Generalised trisections in all dimensions

J. Hyam Rubinstein[a] **and Stephan Tillmann**[b]

[a]School of Mathematics and Statistics, The University of Melbourne, VIC, 3010, Australia; [b]School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia

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This paper describes a generalisation of Heegaard splittings of 3–manifolds and trisections of 4–manifolds to all dimensions, using triangulations as a key tool. In particular, every closed piecewise linear n–manifold can be divided into $k+1$ n–dimensional 1–handlebodies, where $n = 2k+1$ or $n = 2k$, such that intersections of the handlebodies have spines of small dimensions. Several applications, constructions and generalisations of our approach are given.

manifold | multisection | triangulation | colouring | CAT(0) cubing

David Gay and Rob Kirby (1) introduced a beautiful decomposition of an arbitrary smooth, oriented closed 4–manifold, called *trisection*, into three handlebodies glued along their boundaries as follows. Each handlebody is a boundary connected sum of copies of $S^1 \times B^3$, and has boundary a connected sum of copies of $S^1 \times S^2$. The triple intersection of the handlebodies is a closed orientable surface Σ, which divides the boundary of each handlebody into two 3–dimensional handlebodies (and hence is a Heegaard surface). These 3–dimensional handlebodies are precisely the intersections of pairs of the 4–dimensional handlebodies.

In dimensions ≤ 4, there is a bijective correspondence between isotopy classes of smooth and piecewise linear structures (2, 3), but this breaks down in higher dimensions. This paper generalises Gay and Kirby's concept of a trisection to higher dimensions in the piecewise linear category. All manifolds, maps and triangulations are therefore assumed to be piecewise linear unless stated otherwise. Our definition and results apply to any compact smooth manifold by passing to its unique piecewise linear structure (4).

1. Multisections

The definition of a multisection, which generalises both that of a Heegaard splitting of a 3–manifold and that of a trisection of a 4–manifold, focuses on properties of spines. Let N be a compact manifold with non-empty boundary. The subpolyhedron P is a *spine* of N if $P \subset \text{int}(N)$ and N PL collapses onto P.

Definition 1 (Multisection of closed manifold)**.** Let M be a closed, connected, piecewise linear n–manifold. A *multisection* of M is a collection of $k+1$ piecewise linear submanifolds $H_i \subset M$, where $0 \leq i \leq k$ and $n = 2k$ or $n = 2k+1$, subject to the following four conditions:

1. Each H_i has a single 0–handle and a finite number, g_i, of 1–handles, and is PL homeomorphic to a standard piecewise linear n–dimensional 1–handlebody of genus g_i.
2. The handlebodies H_i have pairwise disjoint interior, and $M = \bigcup_i H_i$.
3. The intersection $H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r}$ of any proper subcollection of the handlebodies is a compact submanifold with boundary and of dimension $n-r+1$. Moreover, it has a spine of dimension r, except if $n = 2k$ and $r = k$, then there is a spine of dimension $r-1$.
4. The intersection $H_0 \cap H_1 \cap \ldots \cap H_k$ of all handlebodies is a closed submanifold of M^n of dimension $n-k$, and called the *central submanifold*.

It follows from our definitions that the first condition in the above definition is equivalent to each H_i having spine a graph with Euler characteristic $1 - g_i$. Moreover, each intersection $H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r}$ is connected, where $1 \leq r \leq k+1$.

Nomenclature. A multisection of a 1–manifold is just the 1–manifold. The study of multisections in dimension 2 is the study of separating, simple, closed curves. A multisection of a 3–manifold is a *Heegaard splitting*. A trisection in the sense of Gay and Kirby (1) is a multisection of an orientable 4–manifold with the additional property that the handlebodies H_j have the same genus. It is shown in (5) that a multisection of an orientable 4–manifold can be modified to a trisection in the sense of (1), by stabilising the handlebodies of lower genus to achieve the same genus as the handlebody of highest genus. We will therefore use the term trisection to apply to all multisections in dimension four — if necessary, we will say that they are *balanced* if all handlebodies have the same genus and *unbalanced* otherwise. This allows us to talk about *bisections* ($n = 2, 3$), *trisections* ($n = 4, 5$), *quadrisections* ($n = 6, 7$), etc. without further qualification.

Existence. We recall the classical existence proof of Heegaard splittings (see, for instance, (6)), which motivates our definition in higher dimensions, and provides a model for the existence proofs. Suppose that M is a triangulated, closed, connected 3–manifold, and there is a partition $\{P_0, P_1\}$ of the set of all vertices in the triangulation, such that

Significance Statement

Decomposing a manifold into handles was introduced by Smale, from the study of the critical points of smooth real valued functions. Here we study combinatorial functions from a manifold to a simplex and use them to decompose the manifold into simple building blocks. Given a description of a manifold as the quotient space of a union of n–dimensional simplices, this note constructs multisections, which describe an n–dimensional manifold as a union of $k+1$ n–dimensional handlebodies, where $n = 2k$ or $2k+1$. These handlebodies have disjoint interiors and subcollections intersect in submanifolds with spines of small dimension. The intersection of all the handlebodies is the central submanifold Σ. This submanifold Σ can be chosen to have a special structure called a CAT(0) cubing.

(1_3) for each set P_k, every tetrahedron has a pair of vertices in the set; and

(2_3) the union of all edges with both ends in P_k is a connected graph Γ_k in M.

We can form regular neighbourhoods of each of these graphs Γ_k, which are handlebodies H_0, H_1 respectively, such that the handlebodies meet along their common boundary Σ, which is a surface consisting entirely of quadrilateral disks, one in each tetrahedron, separating the vertices in P_0, P_1 (see Figure 1). Hence Σ is a Heegaard surface in M. A triangulation with the desired properties is obtained as follows. Suppose $|K| \to M$ is a triangulation of M, and take the first barycentric subdivision K' of K. Let P_0 be the set of all vertices of K and barycentres of edges of K; and let P_1 be the set of all barycentres of the triangles and the tetrahedra of K. Then $\{P_0, P_1\}$ is a partition of the vertices of K' satisfying (1_3) and (2_3). Moreover, the vertices of the cubulated surface Σ have degrees 4 or 6, and hence Σ is a non-positively curved cube complex (cf. (7)).

Examples of triangulations of manifolds that satisfy (1_3) and (2_3), but are not barycentric subdivisions, are the standard 2–vertex triangulations of lens spaces (see (8)). See §4 for a strategy to identify triangulations dual to multisections.

The partition $\{P_0, P_1\}$ defines a piecewise linear map $\phi\colon M \to [0,1]$ by $\phi(P_0) = 0$ and $\phi(P_1) = 1$. This is often called a *height function* and we refer to it as a *partition map*. The pre-image $\phi^{-1}(\frac{1}{2})$ is a Heegaard surface Σ for M as described above. The inverse image of any point in the interior of $[0,1]$ is a surface isotopic to Σ. The intersection of this inverse image with any tetrahedron of $\mathcal{T}$ is a quadrilateral disk (2–cube). The inverse image of either endpoint 0 or 1 is a graph and its intersection with any tetrahedron is an edge (1-cube). The division of the closed interval (1–simplex) into two half intervals is the dual decomposition into 1–cubes. An analogous decomposition is exactly what we will use in arbitrary odd dimensions.

In even dimensions, one encounters the problem that a simplex has an odd number of vertices. In this case, one needs to add an additional modification. This is given in detail in the article (9) in this collection. Again, this approach generalises to all even dimensions.

An outline of the general existence result can thus be given as follows. The complete details can be found in (5).

Theorem 2. *Every closed piecewise linear manifold has a multisection.*

Sketch of proof. Suppose M is a closed, connected, piecewise linear manifold of dimension n. Our strategy is to construct a piecewise linear map $\phi\colon M \to \sigma$, where σ is a k–simplex for k satisfying $n = 2k$ or $n = 2k+1$, and to obtain the multisection as the pull back of the dual cubical structure of σ to M. Our map ϕ will have the property that each vertex of σ pulls back to a connected graph, and each top-dimensional cube pulls back to a regular neighbourhood of this graph, a 1–handlebody.

We use triangulations to define ϕ. Since a piecewise linear manifold admits a piecewise linear triangulation $|K| \to M$ (where the link of each simplex in the simplicial complex K is equivalent to a standard piecewise linear sphere) we can and will assume that such a triangulation of M is fixed. Since M is closed, there is a finite number of simplices in the triangulation,

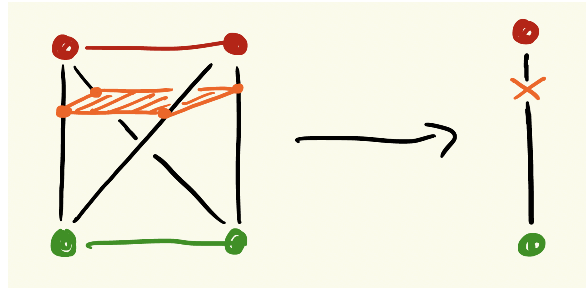

Fig. 1. Partition map for $n = 3$

and ϕ is uniquely determined by a partition of the vertices of the triangulation into $k+1$ sets, and a bijection between the sets in this partition and the vertices of σ. We call such a map $M \to \sigma$ a *partition map.* To ensure that the dual cubical structure of σ pulls back to submanifolds with the required properties, we determine suitable combinatorial properties on the triangulation. In the odd-dimensional case, we show in (5) that the first barycentric subdivision of any triangulation has a suitable partition. Moreover the r–dimensional spine of the intersection of r handlebodies meets each top-dimensional simplex in M in exactly one r–cube. In even dimensions, we obtain an analogous result in (5) after performing bistellar moves on this subdivision. □

We say that a triangulation *supports a multisection* if there is a partition of the vertices defining a partition map $\phi\colon M \to \sigma$ with the property that the pull back of the cubical structure is a multisection. Special properties of triangulations may imply special properties of the supported multisections and vice versa. For instance, special properties of a Heegaard splitting of a 3–manifold are shown in (10) to imply special properties of the dual triangulation. The cornerstone of the modern development of Heegaard splittings is the work of Casson and Gordon (11), and it is a tantalising problem to generalise this to higher dimensions.

2. Examples

An extended set of examples of trisections of 4–manifolds can be found in (1), and of multisections of higher dimensional manifolds in (5). The recent work of Gay (12), Meier, Schirmer and Zupan (13–15) gives some applications and constructions arising from trisections of 4–manifolds and relates them to other structures.

We begin with the ‘standard’ tropical multisection of complex projective space and then give new examples of multisections of a large class of spherical space forms. A *spherical space form* is a quotient space of a finite group of orthogonal transformations acting freely on a sphere. They admit a Riemannian metric of constant positive curvature. See (16) for more information on spherical space forms.

The tropical picture of complex projective space. Consider the map $\mathbb{C}P^n \to \Delta^n$ defined by

$$[\, z_0 \,:\, \dots \,:\, z_n \,] \;\mapsto\; \frac{1}{\sum |z_k|} \,(\, |z_0| \,,\, \dots \,,\, |z_n| \,).$$

The *dual spine* Π^n in Δ^n is the subcomplex of the first barycentric subdivision of Δ^n spanned by the 0–skeleton of the first barycentric subdivision minus the 0–skeleton of Δ^n. This is shown for $n = 2, 3$ in Figure 2. Decomposing along Π^n gives Δ^n a natural *cubical structure* with $n+1$ n–cubes, and the lower-dimensional cubes that we will focus on are the intersections of non-empty collections of these top-dimensional cubes. Each n–cube pulls back to a $2n$–ball in $\mathbb{C}P^n$, and the collection of these balls is a multisection. For example, if $n = 2$, the 2–cubes pull back to 4–balls, each 1–cube pulls back to $S^1 \times D^2$ and the 0–cube pulls back to $S^1 \times S^1$.

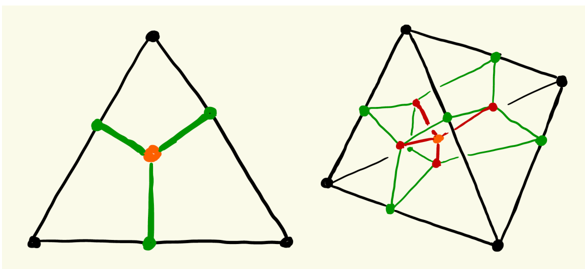

Fig. 2. Dual cubical structure of the 2–simplex and the 3–simplex

Lens spaces. Lens spaces form an important special subclass of spherical space forms.

A *lens space* $L(m : k_1, \dots k_n)$ is obtained as the quotient space of a free linear action of a finite cyclic group on S^{2n-1}, the unit sphere in $\mathbb{C}^n$. To be more specific, the action of $\mathbb{Z}_m$ on S^{2n-1} is given by

$$(z_1, \dots, z_n) \mapsto (z_1 e^{\frac{2\pi i k_1}{m}}, \dots, z_n e^{\frac{2\pi i k_n}{m}}),$$

where k_i, m are relatively prime positive integers, and $(z_1, \dots z_n) \in S^{2n-1}$.

Notice there is an associated S^1–action on $L(m : k_1, \dots k_n)$. Here $[z_1, \dots, z_n] \mapsto [z_1 z, \dots, z_n z]$, where $[z_1, \dots, z_n]$ denotes the orbit of $(z_1, \dots, z_n)$ under the action of $\mathbb{Z}_m$ on S^{2n-1} and $z \in S^1 \subset \mathbb{C}$.

Exceptional fibers $\Gamma_j, 1 \le j \le n$ of this S^1–action are obtained as the sets of points $[0, \dots, 0, z_j, 0, \dots, 0]$, where only one coordinate is non-zero. To find a natural multisection, we use a *Dirichlet* construction, based on the loops $\Gamma_j, 1 \le j \le n$ as cores of the n handlebodies $H_j, 1 \le j \le n$, each of which will then be of the form $S^1 \times B^{2n-2}$. Hence

$$H_j = \{x \in L(m : k_1, \dots k_n) : d(x, \Gamma_j) \le d(x, \Gamma_l), \forall l \ne j\},$$

where d is the standard locally spherical metric induced on the lens space.

This construction is closely related to the tropical multisection of $\mathbb{C}P^n$ in the previous section and the multisections of $\mathbb{R}P^n$ in (5). In particular, the central submanifold is again the n–torus $S^1 \times \cdots \times S^1$ given by

$$\{[z_1, z_2, \dots z_n] : |z_j| = \frac{1}{\sqrt{n}}, 1 \le j \le n\}.$$

This is an orbit of the action of the n–torus on the lens space by coordinate multiplication. The spine of each non-empty intersection of subsets of the handlebodies is a lower dimensional torus, which is a singular orbit of this torus action.

3. Structure results

Non-positively curved cubings from multisections. We work with the combinatorial definition of a non-positively curved cubing (see (7, §2.1)). A *flag complex* is a simplicial complex with the property that each subgraph in the 1–skeleton that is isomorphic to the 1–skeleton of a k–dimensional simplex is in fact the 1–skeleton of a k–dimensional simplex. A cube complex is *non-positively curved* if the link of each vertex is a flag complex. Here, the link of a vertex in a cube complex is the simplicial complex whose h–simplices are the corners of $(h+1)$–cubes adjacent with the vertex. Basic facts are that the barycentric subdivision of any complex is flag, and that the link (in the sense of simplicial complexes) of any simplex in a flag complex is a flag complex.

The partition map $\phi\colon M \to \sigma$ can be used to pull back the dual cubical structure of the target simplex. This gives a natural cell decomposition of the submanifolds in a multisection, with cells of very simple combinatorial types. Each multisection submanifold that is the intersection of $r < k+1$ handlebodies has a spine with a cubing by r–cubes, and the closed, central submanifold $\Sigma = H_0 \cap H_1 \cap \cdots \cap H_k$ has a cubing.

In (5) we show that ϕ can be chosen such that the cubing of Σ satisfies the Gromov link conditions (17), and hence is non-positively curved:

Theorem 3. *Every piecewise linear manifold has a triangulation supporting a multisection, such that the central submanifold has a non-positively curved cubing.*

Since a $(2k+1)$–manifold has a central submanifold of dimension $k+1$, this result produces manifolds with non-positively curved cubings in each dimension. We also remark that our construction yields cubings with precisely one top-dimensional cube in the central submanifold for each top-dimensional simplex in the triangulation of the manifold.

Question 4. What conditions does a non-positively curved cubed k–manifold need to satisfy so that it is PL–homeomorphic to the central submanifold in a multisection of a $(2k+1)$–manifold or a $2k$–manifold?

Uniqueness. There is a natural stabilisation procedure of multisections. In dimension 3, this increases the genus of both 3–dimensional handlebodies, whilst in higher dimensions, this increases the genus of just one of the top-dimensional handlebodies. The Reidemeister-Singer theorem (18, 19) states that any two Heegaard splittings of a 3–manifold have a common stabilisation. Using in an essential way the uniqueness up to isotopy of genus g Heegaard splittings of $\#^k(S^2 \times S^1)$ due to Waldhausen (20), Gay and Kirby (1) show that any two trisections of a 4–manifold have a common stabilisation up to isotopy. This implies that any two multisections of a 4–manifold also have a common stabilisation up to isotopy.

Question 5. Under what conditions is there a common stabilisation for two given multisections of a manifold of dimension at least five?

Our existence proof constructs multisections dual to triangulations. Conversely, up to possibly stabilising the multisection, one can build triangulations dual to multisections. However, stabilisation in higher dimensions adds summands of $S^1 \times S^{n-k-1}$ to the central submanifold, and hence we expect the equivalence relation generated by stabilisation to be finer than PL equivalence of the dual triangulations.

Recursive structure of multisections and generalisations. An important part of multisections is their *recursive structure*. By this we mean that inside a multisection of an n–dimensional manifold, we see a stratification of the boundary of each handlebody into lower dimensional manifolds. For example, for a trisection where $n = 4, 5$, we see partition functions on the boundaries of each handlebody, dividing the boundary into two pieces. For a quadrisection, where $n = 6, 7$, the boundaries of the handlebodies are divided into three pieces. However, the top dimensional pieces are not necessarily handlebodies, whereas all the pieces have spines of low dimension. So these are not multisections in the sense defined above.

The same works in all dimensions. Namely for n–manifolds, with $n = 2k$ or $n = 2k + 1$, the boundaries of the handlebodies have natural divisions into k regions. Each of these regions has a spine of dimension at most two. However, these regions are not necessarily 1–handlebodies.

From the viewpoint of the complexity theory of Martelli (21), such generalised multisections may be a fruitful approach to the study of classes of examples. For instance, a decomposition of a 4–manifold into 4–dimensional 1– or 2–handlebodies is a decomposition into 4–manifolds of complexity 0.

We note a useful result giving a relationship between the properties of having low dimensional spines and connectivity of the intersection submanifolds. This requires the following definition, which applies to all subdivisions of manifolds in the recursive structure. Note that there is no relationship assumed between n and k.

Definition 6 (Generalised multisection of closed manifold)**.** Let M be a closed, connected, piecewise linear n–manifold. A *generalised multisection* of M is a collection of $k + 1$ piecewise linear n–dimensional submanifolds $H_i \subset M$, where $0 \leq i \leq k$, subject to the following three conditions:

1. Each H_i is non-empty and has a spine of codimension at least two.
2. The submanifolds H_i have pairwise disjoint interior, and $M = \bigcup_i H_i$.
3. The intersection $H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r}$ of any proper subcollection of the submanifolds ($r \leq k$) is a compact submanifold with boundary and of dimension $n - r + 1$. Moreover, it has a spine of codimension at least two.

Proposition 7. *Suppose that a closed connected manifold M has a generalised multisection into submanifolds H_i for $0 \leq i \leq k$. Then the intersection $H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_r}$ of* any *collection of the submanifolds is non-empty and connected. In particular, each H_i and the intersection $H_0 \cap H_1 \cap \ldots \cap H_k$ is connected. Moreover, $H_0 \cap H_1 \cap \ldots \cap H_k$ is a closed manifold of dimension $n - k$.*

Proof. The argument is by complete induction on k. To start the induction, suppose M is a manifold of dimension n and $k = 1$. Whence $M = H_0 \cup H_1$. Each component X of H_0 has a spine of codimension 2 and hence connected boundary. Since $M = H_0 \cup H_1$ and $\text{int}(H_0) \cap \text{int}(H_1) = \emptyset$, we have $\partial H_0 = \partial H_1$. Therefore there is a component Y of H_1 such that $\partial X = \partial Y$. But then $X \cup Y$ is a closed n–manifold and hence $M = X \cup Y$. Moreover, $X \cap Y = \partial X$ is a non-empty, closed and connected manifold of dimension $n - 1$. This proves the result for all manifolds M and all generalised multisections into two submanifolds.

Before giving the general induction step, we move to $k = 2$. So assume M is a manifold of dimension n and that it has a generalised multisection into three submanifolds. Whence $M = H_0 \cup H_1 \cup H_2$. Again, each component X of H_0 has connected boundary. If $X \cap H_2 = \emptyset$, then there is a component Y of H_1 such that $\partial X = \partial Y$. But then, as above, $M = X \cup Y$. This contradicts the fact that $H_2 \neq \emptyset$. Hence $X \cap H_2 \neq \emptyset$, and by symmetry $X \cap H_1 \neq \emptyset$. Now ∂X is a closed $(n-1)$–dimensional manifold and $\partial X = (X \cap H_1) \cup (X \cap H_2)$. Each of $X \cap H_1$ and $X \cap H_2$ is non-empty, $(n-1)$–dimensional and has a spine of co-dimension at least two. Moreover, $(X \cap H_1) \cap (X \cap H_2) = X \cap H_1 \cap H_2$, which has dimension $n - 2$, and so $X \cap H_1$ and $X \cap H_2$ have disjoint interior. Hence by induction, each of $X \cap H_1$, $X \cap H_2$ and $X \cap H_1 \cap H_2$ is non-empty, connected and the latter is a closed manifold of dimension $n - 2$. It follows that there is a unique component Y of H_1 and a unique component Z of H_2 such that $X \cap H_1 = X \cap Y$ and $X \cap H_2 = X \cap Z$. In particular, $X \cap Y \cap Z$ is non-empty, connected and closed. It follows that $\partial Y = (X \cap Y) \cup (Y \cap Z)$ and $\partial Z = (X \cap Z) \cup (Y \cap Z)$. Whence $X \cup Y \cup Z$ has empty boundary and hence $M = X \cup Y \cup Z$. This finishes the proof for $k = 2$.

Hence assume the conclusion holds for all manifolds and all multisections into at most k_0 submanifolds. Assume that we are given a generalised multisection of an n–manifold M with $k_0 + 1$ submanifolds $H_0, \ldots, H_{k_0}$. Each component X_0 of H_0 has ∂X_0 connected. As above, we have

$$\partial X_0 = (X_0 \cap H_1) \cup \ldots \cup (X_0 \cap H_{k_0}).$$

This is a generalised multisection of the closed manifold ∂X_0 with at most k_0 submanifolds. By the induction hypothesis, all components in the above decomposition are connected, hence there are unique components X_i of H_i such that $X_0 \cap H_i = X_0 \cap X_i$, where we put $X_i = \emptyset$ if $X_0 \cap H_i = \emptyset$. Whence

$$\partial X_0 = (X_0 \cap X_1) \cup \ldots \cup (X_0 \cap X_{k_0}).$$

Since the non-empty submanifold $X_0 \cap X_i \cap X_j$ is contained in $X_i \cap X_j$, it follows by uniqueness that

$$\partial X_i = (X_i \cap X_0) \cup (X_i \cap X_1) \cup \ldots \cup (X_i \cap X_{k_0}),$$

where we omit $X_i \cap X_i$ from the union. It follows that $X_0 \cup X_1 \cup \ldots \cup X_{k_0}$ has empty boundary, and hence equals M. In particular, each $X_i \neq \emptyset$. This completes the proof. □

4. Constructing multisections

We first explain how the symmetric representations of (22) can be used to construct multisections. A number of applications of this approach are given in (5). Here we highlight *generalised multisections* and *twisted multisections*. The interested reader can find multisections of *connected sums* and *products*, a *Dirichlet construction* and the case of *manifolds with non-empty boundary* in (5).

As a second, new construction, we describe another type of twisted multisections. These arise as generalisations of certain one-sided Heegaard splittings.

Constructing multisections using symmetric representations. Given a triangulated n–manifold $|K| \to M$ with the property that the degree of each $(n-2)$–simplex is even, the authors defined a *symmetric representation* $\pi_1(M) \to \mathrm{Sym}(n+1)$ in (22) as follows. Pick one n–simplex as a base, choose a bijection between its corners and $\{1,\dots,n+1\}$ and then *reflect* this labelling across its codimension-one faces to the adjacent n–simplices. This induced labelling is propagated further and if one returns to the base simplex, one obtains a permutation of the vertex labels. Since the dual 1–skeleton carries the fundamental group, it can be shown that this gives a homomorphism $\pi_1(M) \to \mathrm{Sym}(n+1)$. See (22, §2.3) for the details. For example, the symmetric representation associated to any barycentric subdivision is trivial, since the labels correspond to the dimension of the simplex containing that vertex in its interior, but there may be more efficient even triangulations with this property.

The symmetric representation can also be used to propagate partitions of the vertices of the base simplex; this is done in (22, §2.5) for partitions into two sets, but extends to arbitrary partitions. One then obtains an *induced representation*, usually into a symmetric group of larger degree. The aim in (22) was to obtain information on the topology of a manifold from a non-trivial symmetric representation arising from a triangulation with few vertices. Our needs in this paper are different: we wish to use the symmetric representations to identify triangulations to which we can apply our constructions without barycentric subdivision. So we either want the *orbits* of the vertices under the symmetric representation to give a partition satisfying the conditions in our constructions; or we ask for partitions of the vertices with the property that the induced representation is trivial.

The main properties to check for a given partition of the vertices are that the graphs spanned by the partition sets are connected, and, in even dimensions, that the dimension of the spine drops when intersecting all but one of the handlebodies.

The following are two applications of this approach.

Generalised multisections. Suppose that M is a triangulated n–manifold with an even triangulation with trivial symmetric representation. As above, given any triangulation, the first barycentric subdivision has this property. We can define some special generalised multisections as follows.

Suppose that $n = 3k+2$. Assume that we have partition sets $P_0, P_1, \dots P_k$ where the sets meet every n-simplex in three vertices. We then map each n–simplex to the k–simplex by mapping each partition set to a vertex of this k–simplex. It is then easy to verify that we obtain a division of M into $k+1$ regions, and each region has a 2–dimensional spine, given by the union of all the 2–simplices in each n–simplex with all vertices in the same partition set. In this case, the manifold Σ which is the intersection of all the handlebodies, is closed of dimension $2k+2$. Again we can arrange that the induced cubing of Σ is non-positively curved and each intersection of a proper subcollection has a spine of low dimension.

Another interesting example is to have two partition sets of size $k', k^\star$ of the vertices of each n–simplex, so that $k' + k^\star = n+1$. We assume that both $k' > 1, k^\star > 1$. The induced decomposition is a bisection into two regions with spines of dimension $k', k^\star$. Given a handle decomposition of M, this is similar to a hypersurface which is the boundary of the region containing all the i–handles for $0 \le i \le k'$.

Finally a very specific example is a 6–manifold M with three partition sets of respective sizes $2, 2, 3$. This induces a trisection of M into three regions, where two are handlebodies and the third has a spine of dimension 2.

Twisted multisections. Suppose a closed PL n–manifold has an even triangulation with a non-trivial symmetric representation. Assume also that the symmetry preserves our standard partition of the vertices, i.e. every symmetry mapping produces a permutation of the partition sets of vertices. Then there is an associated 'twisted' multisection, which we illustrate with a simple example—the general construction then becomes clear.

Assume M is a 5–manifold that admits an even triangulation with a symmetric representation with image $\mathbb{Z}_3$. Also assume this symmetry is a permutation of the form $(012)(345)$ of the labelling of the vertices. In this case, we choose as partition sets $\{0,3\}, \{1,4\}, \{2,5\}$. Then these are permuted under the action of the symmetric mapping. The edges joining these three pairs of vertex sets form a connected graph Γ.

A regular neighbourhood of Γ then forms a single handlebody H whose boundary is glued to itself to form M. The handlebody H lifts to three handlebodies in a regular 3–fold covering space $\widetilde{M}$ of M and these give a standard trisection of $\widetilde{M}$. The covering transformation group $\mathbb{Z}_3$ permutes the handlebodies and preserves the central submanifold. If the initial triangulation is flag, then the lifted triangulation is flag and hence the central submanifold has a non-positively curved cubing on which the covering transformation group acts isometrically. Hence the quotient, which embeds in ∂H, also has a non-positively curved cubing.

Twisted multisections of some other spherical space forms. We define a natural generalisation of the one-sided Heegaard splittings of 3-dimensional spherical space forms which have fundamental groups that are either dihedral or binary dihedral by cyclic groups, discussed in (23). Such splittings are given by embedded Klein bottles with complements open solid tori. One-sided Heegaard splittings are examples of twisted multisections in dimension 3.

Consider the unit sphere S^{4n-1} in $\mathbb{H}^n$, where $\mathbb{H}$ is the quaternions. Let G be a suitable direct product of a finite subgroup of the unit quaternions that is dihedral or binary dihedral and a relatively prime order cyclic subgroup. In particular we require that joint left and right multiplication respectively of these factors of G on the unit quaternions defines a free action.

Now use the above to define a diagonal action of G on the n quaternionic factors in $S^{4n-1} \subset \mathbb{H}^n$. We claim there is a twisted multisection of the spherical space form S^{4n-1}/G, consisting of n copies of $S^1 \times B^{4n-2}$ glued together.

An easy way to see how this multisection is constructed is to pass to the 2–fold cover of S^{4n-1}/G by a lens space, using a normal cyclic subgroup of G of index 2. The multisection of this lens space as described in the previous subsection is easily seen to be invariant under the covering transformation. In fact, this covering involution interchanges pairs of exceptional fibers, and hence in the quotient space there are n loops which are projections of the $2n$ exceptional fibers.

The multisection of the lens space is obtained by a Dirichlet construction from the exceptional fibers and hence all the components, i.e intersections of families of neighbourhoods of exceptional fibers, are invariant or interchanged under the involution. So it is easy to verify that there is an induced twisted multisection of S^{4n-1}/G as claimed.

5. Category of $(k+1)$–coloured structures

A manifold M admits a $(k+1)$–colouring if it has a triangulation where the vertices are partitioned into $k+1$ sets $P_0, P_1, \dots P_k$ so that every top dimensional simplex has either one or two vertices in each set P_i.

If two manifolds M, N have triangulations which both admit $(k+1)$–colourings, then a colour preserving mapping $\phi : M \to N$ is a simplicial map which takes the partition sets $P_0, P_1, \dots P_k$ of the vertices of M to the partition sets $P'_0, P'_1, \dots P'_k$ of the vertices of N.

There is clearly a category of $(k+1)$–coloured structures defined this way. Note that the basic construction of a multisection arises from a colour preserving mapping $\phi : M \to \sigma$ where σ is the $(k+1)$–simplex with the trivial partition consisting of one vertex in each partition set.

We can also specialise to $(k+1)$–coloured structures which induce multisections. The corresponding colour preserving map then takes the multisection of the domain to the multisection of the range.

Waldhausen (24) showed that given any degree one mapping ϕ between 3–manifolds M, N and a Heegaard splitting Σ for N, the map ϕ can be homotoped so that $\phi^{-1}(\Sigma)$ is a Heegaard splitting for M. Moreover after the homotopy, the map ϕ can be put into a standard form. This implies there are 2–coloured triangulations of both M, N and a colour preserving map between them in the homotopy class of ϕ.

Question 8. Given a degree one mapping ϕ between 4–manifolds, M, N and a trisection of N is there a trisection of M so that ϕ can be homotoped to a map taking one trisection to the other? In particular, is there a colour preserving map from M to N with respect to 3–coloured triangulations of M, N, where the 3–coloured triangulation of N induces the given trisection of N?

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1. Gay D, Kirby R (2016) Trisecting 4-manifolds. *Geom. Topol.* 20(6):3097–3132.
2. Cairns SS (1944) Introduction of a Riemannian geometry on a triangulable 4-manifold. *Ann. of Math. (2)* 45:218–219.
3. Cairns SS (1961) The manifold smoothing problem. *Bull. Amer. Math. Soc.* 67:237–238.
4. Whitehead JHC (1940) On C^1-complexes. *Ann. of Math. (2)* 41:809–824.
5. Hyam Rubinstein J, Tillmann S (2016) Multisections of piecewise linear manifolds. *ArXiv e-prints.*
6. Hempel J (2004) *3-manifolds.* (AMS Chelsea Publishing, Providence, RI), pp. xii+195. Reprint of the 1976 original.
7. Wise DT (2012) *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics. (Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI) Vol. 117, pp. xiv+141.
8. Seifert H, Threlfall W (1980) A textbook of topology. Transl. by Michael A. Goldman. Seifert, H.: Topology of 3-dimensional fibered spaces. Transl. by Wolfgang Heil. Ed. by Joan S. Birman and Julian Eisner. (Pure and Applied Mathematics, 89. New York etc.: Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers. XVI, 437 p. $ 39.50 (1980).).
9. Bell M, Hass J, Rubinstein JH, Tillmann S (2017) Computing trisections of 4-manifolds. *preprint.*
10. Hodgson CD, Rubinstein JH, Segerman H, Tillmann S (2015) Triangulations of 3-manifolds with essential edges. *Ann. Fac. Sci. Toulouse Math. (6)* 24(5):1103–1145.
11. Casson AJ, Gordon CM (1987) Reducing Heegaard splittings. *Topology Appl.* 27(3):275–283.
12. Gay DT (2016) Trisections of Lefschetz pencils. *Algebr. Geom. Topol.* 16(6):3523–3531.
13. Meier J, Schirmer T, Zupan A (2016) Classification of trisections and the generalized property R conjecture. *Proc. Amer. Math. Soc.* 144(11):4983–4997.
14. Meier J, Zupan A (2017) Genus-two trisections are standard. *Geom. Topol.* 21(3):1583–1630.
15. Meier J, Zupan A (2017) Bridge trisections of knotted surfaces in S^4. *Trans. Amer. Math. Soc.* 369(10):7343–7386.
16. Wolf JA (2011) *Spaces of constant curvature.* (AMS Chelsea Publishing, Providence, RI), Sixth edition, pp. xviii+424.
17. Gromov M (1987) Hyperbolic groups in *Essays in group theory*, Math. Sci. Res. Inst. Publ. (Springer, New York) Vol. 8, pp. 75–263.
18. Reidemeister K (1933) Zur dreidimensionalen Topologie. *Abh. Math. Sem. Univ. Hamburg* 9(1):189–194.
19. Singer J (1933) Three-dimensional manifolds and their Heegaard diagrams. *Trans. Amer. Math. Soc.* 35(1):88–111.
20. Waldhausen F (1968) Heegaard-Zerlegungen der 3-Sphäre. *Topology* 7:195–203.
21. Martelli B (2010) Complexity of PL manifolds. *Algebr. Geom. Topol.* 10(2):1107–1164.
22. Rubinstein JH, Tillmann S (2015) Even triangulations of n-dimensional pseudo-manifolds. *Algebr. Geom. Topol.* 15(5):2949–2984.
23. Rubinstein JH (1979) On 3-manifolds that have finite fundamental group and contain Klein bottles. *Trans. Amer. Math. Soc.* 251:129–137.
24. Waldhausen F (1970) On mappings of handlebodies and of Heegaard splittings in *Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969).* (Markham, Chicago, Ill.), pp. 205–211.