, \pi_m(\sigma(\widetilde{\psi}^{m,\epsilon})e_i)\gamma)\,ds] = 0.\end{aligned} \tag{5.3}$$

It remains to take the limit $m\to\infty$ in equalities (5.2) and (5.3). By a priori estimates (3.3),(3.1), all the terms in equalities (5.2) and (5.3) are uniformly integrable w.r.t. ω. Thus it is enough to show convergence $\mathbb{P}$-a.s. Note that for any test function $\phi\in C^{\infty}(\mathbb{S}^1)$ the drift term $\left(\int\limits_0^t \pi_m(A^{\epsilon}(\widetilde{\psi}^{m,\epsilon}))\,ds,\phi\right)_H$ can be rewritten as follows

$$\left(\int\limits_0^t \pi_m(A^{\epsilon}(\widetilde{\psi}^{m,\epsilon}))\,ds,\phi\right) = -\frac{M}{2}\int\limits_0^t \left(\frac{\pi_m\phi_x}{\epsilon}, G\left(\frac{\widetilde{\psi}^{m,\epsilon}_x}{\epsilon}\right)\right)ds + \int\limits_0^t \left(\widetilde{\psi}^{m,\epsilon}, A^*\phi\right)ds, \tag{5.4}$$

where G is given by (2.4). Indeed, representation (5.4) follows from integration by parts. Consequently, the convergence of the first RHS term in (5.4) follows from the global Lipshitz property of the function G. Similarly, we can show the convergence of quadratic variation. Now, the existence of weak solution follows from the Representation Theorem for martingales (Theorem 8.2, p. 220 [2]). The weak solution is a strong one by the regularity property (5.1) and integration by parts formula. The identity (4.1) follows from identity (3.17). □

Proof of Theorem 4.1. We can represent ψ^{ϵ} as follows

$$\psi^{\epsilon} = \left(\psi^{\epsilon} - \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\psi^{\epsilon}\,dx\right) + \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\psi^{\epsilon}\,dx.$$

Let

$$\chi(x) := \int\limits_0^x \phi(y)\,dy - \frac{x}{2\pi}\int\limits_0^{2\pi}\phi(y)\,dy, \quad x\in[0,2\pi).$$

Note that $\chi_x = \phi - \frac{1}{2\pi}\int\limits_0^{2\pi} \phi(y)\,dy$. Consequently, we have by integration by parts that

$$\Big|\int\limits_0^T\int\limits_{\mathbb{S}^1} (\psi^\epsilon - \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\psi^\epsilon\,dx)\phi\,dx\,ds\Big|$$
$$= \left|\int\limits_0^T\int\limits_{\mathbb{S}^1}\left(\psi^\epsilon - \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\psi^\epsilon\,dx\right)\chi_x\,dx\,ds\right|$$
$$= \left|\int\limits_0^T\int\limits_{\mathbb{S}^1}\psi^\epsilon_x\chi\,dx\,ds\right| \le ||\chi||_{L^\infty([0,T]\times\mathbb{S}^1)}\int\limits_0^T\int\limits_{\mathbb{S}^1}|\psi^\epsilon_x|\,dx\,ds$$

which converges to 0 by Proposition 4.1. Hence it remains to find the limit of

$$M^\epsilon(t) := \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\psi^\epsilon\,dx, \quad t \ge 0\,,$$

for ϵ converging to zero. First, let us note that we have the following representation of M^ϵ:

$$M^\epsilon(t) = \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\psi_0(x)\,dx + \frac{1}{2\pi}\int\limits_0^t\int\limits_{\mathbb{S}^1} g\left(\frac{\psi^\epsilon_x}{\epsilon}\right)\,dx\,dW^Q_s, \quad t \ge 0, \tag{5.5}$$

where we have used Assumption (4.2) to cancel the drift part. Thus we find that M_ϵ is a square integrable martingale and by the Burkholder-Davis-Gundy inequality

$$\sup_{\epsilon>0}\mathbb{E}\sup_{t\in[0,T]}|M^\epsilon(t)|^p < \infty, \quad p \ge 1.$$

Furthermore, we can deduce from representation (5.5) that

$$\sup_{\epsilon>0}\mathbb{E}|M^\epsilon|^p_{W^{\alpha,p}([0,T],\mathbb{R})} < \infty, \quad \alpha \in \left(0,\frac{1}{2}\right),\ p > 1.$$

Hence, by the compact embedding theorem, the family of martingales $\{M^\epsilon;\, \epsilon \in (0,1)\}$ is tight in $C([0,T],\mathbb{R})$. Consequently, by the Prokhorov Theorem there exists a sequence $\epsilon_l \searrow 0$ such that M^{ϵ_l} converges in law to the process ψ in $C([0,T],\mathbb{R})$. In particular,

$$\int\limits_0^T M^{\epsilon_l}(s)\int\limits_{\mathbb{S}^1}\phi(s,x)\,dx\,ds \xrightarrow{l\to\infty} \int\limits_0^T \psi(s)\int\limits_{\mathbb{S}^1}\phi(s,x)\,dx\,ds, \quad \text{in law.}$$

The process ψ is a square integrable martingale (see, for instance Proposition 1.12, Chapter 9 of [6]) with expectation

$$\mathbb{E}\psi(t) = \frac{1}{2\pi}\int\limits_{\mathbb{S}^1}\psi_0(x)\,dx\,.$$

□

6. Example and counterexample

Here we will present example when the regularisation effect holds although g does not satisfy conditions of the Proposition 4.1. Furthermore, we will give a counterexample showing that in the linear case we don't have this effect.

Example 6.1. Assume that g is linear: $g(z) = z$. In this case Assumption 2 is not satisfied, and homogenisation doesn't hold as following elementary example shows. Let $A = \partial^2_{xx}$ with periodic boundary conditions and assume that noise $W^Q = \beta$ is a one dimensional Wiener process. Then system (2.3) has a unique solution of the form

$$\psi^\epsilon(t,x) = \psi_0\left(x + \frac{\beta(t)}{\epsilon}\right), \quad t \geq 0,\ x \in \mathbb{S}^1 .$$

Consequently, the integral $\int_0^t \mathbb{E}|\psi_x^\epsilon|_{L^1}\, ds$ does not depend on ϵ.

Example 6.2. Assume that $g(z) = \sin z$.

Proposition 6.1. *Assume that $g(z) = \sin z$ and A satisfies Assumption* (2.2)*. Then there exist a strong martingale solution ψ^ϵ of the system* (2.3) *and a constant $C = C(t, \alpha, \beta, M, |g|_{L^\infty}, \psi_0) > 0$ such that*

$$\int_0^t \mathbb{E}|\psi_x^\epsilon|_{L^2}^2\, ds \leq C\epsilon^2. \tag{6.1}$$

In particular, we have that

$$\limsup_{\epsilon\to 0} \int_0^t \mathbb{E}|\psi_x^\epsilon|_{L^2}^2\, ds = 0.$$

Proof. A priori estimate (6.1) follows directly from the Itô formula. In this case it is not necessary to use decomposition of integration interval as in Proposition 4.1 and, consequently, we don't need conditions $g' \in L^2(\mathbb{R})$,(3.15). In the same time, function G is globally Lipshitz and the proof of convergence is exactly the same as in Proposition 4.1. $\square$

7. Appendix

In the appendix we formally calculate the Stratonovich correction term for equation (1.1). From (1.1) we have

$$\begin{aligned}
\frac{1}{2}\left\langle g\left(\frac{\psi_x^\epsilon}{\epsilon}\right), W^Q\right\rangle_t &= \frac{1}{2}\left\langle \int_0^{\cdot} \frac{1}{\epsilon} g'\left(\frac{\psi_x^\epsilon}{\epsilon}\right) d\psi_x^\epsilon, W^Q\right\rangle_t \\
&= \frac{1}{2}\left\langle \int_0^{\cdot} \frac{1}{\epsilon} g'\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\left(g'\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\frac{\psi_{xx}^\epsilon}{\epsilon} dW^Q + g\left(\frac{\psi_x^\epsilon}{\epsilon}\right) dW_x^Q\right), W^Q\right\rangle_t \\
&= \int_0^t \frac{1}{2\epsilon^2}\left|g'\right|^2\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\psi_{xx}^\epsilon \rho^Q(x)\, ds + \int_0^t \frac{1}{4\epsilon} g' g\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\rho_x^Q(x)\, ds,
\end{aligned} \tag{7.1}$$

where

$$\rho^Q = \sum_{n=1}^{\infty} q_k^2 e_k^2.$$

Note that we can rewrite ρ^Q as follows

$$\rho^Q(x) = \frac{q_1^2}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(q_{2n+1}^2\cos^2 nx + q_{2n}^2 \sin^2 nx\right) = \frac{1}{2\pi}\sum_{n=1}^{\infty} q_n^2 + \sum_{n=1}^{\infty}(q_{2n+1}^2 - q_{2n}^2)\cos 2nx .$$

Consequently, condition (2.1) implies that

$$M := \rho^Q = \frac{1}{2\pi}\sum_{n=1}^{\infty} q_n^2, \quad \rho_x^Q = 0. \tag{7.2}$$

Combining (7.1) and (7.2) we find that

$$\frac{1}{2}\left\langle g\left(\frac{\psi_x^\epsilon}{\epsilon}\right), W^Q\right\rangle_t = \frac{M}{2\epsilon^2}\int_0^t \left|g'\right|^2\left(\frac{\psi_x^\epsilon}{\epsilon}\right)\psi_{xx}^\epsilon\, ds\,. \tag{7.3}$$

Acknowledgment: We thank Z. Brzeźniak and M. Röckner for helpful discussions. This work was supported by the ARC Discovery grant DP160101755 and financed in part by the CAPES (Brasil)– Finance code 001.

References

[1] Berkov, D., *Magnetization dynamics including thermal fluctuations.* In: Kronmüller, H., Parkin, S. (eds.) Handbook of Magnetism and Advanced Magnetic Materials, vol. 2. Wiley, Chichester, 2007.

[2] Da Prato, G., Zabczyk, J. Stochastic equations in infinite dimensions. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.

[3] Flandoli, F. Random perturbation of PDEs and fluid dynamic models. Lecture Notes in Mathematics, Vol. 2015. Springer, Heidelberg, 2011.

[4] Flandoli, F., Gatarek, D., *Martingale and stationary solutions for stochastic Navier-Stokes equations.* Probab. Theory Related Fields **102**, no. 3, 367-391 (1995).

[5] Flandoli, F., Gubinelli, M., Priola, E., *Well-posedness of the transport equation by stochastic perturbation.* Invent. Math. **180**, no. 1, 1–53 (2010).

[6] Jacod, J., Shiryaev, A. N. Limit theorems for stochastic processes. Second edition. Grundlehren der Mathematischen Wissenschaften, 288. Springer-Verlag, Berlin, 2003.

[7] Ikeda, N., Watanabe, S., Stochastic Differential Equations and Diffusion Processes. North-Holland, Amsterdam, 1981.

[8] Gassiat, P., Gess, B., *Regularization by noise for stochastic Hamilton-Jacobi equations.* arXiv:1609.07074

[9] Gess, B., Souganidis, Panagiotis E., *Long-time behavior, invariant measures, and regularizing effects for stochastic scalar conservation laws.* Comm. Pure Appl. Math. **70**, no. 8, 1562–1597 (2017).

[10] Gess, B., Souganidis, Panagiotis E., *Stochastic non-isotropic degenerate parabolic-hyperbolic equations.* Stochastic Process. Appl. **127**, no. 9, 2961–3004 (2017).