MONOMIAL BASES AND BRANCHING RULES

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ABSTRACT. Following a question of Vinberg, a general method to construct monomial bases in finite-dimensional irreducible representations of a reductive Lie algebra $\mathfrak{g}$ was developed in a series of papers by Feigin, Fourier, and Littelmann. Relying on this method, we construct monomial bases of multiplicity spaces associated with the restriction of the representation to a reductive subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$. As an application, we produce monomial bases for representations of the general linear and symplectic Lie algebras associated with natural chains of subalgebras. We also show that our basis in type A is related to both the Gelfand–Tsetlin basis and the Littelmann basis via triangular transition matrices which implies that the triangularity property extends to the matrix connecting the Gelfand–Tsetlin and canonical bases. A similar relationship holds between our basis in type C and a suitably modified version of the basis constructed earlier by the first author.

INTRODUCTION

A general method to construct monomial bases in finite-dimensional irreducible representations of a reductive Lie algebra $\mathfrak{g}$ has been developed in a series of papers by E. Feigin, G. Fourier, and P. Littelmann [5, 6, 7] following a question and initial examples of E. Vinberg. In accordance with this method, one chooses a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and a basis $\{f_1, \dots, f_N\}$ of the nilpotent Lie algebra $\mathfrak{n}^-$ consisting of root vectors. Let $V(\lambda)$ be a finite-dimensional irreducible $\mathfrak{g}$-module and let $\xi \in V(\lambda)$ be a highest weight vector. By introducing special orderings on monomials in the basis elements f_i it is possible to specify conditions on the powers α_i so that the vectors

$$f_1^{\alpha_1} \dots f_N^{\alpha_N} \xi$$

form a basis of $V(\lambda)$. Such conditions are given in an explicit form for types A and C in [5] and [6], respectively. A unified approach is presented in [7].

One of the features of the initial solutions [5, 6] is that a *homogeneous* order on the monomials was used, which means that the degrees are compared first. In such a setup, the sequence of factors is not significant. By now there is a tremendous development in the area, with both geometric and combinatorial applications, and numerous variations have been studied, see e.g. [3, 4] and references therein. Of particular interesest and importance are connections with the Littelmann bases [11] and with the PBW-type versions of the canonical basis of Lusztig [12, 13], see [4, Sec. 11&12].

Our goal in this paper is to adjust the FFLV method to construct bases of the multiplicity spaces associated with the restriction of $V(\lambda)$ to a reductive subalgebra $\mathfrak{g}_0$. Given a finite-dimensional irreducible $\mathfrak{g}_0$-module $V'(\mu)$, the corresponding *multiplicity space* is defined by

$$U(\lambda,\mu) = \mathrm{Hom}_{\mathfrak{g}_0}\big(V'(\mu), V(\lambda)\big).$$

Note that $U(\lambda,\mu)$ is isomorphic to the subspace $V(\lambda)^+_\mu$ of $\mathfrak{g}_0$-highest weight vectors in $V(\lambda)$ of weight μ and we have a vector space decomposition

$$V(\lambda) \cong \bigoplus_{\mu} V(\lambda)^+_\mu \otimes V'(\mu). \tag{0·1}$$

Hence, if some bases of the spaces $V(\lambda)^+_\mu$ and $V'(\mu)$ are produced, then the decomposition (0·1) yields the natural tensor product basis of $V(\lambda)$. The celebrated Gelfand–Tsetlin bases [8, 9] for representations of the general linear and orthogonal Lie algebras are obtained by iterating this procedure and applying it to the subalgebras of the chains

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n \qquad \text{and} \qquad \mathfrak{o}_2 \subset \mathfrak{o}_3 \subset \cdots \subset \mathfrak{o}_N.$$

The multiplicity spaces $V(\lambda)^+_\mu$ corresponding to the pairs of orthogonal and symplectic Lie algebras $\mathfrak{o}_{N-2} \subset \mathfrak{o}_N$ and $\mathfrak{sp}_{2n-2} \subset \mathfrak{sp}_{2n}$ turned out to carry representations of certain quantum algebras originally introduced by Olshanski [17] and which are known as *twisted Yangians*. The Yangian representation theory together with the theory of *Mickelsson algebras* developed in the work by Zhelobenko [19, 20, 21] have lead to a construction of bases of the Gelfand–Tsetlin type for representations of the orthogonal and symplectic Lie algebras; see review paper [15] and book [16, Ch. 9] for a detailed exposition of these results, as well as a discussion of various approaches to constructions of Gelfand–Tsetlin-type bases in the literature.

The Zhelobenko theory allows one to describe the multiplicity spaces $V(\lambda)^+_\mu$ corresponding to the pair $\mathfrak{g}_0 \subset \mathfrak{g}$ as linear spans of *lowering operators* obtained via the action of the *extremal projector* p associated with the Lie algebra $\mathfrak{g}_0$. Our main general result provides precise choices of those operators to form a basis of $V(\lambda)^+_\mu$. These choices are made in the spirit of the FFLV method and rely on some special monomial order. In more detail, we will assume that $\mathfrak{g}_0 \subset \mathfrak{g}$ is a reductive subalgebra normalised by $\mathfrak{h}$. Then $\mathfrak{g}_0$ inherits the triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^+ \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^-$ with $\mathfrak{n}_0^\pm = \mathfrak{n}^\pm \cap \mathfrak{g}_0$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$. Let $\mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{r}$ be an $\mathfrak{h}$-stable vector space decomposition. We describe a family of admissible monomials $m \in \mathcal{U}(\mathfrak{r})$ such that the elements $pm\xi$ form a basis of the multiplicity space $V(\lambda)^+_\mu$.

Then we apply these results to produce monomial bases for representations of $\mathfrak{gl}_n$ and $\mathfrak{sp}_{2n}$ which thus provide new answers to the original question of Vinberg, different from those obtained in [5, 6]. Recall that finite-dimensional irreducible representations of $\mathfrak{gl}_n$ are parameterised by their highest weights $\lambda = (\lambda_1, \dots, \lambda_n)$ which are n-tuples of complex numbers satisfying the conditions $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for all $i = 1, \dots, n-1$. The corresponding representation $L(\lambda)$ is generated by a nonzero vector ξ such that

$$
\begin{aligned}
&E_{ij}\,\xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n, \qquad \text{and}\\
&E_{ii}\,\xi = \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n,
\end{aligned}
\tag{0·2}
$$

where the E_{ij} denote the standard basis elements of $\mathfrak{gl}_n$. A *Gelfand–Tsetlin pattern* Λ associated with λ is an array of row vectors

$$
\begin{array}{ccccccccc}
\lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\
& \lambda_{n-1\,1} & & \cdots & & \lambda_{n-1\,n-1} & \\
& & \cdots & \cdots & \cdots & & \\
& & \lambda_{21} & & \lambda_{22} & & \\
& & & \lambda_{11} & & &
\end{array}
$$

where the upper row coincides with λ and the following conditions hold

$$
\lambda_{k\,i} - \lambda_{k-1\,i} \in \mathbb{Z}_+, \qquad \lambda_{k-1\,i} - \lambda_{k\,i+1} \in \mathbb{Z}_+, \qquad i = 1, \dots, k-1
\tag{0·3}
$$

for each $k = 2, \dots, n$.

Theorem A. *The vectors*

$$
\pi_\Lambda = E_{21}^{\lambda_{21}-\lambda_{11}} E_{31}^{\lambda_{31}-\lambda_{21}} E_{32}^{\lambda_{32}-\lambda_{22}} \dots E_{n1}^{\lambda_{n1}-\lambda_{n-1\,1}} \dots E_{n\,n-1}^{\lambda_{n\,n-1}-\lambda_{n-1\,n-1}}\,\xi
$$

parameterised by all Gelfand–Tsetlin patterns Λ associated with λ form a basis of $L(\lambda)$.

Theorem A will be proved as an application of our general results on monomial bases of multiplicity spaces (Section 1.4). On the other hand, it can also be derived by using the methods of the Mickelsson algebra theory of Zhelobenko [19, 20, 21]. With a slightly different, but combinatorially equivalent, description this basis appeared in [18]. We will show in Section 3 that the basis π_Λ is related to the Gelfand–Tsetlin basis of $L(\lambda)$ via a triangular transition matrix essentially repeating the argument used in [19, Theorem 7] and [20, Lemma 2]. Moreover, a triangular transition matrix turns out to relate the basis π_Λ with the monomial basis constructed by Littelmann [10], see Section 3.1 below. This implies that the transition matrix between the Gelfand–Tsetlin basis and the canonical basis is also triangular; see Remark 3.3 and Corollary 3.5.

We will regard the symplectic Lie algebra $\mathfrak{sp}_{2n}$ as a subalgebra of $\mathfrak{gl}_{2n}$ and we will number the rows and columns of $2n \times 2n$ matrices with the indices $-n, \dots, -1, 1, \dots, n$. Accordingly, the zero value will be omitted in the summation or product formulas. The Lie algebra $\mathfrak{sp}_{2n}$ is spanned by the elements F_{ij} with $-n \leqslant i, j \leqslant n$, defined by

$$
F_{ij} = E_{ij} - \operatorname{sgn} i \, \operatorname{sgn} j \, E_{-j,-i}.
\tag{0·4}
$$

For any n-tuple of nonpositive integers $\lambda = (\lambda_1, \dots, \lambda_n)$ satisfying the conditions

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_n$$

the finite-dimensional irreducible representation $V(\lambda)$ of the Lie algebra $\mathfrak{sp}_{2n}$ with the highest weight λ is generated by a nonzero vector ξ such that

$$\begin{aligned} F_{ij}\,\xi &= 0 \qquad \text{for} \quad -n \leqslant i < j \leqslant n, \qquad \text{and} \\ F_{ii}\,\xi &= \lambda_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n. \end{aligned} \tag{0·5}$$

Define a *type* C *pattern* Λ associated with λ as an array of the form

$$\begin{array}{ccccccccccc}
& \lambda_{n\,1} & & \lambda_{n\,2} & & \lambda_{n\,3} & & \cdots & & \lambda_{n\,n} \\
\lambda'_{n\,1} & & \lambda'_{n\,2} & & \lambda'_{n\,3} & & \cdots & & \lambda'_{n\,n} & \\
& \lambda_{n-1\,1} & & \lambda_{n-1\,2} & & \cdots & & \lambda_{n-1\,n-1} & & \\
\lambda'_{n-1\,1} & & \lambda'_{n-1\,2} & & \cdots & & \lambda'_{n-1\,n-1} & & & \\
& \cdots & & \cdots & & \cdots & & & & \\
& \lambda_{1\,1} & & & & & & & & \\
\lambda'_{1\,1} & & & & & & & & &
\end{array}$$

such that $\lambda_{n\,i} = \lambda_i$ for $i = 1, \dots, n$, the remaining entries are all nonpositive integers and the following inequalities hold:

$$\lambda'_{k\,1} \geqslant \lambda_{k\,1} \geqslant \lambda'_{k\,2} \geqslant \lambda_{k\,2} \geqslant \dots \geqslant \lambda'_{k\,k-1} \geqslant \lambda_{k\,k-1} \geqslant \lambda'_{k\,k} \geqslant \lambda_{k\,k}$$

for $k = 1, \dots, n$, and

$$\lambda'_{k\,1} \geqslant \lambda_{k-1\,1} \geqslant \lambda'_{k\,2} \geqslant \lambda_{k-1\,2} \geqslant \dots \geqslant \lambda'_{k\,k-1} \geqslant \lambda_{k-1\,k-1} \geqslant \lambda'_{k\,k}$$

for $k = 2, \dots, n$.

Theorem B. *The vectors*

$$\theta_\Lambda = \prod_{k=1,\dots,n}^{\longrightarrow} \left(F_{k,-k}^{-\lambda'_{k\,1}} \prod_{i=1}^{k-1} F_{k,-i}^{\lambda_{k-1\,i} - \lambda'_{k\,i+1}} F_{-i,-k}^{\lambda_{k\,i} - \lambda'_{k\,i+1}} \right) \xi$$

parameterised by all type C *patterns* Λ *associated with* λ *form a basis of* $V(\lambda)$.

The proof of Theorem B will be given in Section 2.1, it will be derived from Theorem A and our general results on monomial bases of multiplicity spaces. In Section 2.2, we present another basis of $U(\lambda, \mu)$ with somewhat more complicated conditions on the exponents of the monomials, which can be extended inductively to a basis of $V(\lambda)$. Furthermore, in Section 4, we will produce a certain modified version ζ_Λ of the basis of $V(\lambda)$

constructed in [14] and derive explicit formulas for the action of generators of the Lie algebra $\mathfrak{sp}_{2n}$ in this basis. Then we will demonstrate in Section 5 that the bases θ_Λ and ζ_Λ are related via a triangular transition matrix. This also gives another proof of Theorem B.

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1. THE FFLV APPROACH TO THE BRANCHING PROBLEM

Let $\mathfrak{g}$ be a complex reductive Lie algebra and $V(\lambda)$ be an irreducible finite-dimensional $\mathfrak{g}$-module. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The Lie algebra $\mathfrak{n}^-$ has a standard basis consisting of root vectors $\{f_1, \dots, f_N\}$. We choose a total *monomial order* on the monomials $m \in \mathcal{S}(\mathfrak{n}^-)$ in this basis. Recall that by the definition any monomial order is compatible with multiplication, i.e., satisfies the following two conditions:

- $\diamond$ $1 \leqslant m$ for each monomial m,
- $\diamond$ if $m_1 \leqslant m_2$ and $m_3 \leqslant m_4$, then $m_1 m_3 \leqslant m_2 m_4$.

The order leads to a filtration on $V(\lambda)$. Let $v_\lambda \in V(\lambda)$ denote a highest weight vector. By fixing a sequence of the factors of $m \in \mathcal{S}(\mathfrak{n}^-)$, we identify m with an element of $\mathcal{U}(\mathfrak{n}^-)$ and say that a monomial $\tilde{m} \in \mathcal{S}(\mathfrak{g})$ is *essential* if $\tilde{m} v_\lambda$ does not lie in the linear span of $\{m v_\lambda\}$ with $m < \tilde{m}$. Let $\mathcal{E}s(V(\lambda)) = \mathcal{E}s(\lambda)$ denote the set of essential monomials related to $V(\lambda)$. By the construction, $\{m v_\lambda \mid m \in \mathcal{E}s(\lambda)\}$ is a basis of $V(\lambda)$, for more details see [7].

For any two finite-dimensional irreducible $\mathfrak{g}$-modules $V(\lambda)$ and $V(\lambda')$, one has the inclusion

$$\mathcal{E}s(\lambda)\mathcal{E}s(\lambda') \subset \mathcal{E}s(\lambda + \lambda'), \tag{1·1}$$

see [7, Prop. 2.11]. The proof of that proposition works for any, not necessary homogeneous, monomial order. However, the authors remark in the proof that they are using a homogeneous order and therefore can assume that the root vectors commute. For completeness, we briefly outline the argument.

Suppose that $m = \prod_{i=1}^N f_i^{a_i}$ is essential for λ and $m' = \prod_{i=1}^N f_i^{a'_i}$ is essential for λ'. Set $\tilde{m} = mm'$ in $\mathcal{S}(\mathfrak{n}^-)$. As an element of $\mathcal{U}(\mathfrak{n}^-)$, the monomial $\tilde{m}$ is equal to the product $f_N^{a_N + a'_N} \dots f_1^{a_1 + a'_1}$. Let $\boldsymbol{v} = v_\lambda \otimes v_{\lambda'}$ be a highest weight vector of $V_{\lambda+\lambda'} \subset V_\lambda \otimes V_{\lambda'}$. Then we have

$$\tilde{m}\boldsymbol{v} \in \prod_{i=1}^N \binom{a_i + a'_i}{a_i} m v_\lambda \otimes m' v_{\lambda'} \;+\; \left(V_\lambda \otimes \langle \hat{m} v_{\lambda'} \mid \hat{m} < m' \rangle_{\mathbb{C}} \,\oplus\, \langle \hat{m} v_\lambda \mid \hat{m} < m \rangle_{\mathbb{C}} \otimes V_{\lambda'}\right).$$

From this one can conclude that $\tilde{m} \in \mathcal{E}s(\lambda + \lambda')$.

The main novelty of our approach to the branching problem is that we combine the FFLV method with the more classical theory of Zhelobenko. In particular, the *extremal projector* will be playing a major role.

1.1. **The extremal projector.** Let Δ^+ be the set of positive roots of $\mathfrak{g}$ which is determined by the triangular decomposition so that $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) is spanned by the root vectors e_α (resp., f_α) with $\alpha \in \Delta^+$. Consider the $\mathfrak{sl}_2$-triples $\{f_\alpha, h_\alpha, e_\alpha\} \subset \mathfrak{g}$ and assume that the roots are normalised to satisfy the condition $\alpha(h_\alpha) = 2$. Set

$$p_\alpha = 1 + \sum_{k=1}^{\infty} f_\alpha^k e_\alpha^k \, \frac{(-1)^k}{k!(h_\alpha + \rho(h_\alpha) + 1) \ldots (h_\alpha + \rho(h_\alpha) + k)},$$

ρ is the half sum of the positive roots. This expression is regarded as an element of the algebra of formal series of monomials

$$f_{\alpha_1}^{r_1} \ldots f_{\alpha_N}^{r_N} e_{\alpha_N}^{k_N} \ldots e_{\alpha_1}^{k_1} \quad \text{with} \quad (k_1 - r_1)\alpha_1 + \ldots + (k_N - r_N)\alpha_N = 0$$

with coefficients in the field of fractions of the commutative algebra $\mathcal{U}(\mathfrak{h})$. Choose a numbering of positive roots, $\alpha_1, \ldots, \alpha_N$. A total order on Δ^+ is said to be *normal* if either $\alpha < \alpha + \beta < \beta$ or $\beta < \alpha + \beta < \alpha$ for each pair of positive roots α, β such that $\alpha + \beta \in \Delta$. Choose a normal order $\alpha_1 < \cdots < \alpha_N$ and set

$$p = p_{\alpha_1} \ldots p_{\alpha_N}.$$

The element p is called the *extremal projector*. It was introduced by Asherova, Smirnov, and Tolstoy in [1]. The projector is independent of the choice of a normal order. A more detailed description of its properties can be found in the work by Zhelobenko [20, 21]. In particular, p is characterised by the properties $p^2 = p$ and

$$e_\alpha p = p f_\alpha = 0 \qquad \text{for all} \qquad \alpha \in \Delta^+. \tag{1·2}$$

1.2. **The specifics of branching.** A subalgebra $\mathfrak{q} \subset \mathfrak{g}$ is a *reductive subalgebra* if $\mathfrak{q}$ is reductive and the centre of $\mathfrak{q}$ consists of $\mathrm{ad}_{\mathfrak{g}}$-semisimple elements.

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a reductive subalgebra normalised by $\mathfrak{h}$. Then $\mathfrak{g}_0$ inherits the triangular decomposition, $\mathfrak{g}_0 = \mathfrak{n}_0^+ \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^-$, where $\mathfrak{n}_0^\pm = \mathfrak{n}^\pm \cap \mathfrak{g}_0$. In order to see the branching rules $\mathfrak{g} \downarrow \mathfrak{g}_0$, we need a certain special monomial order. Let $\mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{r}$ be the $\mathfrak{h}$-stable decomposition. Write $m = m_0 m_1$, where $m_0 \in \mathcal{S}(\mathfrak{n}_0^-)$ and $m_1 \in \mathcal{S}(\mathfrak{r})$. Having two monomials $m = m_0 m_1$ and $m' = m_0' m_1'$, we first compare m_1 with m_1' and if $m_1 < m_1'$, then $m < m'$. If $m_1 = m_1'$, then we compare m_0 with m_0'. The order on the $\mathcal{S}(\mathfrak{n}_0^-)$-factors is of no particular importance. When identifying $m_0 m_1 \in \mathcal{S}(\mathfrak{n}^-)$ with an element of $\mathcal{U}(\mathfrak{n}^-)$, we take a monomial from $\mathcal{U}(\mathfrak{n}_0^-)\mathcal{U}(\mathfrak{r})$.

Let $m_1 \in \mathcal{U}_+(\mathfrak{r})$ be a monomial having our chosen sequence of factors. The most crucial restriction on the monomial order is that

$$(1{\cdot}3) \qquad x m_1 v_\lambda = [x, m_1] v_\lambda \in \big\langle \tilde{m} v_\lambda \mid \tilde{m} \in \mathcal{S}(\mathfrak{n}^-), \tilde{m} < m_1 \big\rangle_{\mathbb{C}}$$

for each dominant weight λ and each $x \in \mathfrak{n}_0^+$. We will assume that it is satisfied. If $\tilde{m} < m_1$ and $m_1 \in \mathcal{S}(\mathfrak{r})$, then $\tilde{m} = \tilde{m}_0 \tilde{m}_1$, where $\tilde{m}_1 < m_1$. Therefore (1·3) implies that

$$(1{\cdot}4) \qquad X m_1 v_\lambda \in \big\langle \tilde{m} v_\lambda \mid \tilde{m} \in \mathcal{S}(\mathfrak{n}^-), \tilde{m} < m_1 \big\rangle_{\mathbb{C}}$$

for each dominant weight λ and each $X \in \mathcal{U}(\mathfrak{g}_0)\mathfrak{n}_0^+$.

Let p be the extremal projector associated with $\mathfrak{g}_0$. Set $N' = \dim \mathfrak{n}_0^-$. Suppose that $w \in V(\lambda)$ is a weight vector such that pw is well-defined. Then pw is equal to w plus a finite linear combination of expressions

$$f_{\alpha_1}^{r_1} \dots f_{\alpha_{N'}}^{r_{N'}} e_{\alpha_{N'}}^{k_{N'}} \dots e_{\alpha_1}^{k_1} w,$$

where $k_1 + \ldots + k_{N'} > 0$. By (1·4), we have

$$(1{\cdot}5) \qquad p m_1 v_\lambda \in m_1 v_\lambda + \langle \tilde{m} v_\lambda \mid \tilde{m} < m_1 \rangle_{\mathbb{C}}$$

whenever pm_1v_λ is well-defined (that is, the values of the denominators occurring in pm_1v_λ are not zero).

Proposition 1.1. *Keep the above notation and the assumptions on the monomial order. Then pm_1v_λ is well-defined for each $m_1 \in \mathcal{E}s(\lambda) \cap \mathcal{S}(\mathfrak{r})$ and the set of vectors*

$$\{pm_1v_\lambda \mid m_1 \in \mathcal{E}s(\lambda) \cap \mathcal{S}(\mathfrak{r})\}$$

is a basis of the subspace $V(\lambda)^+ = \bigoplus_\mu V(\lambda)^+_\mu$.

Proof. One observes easily that $V(\lambda)^+$ is spanned by pmv_λ, where $m \in \mathcal{E}s(\lambda)$ and the $\mathfrak{h}_0$-weight of mv_λ is dominant for $\mathfrak{g}_0$. If $m \notin \mathcal{S}(\mathfrak{r})$, then $pm = 0$ by (1·2) and our assumption on the sequence of factors in $\mathcal{U}(\mathfrak{n}^-)$. It remains to prove that the vectors in question are well-defined and linearly independent.

Assume that pm_1v_λ is not well-defined for some $m_1 \in \mathcal{E}s(\lambda) \cap \mathcal{S}(\mathfrak{r})$. Then the weight of $u = m_1v_\lambda$ is not dominant for $\mathfrak{g}_0$. Let $\{e, h, f\} \subset \mathfrak{g}_0$ be an $\mathfrak{sl}_2$-triple such that $e \in \mathfrak{n}_0^+$ is a simple root vector and $hu = -du$ for some $d > 0$. By the standard $\mathfrak{sl}_2$-theory, which includes classification of the finite-dimensional $\mathfrak{sl}_2$-modules, there is $k = d + 2k'$ such that u lies in $\bigoplus_{t=d}^{k} \mathcal{S}^t\mathbb{C}^2$ up to an isomorphism. Therefore one can find elements $a(t) \in \mathbb{C}$ such that

$$u = \sum_{t=d}^{d+k'} a(t) f^t e^t u.$$

Here each $e^t m_1 v_\lambda$, and hence also each $f^t e^t m_1 v_\lambda$, lies in $\langle \tilde{m} v_\lambda \mid \tilde{m} < m_1 \rangle_{\mathbb{C}}$, see (1·4). Therefore m_1 is not essential for λ.

Assume finally that a non-trivial linear combination of pm_1v_λ with $m_1 \in \mathcal{E}s(\lambda) \cap \mathcal{S}(\mathfrak{r})$ is equal to zero. Then by (1·5), the largest monomial appearing in it with a non-zero coefficient is not essential for λ. $\square$

The inclusion (1·1) justifies the following definition.

Definition 1.2. The subset

$$\Gamma = \Gamma_{\mathfrak{g}\downarrow\mathfrak{g}_0} := \{(\lambda, m_1) \mid m_1 \in \mathcal{E}s(\lambda) \cap \mathcal{S}(\mathfrak{r})\} \subset \mathfrak{h}^* \times \mathcal{S}(\mathfrak{r}), \quad \text{where } \lambda \text{ is dominant,}$$

is called the *branching semigroup* of $\mathfrak{g} \downarrow \mathfrak{g}_0$. Set also $\Gamma(\lambda) = \{m_1 \mid (\lambda, m_1) \in \Gamma\}$.

Note that the above objects depend on the basis of $\mathfrak{n}_-$, on the monomial order, and on the sequence of factors in $\mathcal{U}(\mathfrak{n}^-)$. A standard procedure for calculating Γ is to consider first small values of λ, like the fundamental weights ϖ_i, obtain enough elements in $\Gamma(\lambda)$, and then compare the cardinality with the dimension of $V(\lambda)^+$. However, this approach can produce a description of Γ only if the semigroup is finitely generated.

Example 1.3. As we will see below, the semigroup $\Gamma = \Gamma_{\mathfrak{sl}_n\downarrow\mathfrak{gl}_{n-1}}$ is generated by the pairs (ϖ_i, m_1) with $m_1 \in \Gamma(\varpi_i)$ and $1 \leqslant i < n$.

1.3. **Inductive bases for $V(\lambda)$.** Next we show how branching rules lead to constructions of FFLV-type bases.

Proposition 1.4. *We have $m_0m_1 \in \mathcal{E}s(\lambda)$ if and only if $m_1 \in \Gamma_{\mathfrak{g}\downarrow\mathfrak{g}_0}(\lambda)$ and $m_0 \in \mathcal{E}s(\mu)$, where $\mu = \mu(m_1v_\lambda)$ is the weight of m_1v_λ w.r.t. $\mathfrak{h}_0$.*

Proof. Suppose first that $m_0m_1 \in \mathcal{E}s(\lambda)$. If m_1 is not essential for λ, then

$$m_1v_\lambda = \sum_k A(k)a_0(k)a_1(k)v_\lambda$$

for some $A(k) \in \mathbb{C}$, some monomials $a_0(k) \in \mathcal{U}(\mathfrak{n}_0^-)$ and $a_1(k) \in \mathcal{U}(\mathfrak{r})$, and $a_1(k) < m_1$ for all k. In this case $m_0a_0(k)a_1(k) < m_0m_1$ for each k and hence m_0m_1 is not essential, a contradiction.

If $m_0 \notin \mathcal{E}s(\mu)$, then

$$m_0pm_1v_\lambda = \sum_k B(k)b_0(k)pm_1v_\lambda$$

for some $B(k) \in \mathbb{C}$, some monomials $b_0(k) \in \mathcal{U}(\mathfrak{n}_0^-)$, and we have $b_0(k) < m_0$ for each k. Since m_1v_λ is the leading term of pm_1v_λ by (1·5), we conclude that m_0m_1 is not essential, a contradiction.

Now we know that

$$|\mathcal{E}s(\lambda)| \leqslant \sum_{m_1\in\Gamma(\lambda)} |\mathcal{E}s(\mu(m_1v_\lambda))| = \dim V(\lambda).$$

Since also $|\mathcal{E}s(\lambda)| = \dim V(\lambda)$, we can conclude that each product $m_0 m_1$, where m_1 and m_0 are essential for λ and μ, respectively, is essential for λ. This completes the proof. □

Remark 1.5. One can also give a direct proof of the inclusion $\mathcal{E}s(\mu)\Gamma_{\mathfrak{g}\downarrow\mathfrak{g}_0}(\lambda) \subset \mathcal{E}s(\lambda)$ avoiding dimension reasons.

1.4. **The Gelfand–Tsetlin order in type** A. Here we show how the FFLV method leads to a construction of the basis described in Theorem A. A slightly different description of the same basis is presented in [18], where some PBW-type bases related to crystal graphs are considered. Our proof involves neither the canonical basis of Lusztig nor quiver representations.

Take $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{g}_0 = \mathfrak{gl}_{n-1}$ that is the span of E_{ij} with $1 \leqslant i, j < n$. Then $\mathfrak{r}$ is the linear span of $E_{n\,k}$ with $1 \leqslant k < n$. Note that $[\mathfrak{r}, \mathfrak{r}] = 0$. Hence the sequence of factors in $m_1 \in \mathcal{U}(\mathfrak{n}^-)$ is of no significance. The $\mathfrak{h}_0$-weights of $E_{n\,k}$ with $1 \leqslant k < n$ are linearly independent. If $m_1 \neq \tilde{m}_1$ and $pm_1v_\lambda \neq 0$, then $pm_1v_\lambda \neq p\tilde{m}_1v_\lambda$. The branching $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$ is *multiplicity free*, which is the key point of [8]. Given a highest weight μ such that $U(\lambda, \mu) \neq 0$, there is a unique way to write the corresponding $m_1 \in \mathcal{E}s(\lambda)$, which exists by Proposition 1.1. Since the branching rules are well-known, we description of $\Gamma(\lambda)$ results from Proposition 1.1 immediately. Write $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_k - \lambda_{k+1} \in \mathbb{Z}_+$ for $k = 1, \ldots, n-1$.

Corollary 1.6. *For each monomial order satisfying the assumptions of Section 1.2,*

$$\Gamma_{\mathfrak{gl}_n\downarrow\mathfrak{gl}_{n-1}}(\lambda) = \{E_{n\,1}^{\alpha_1} \ldots E_{n\,n-1}^{\alpha_{n-1}} \mid \alpha_k \leqslant \lambda_k - \lambda_{k+1}\}.$$

Hence, the semigroup $\Gamma_{\mathfrak{gl}_n\downarrow\mathfrak{gl}_{n-1}}$ is generated by the sets $\{(\varpi_k, 1), (\varpi_k, E_{n\,k})\}$ for $1 \leqslant k < n$; that is, by the 1-dimensional representations of $\mathfrak{gl}_n$ together with $\Gamma(\varpi_k)$ for $1 \leqslant k < n$.

An example of a suitable, i.e., satisfying (1·3), monomial order on $\mathcal{S}(\mathfrak{r})$ is the lexicographical order on $E_{n\,1}^{\alpha_1} \ldots E_{n\,n-1}^{\alpha_{n-1}}$, which is also the *right* lexicographical order on the tuples $(\alpha_{n-1}, \ldots, \alpha_1)$.

The elements of $\Gamma_{\mathfrak{gl}_n\downarrow\mathfrak{gl}_{n-1}}(\lambda)$ can be parameterised by the Gelfand–Tsetlin patterns Λ, as defined in (0·3). Each such Λ corresponds to the monomial

$$m_1(\Lambda) = E_{n\,1}^{\lambda_{n\,1} - \lambda_{n-1\,1}} \ldots E_{n\,n-1}^{\lambda_{n\,n-1} - \lambda_{n-1\,n-1}}.$$

Arguing inductively with the use of Proposition 1.4, we restrict $L(\lambda)$ further to $\mathfrak{gl}_{n-2}$, $\mathfrak{gl}_{n-3}$, and so on. Taking the sequence of factors

$$E_{2\,1}^{\alpha_{2,1}} E_{3\,1}^{\alpha_{3,1}} E_{3\,2}^{\alpha_{3,2}} \ldots E_{n\,1}^{\alpha_{n,1}} \ldots E_{n\,n-1}^{\alpha_{n,n-1}}$$

in $\mathcal{U}(\mathfrak{g})$ and the lexicographical order at each step we obtain the basis of Theorem A. An alternative way to express this basis is to write

$$\mathcal{E}s(\lambda) = \left\{ \prod E_{i\,j}^{\alpha_{i,j}} \mid \alpha_{i,j} \leqslant \lambda_j - \lambda_{j+1} + \sum_{k=i+1}^{n} (\alpha_{k,j+1} - \alpha_{k,j}) \right\}.$$

This is the set of inequalities given in [18, Introduction]. The same inequalities are used in [2, Sec. 6] for a description of a different, but related, basis.

The inductive argument shows also that the semigroup $\Gamma = \Gamma_{\mathfrak{sl}_n \downarrow \{0\}}$ is generated by $\Gamma(\varpi_k)$ with $1 \leqslant k < n$.

The next example is crucial for the symplectic case.

Example 1.7. Consider $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_{n+1}$ embedded as the middle $(n-1) \times (n-1)$-square. For elements of $\mathcal{U}(\mathfrak{r})$, we are using the following sequence of root vectors:

$$\prod_{k=2}^{n} E_{n+1\,k}^{\alpha_{n+1,k}} \prod_{k=2}^{n+1} E_{k\,1}^{\alpha_{k,1}}.$$

The monomial order is given by the right lexicographical order on the tuples

$$(\alpha_{n+1,n}, \dots, \alpha_{n+1,2}, \alpha_{2,1}, \dots, \alpha_{n+1,1}).$$

Here $\Gamma_{\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\lambda)$ is equal to

$$\left\{ \prod_{k=2}^{n} E_{n+1\,k}^{\alpha_{n+1,k}} \prod_{k=2}^{n+1} E_{k\,1}^{\alpha_{k,1}} \mid \alpha_{k+1,1} \leqslant \lambda_k - \lambda_{k+1} \quad \text{and} \quad \alpha_{n+1,k} \leqslant \lambda_k - \lambda_{k+1} + \alpha_{k,1} - \alpha_{k+1,1} \right\}.$$

This branching semigroup is generated by 1-dimensional representations of $\mathfrak{gl}_{n+1}$ and by the essential monomials of the fundamental weights. Record that

$$\begin{aligned} &\Gamma_{\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\varpi_1) = \{1, E_{2\,1}, E_{n+1\,2} E_{2\,1}\}; \\ &\Gamma_{\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\varpi_k) = \{1, E_{k+1\,1}, E_{n+1\,k}, E_{n+1\,k+1} E_{k+1\,1}\} \;\text{ if }\; 2 \leqslant k < n; \\ &\Gamma_{\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\varpi_n) = \{1, E_{n+1\,1}, E_{n+1\,n}\}. \end{aligned} \tag{1·6}$$

2. Symplectic branching rules

In this section we take $\mathfrak{g} = \mathfrak{sp}_{2n}$ and use the presentation of the symplectic Lie algebra defined in the Introduction. The subalgebra $\mathfrak{g}_0 = \mathfrak{sp}_{2n-2}$ is spanned by the elements F_{ij} with $-n+1 \leqslant i, j \leqslant n-1$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ with the basis $\{F_{11}, \dots, F_{nn}\}$, while the subalgebra $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) is spanned by the elements F_{ij} with $i < j$ (resp., $i > j$). We have a vector space decomposition $\mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{r}$, where $\mathfrak{r} = \langle F_{n\,i} \mid i < n \rangle_{\mathbb{C}}$ is a Heisenberg Lie algebra and $[\mathfrak{r}, \mathfrak{r}]$ is spanned by $F_{n,-n}$. The elements from different pairs $(F_{n,i}, F_{i,-n})$, $(F_{n,j}, F_{j,-n})$ commute with each other and $[F_{n,i}, F_{i,-n}] = F_{n,-n}$, where $F_{n,-n}$ is a central element of $\mathfrak{r}$.

2.1. **The Gelfand–Tsetlin-type order in the symplectic case.** We will describe a rather elaborate monomial order on $\mathcal{S}(\mathfrak{r})$ suggested by the structure of the branching semigroup of Example 1.7.

Definition 2.1. Define a monomial order on $\mathcal{S}(\mathfrak{r})$ by the following rule. The monomial

$$F_{n,-n}^{\alpha_1}F_{n,-n+1}^{\alpha_2}\dots F_{n,-1}^{\alpha_n}F_{n,1}^{\alpha_{n+1}}\dots F_{n,n-1}^{\alpha_{2n-1}} \tag{2·1}$$

given by $\bar\alpha=(\alpha_1,\dots,\alpha_{2n-1})$ is smaller than the monomial given by $\bar\alpha'=(\alpha'_1,\dots,\alpha'_{2n-1})$ if and only if:

either $\sum\limits_{i=1}^{n}\alpha_i<\sum\limits_{i=1}^{n}\alpha'_i$; or $\sum\limits_{i=1}^{n}\alpha_i=\sum\limits_{i=1}^{n}\alpha'_i$ and

either $\alpha_1<\alpha'_1$; or $\alpha_1=\alpha'_1$ and the $(n-1)$-tuple

$\left(\sum\limits_{i=2}^{n+1}\alpha_i,\sum\limits_{i=2}^{n+2}\alpha_i,\dots,\sum\limits_{i=2}^{2n-1}\alpha_i\right)$ is smaller than or equal to $\left(\sum\limits_{i=2}^{n+1}\alpha'_i,\sum\limits_{i=2}^{n+2}\alpha'_i,\dots,\sum\limits_{i=2}^{2n-1}\alpha'_i\right)$ in the lexicographical order, whereat if they are equal, then

$(\alpha_2,\dots,\alpha_n)<(\alpha'_2,\dots,\alpha'_n)$ in the lexicographical order.

Lemma 2.2. *Choose the sequence of factors in $\mathcal{U}(\mathfrak{r})$ as in (2.1). Then the monomial order of Definition 2.1 satisfies (1·3).*

Proof. Take $x\in\mathfrak{n}_0^+$ and $m_1\in\mathcal{U}(\mathfrak{r})$ given by $\bar\alpha$ as in (2·1) . Then $[x,m_1]$ is a linear combination of monomials such that in each of them, one $F_{n,i}$ with $i>-n$ is replaced by $[x,F_{n,i}]$. Note that $[x,F_{n,i}]\in\langle F_{n,k}\mid k>i\rangle_{\mathbb{C}}$. The next step is to bring each new $F_{n,j}$ with $j>i$ to its place. In $\mathcal{S}(\mathfrak{r})$, we have $F_{n,j}\frac{m}{F_{n,i}}<m$ if m is divisible by $F_{n,i}$.

It may happen that the new element $F_{n,j}$ has to change places with $F_{n,-j}$. In that case we have to consider also the monomial $\tilde m_1=F_{n,-n}\frac{m}{F_{n,i}F_{n,-j}}\in\mathcal{S}(\mathfrak{r})$. Fortunately, this problem appears only if $i<0$ and $j>0$. Here the sum $\alpha_1+\ldots+\alpha_n$ gets smaller when we pass from m_1 to $\tilde m_1$. Therefore $\tilde m_1<m_1$. □

Let $\widetilde\Gamma$ be the branching semigroup of $\mathfrak{g}\downarrow\mathfrak{g}_0$ defined by the sequence of root vectors as in (2.1) and the monomial order of Definition 2.1.

Theorem 2.3. *The semigroup $\widetilde\Gamma$ is generated by the pairs (ϖ_i,m_1), where ϖ_i is a fundamental weight and $m_1\in\widetilde\Gamma(\varpi_i)$. Under a suitable identification, $\widetilde\Gamma$ is defined by the same inequalities as the semigroup $\Gamma_{\mathfrak{sl}_{n+1}\downarrow\mathfrak{gl}_{n-1}}$.*

Proof. We use the bijection between the sets

$$\{F_{n,k}\mid -n\leqslant k<n,\quad k\neq 0\}\quad\text{and}\quad\{E_{n+1\,k},E_{t\,1}\mid 1\leqslant k\leqslant n,\quad 2\leqslant t\leqslant n\}$$

which takes $F_{n,-n}$ to $E_{n+1\,1}$, the vector $F_{n,-k}$ with $1 \leqslant k < n$ to $E_{n+1\,n-k+1}$, and $F_{n,k}$ to $E_{n+1-k\,1}$. Using the same letters, ϖ_i, for the fundamental weights of both $\mathfrak{sp}_{2n}$ and $\mathfrak{sl}_{n+1}$, we identify also the highest weights $\lambda = \sum c_i\varpi_i$ of $\mathfrak{sp}_{2n}$ and $\mathfrak{sl}_{n+1}$. Then the standard branching theory assures that $|\widetilde{\Gamma}(\lambda)| = |\Gamma_{\mathfrak{sl}_{n+1}\downarrow\mathfrak{gl}_{n-1}}(\lambda)|$, see e.g. [15] and patterns in the Introduction. Since we have the property $\widetilde{\Gamma}(\lambda)\widetilde{\Gamma}(\lambda') \subset \widetilde{\Gamma}(\lambda+\lambda')$, see (1·1), it remains to show that the image of each $\widetilde{\Gamma}(\varpi_k)$ is exactly $\Gamma_{\mathfrak{sl}_{n+1}\downarrow\mathfrak{gl}_{n-1}}(\varpi_k)$. The latter is presented in (1·6). Let $\boldsymbol{v}_r \in V(\varpi_r)$ be a highest weight vector.

Take ϖ_1. Here $|\Gamma(\varpi_1)| = 3$. Notice that $F_{n,n-1}\boldsymbol{v}_1 \neq 0$ is a highest weight vector of $\mathfrak{g}_0$ and that $F_{n,n-1}$ is the smallest root vector in the monomial order. Therefore $F_{n,n-1} \in \mathcal{E}s(\varpi_1)$. This root vector is mapped to $E_{2\,1} \in \Gamma_{\mathfrak{sl}_{n+1}\downarrow\mathfrak{gl}_{n-1}}(\varpi_1)$. It remains to take care of the second copy of the trivial representation, which one obtains by applying either $F_{n,-n}$ or $F_{k,-n}F_{n,k}$ with $1 \leqslant k < n$ to $\boldsymbol{v}_1$. The smallest monomial here is $F_{n-1,-n}F_{n,n-1}$. Since $F_{n-1,-n}$ is mapped to $E_{n+1\,2}$, we see that the image of $\Gamma(\varpi_1)$ is exactly $\Gamma_{\mathfrak{sl}_{n+1}\downarrow\mathfrak{gl}_{n-1}}(\varpi_1)$.

Take next ϖ_k with $2 \leqslant k < n$. Here $|\Gamma(\varpi_k)| = 4$. The monomials of degree 1 in $\Gamma(\varpi_k)$ are $F_{k-n,-n}$ and $F_{n,k-n-1}$. The root vector $F_{n,-n}$ does not appear, because it can be replaced by $F_{n,k-n-1}F_{k-n-1,-n}$, which is smaller. We have also $F_{i,-n}\boldsymbol{v}_k = 0$ if $i < k-n-1$. Therefore, in degree 2 we have to choose the smallest monomial among $F_{n,t}F_{t,-n}$ with $k-n-1 \leqslant t \leqslant -1$. This is exactly $F_{n,k-n-1}F_{k-n-1,-n}$. Therefore the image of $\Gamma(\varpi_k)$ is $\Gamma_{\mathfrak{sl}_{n+1}\downarrow\mathfrak{gl}_{n-1}}(\varpi_k)$.

Finally take ϖ_n, where we have $\widetilde{\Gamma}(\varpi_n) = \{1, F_{n,-n}, F_{1,-n}\}$. Note that $F_{n,-n}$ is mapped to $E_{n+1\,1}$ and $F_{1,-n}$ to $E_{n+1\,n}$. This finishes the proof. $\square$

If $\lambda = \sum\limits_{k=1}^{n} c_k\varpi_k$ is presented by a tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ as in the Introduction, cf. (0·5), then $c_1 = \lambda_{n-1} - \lambda_n$, for $2 \leqslant k < n$, we have $c_k = \lambda_{n-k} - \lambda_{n-k-1}$, and $c_n = -\lambda_1$. Consistently, we write $\mu = (\mu_1, \ldots, \mu_{n-1})$ with $0 \geqslant \mu_1 \geqslant \ldots \geqslant \mu_{n-1}$. Taking this into account and using bijections between the branching semigroups and the corresponding patterns (Gelfand–Tsetlin patterns and type C patterns), we obtain the following statement.

Corollary 2.4. *The vector space $V(\lambda)^+_\mu$ has a basis*

$$\left\{ pF_{n,-n}^{-\nu_1}F_{n,-n+1}^{\mu_{n-1}-\nu_n}F_{n,n-1}^{\lambda_{n-1}-\nu_n}\ldots F_{n,-k}^{\mu_k-\nu_{k+1}}F_{n,k}^{\lambda_k-\nu_{k+1}}\ldots F_{n,-1}^{\mu_1-\nu_2}F_{n,1}^{\lambda_1-\nu_2}v_\lambda \right\},$$

parameterised by the n-tuples $\nu = (\nu_1, \ldots, \nu_n)$ satisfying the betweenness conditions

$$\begin{aligned} &0 \geqslant \nu_1 \geqslant \lambda_1 \geqslant \nu_2 \geqslant \lambda_2 \geqslant \cdots \geqslant \nu_{n-1} \geqslant \lambda_{n-1} \geqslant \nu_n \geqslant \lambda_n, \\ &0 \geqslant \nu_1 \geqslant \mu_1 \geqslant \nu_2 \geqslant \mu_2 \geqslant \cdots \geqslant \nu_{n-1} \geqslant \mu_{n-1} \geqslant \nu_n. \end{aligned} \tag{2·2}$$

Going inductively through the chain of subalgebras

$$\mathfrak{sp}_2 \subset \ldots \subset \mathfrak{sp}_{2n-2} \subset \mathfrak{sp}_{2n} \tag{2·3}$$

and using Proposition 1.4 at each step, we obtain the basis of Theorem B. The chain defines also the branching semigroup $\tilde{\Gamma}_{\mathfrak{sp}_{2n}\downarrow\{0\}}$, where the order of Definition 2.1 is used at each step.

Remark 2.5. Arguing inductively, one shows that $\tilde{\Gamma}_{\mathfrak{sp}_{2n}\downarrow\{0\}}$ is generated by $\tilde{\Gamma}_{\mathfrak{sp}_{2n}\downarrow\{0\}}(\varpi_k)$ with $1 \leqslant k \leqslant n$. The saturation property can be checked inductively as well. It holds at each step because of the bijection between $\tilde{\Gamma}_{\mathfrak{sp}_{2n}\downarrow\mathfrak{sp}_{2n-2}}$ and $\Gamma_{\mathfrak{sl}_{n+1}\downarrow\mathfrak{sl}_{n-1}}$. Therefore $\tilde{\Gamma}_{\mathfrak{sp}_{2n}\downarrow\{0\}}$ is saturated. In this situation, there is a nice toric degeneration of the complete flag variety [7, Sec. 10], [4, Sec. 15].

2.2. **A different, more natural, order.** In this section, we use different indices for the matrix realisation of $\mathfrak{g} = \mathfrak{sp}_{2n}$. Now $\mathfrak{g}$ is the linear span of $F_{i\,j}$ with $i, j \in \{1, \ldots, 2n\}$, where

$$F_{i\,j} = E_{i\,j} - \varepsilon_i\,\varepsilon_j\,E_{j'\,i'}, \qquad i' = 2n - i + 1, \tag{2·4}$$

$\varepsilon_i = 1$ for $i \leqslant n$, and $\varepsilon_i = -1$ for $i > n$. The subalgebra $\mathfrak{g}_0 = \mathfrak{sp}_{2n-2}$ is spanned by the elements $F_{i\,j}$ with $i, j \in \{2, \ldots, 2n-1\}$. We have $\mathfrak{r} = \langle F_{2n\,k} \mid 1 \leqslant k < 2n\rangle_{\mathbb{C}}$.

Fix highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ for $\mathfrak{g}$ and $\mu = (\mu_2, \ldots, \mu_n)$ for $\mathfrak{g}_0$, where we suppose that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0$ and $\mu_2 \geqslant \mu_3 \geqslant \ldots \geqslant \mu_n \geqslant 0$, and that the multiplicity space $U(\lambda, \mu)$ is nonzero.

Set $a_i = |\lambda_i - \mu_i|$ for $i \geqslant 2$ and define the monomial $Y(\mu) = y_n^{a_n} \ldots y_2^{a_2}$ by the rule:

$$y_i = F_{i\,1} \ \text{ if } \ \lambda_i \leqslant \mu_i \quad \text{ and } \quad y_i = F_{2n\,i} \ \text{ if } \ \lambda_i > \mu_i.$$

Now use a non-zero vector $\xi_\mu \in V(\lambda)^+_\mu$ defined in the formula [16, (9.69)]. Writing the formula in our notation, we get

$$F_{1\,1}\xi_\mu = \Big(\lambda_1 - \sum_{i=2}^{n}(2\max(\lambda_i, \mu_i) - \lambda_i - \mu_i)\Big)\xi_\mu = \Big(\lambda_1 - \sum_{i=2}^{n} a_i\Big)\xi_\mu.$$

Hence, the $\mathfrak{h}$-weight of ξ_μ coincides with that of the vector $Y(\mu)v_\lambda$.

Remark 2.6. By the weight considerations, we have $\xi_\mu = pY(\mu)v_\lambda$ and at the same time $\xi_\mu = \prod_{i=2}^{n}(py_i)^{a_i}v_\lambda$, up to a non-zero scalar factor.

We would like to find inequalities for $b, b_n, \ldots, b_2$ such that the corresponding vectors

$$pF_{2n\,1}^{b}(F_{2n\,n}F_{n\,1})^{b_n} \ldots (F_{2n\,2}F_{2\,1})^{b_2}Y(\mu)v_\lambda$$

form a basis of $V(\lambda)^+_\mu$. For this purpose, the most natural monomial order on $\mathcal{S}(\mathfrak{r})$ is suitable.

For a vector $\bar{\alpha} = (\alpha_2, \ldots, \alpha_{2n-1})$, set $|\bar{\alpha}| = \sum_{k=2}^{2n-1}\alpha_k$.

Definition 2.7. We say that $F_{2n\,1}^{\alpha_{2n}} F_{2n\,2}^{\alpha_2} \dots F_{2n\,2n-1}^{\alpha_{2n-1}} < F_{2n\,1}^{\beta_{2n}} F_{2n\,2}^{\beta_2} \dots F_{2n\,2n-1}^{\beta_{2n-1}}$ if and only if either $|\bar\alpha| < |\bar\beta|$ or $|\bar\alpha| = |\bar\beta|$ and there is k such that $2 \leqslant k \leqslant 2n$ and

$$\alpha_k < \beta_k, \ \ \alpha_i = \beta_i \ \text{ for all } \ i < k.$$

A few remarks on the definition are due.
(1) Since we are comparing the degrees first, the sequence of factors of $m_1 \in \mathcal{U}(\mathfrak{r})$ is not significant for being essential.
(2) The chosen order satisfies (1·3). Therefore, by Proposition 1.1, the subspace $V(\lambda)^+$ has a basis $\{pm_1 v_\lambda \mid m_1 \in \mathcal{E}s(\lambda) \cap \mathcal{S}(\mathfrak{r})\}$.

Lemma 2.8. *We have*

$$\begin{aligned} &\Gamma(\varpi_1) = \{1, F_{2n\,1}, F_{2\,1}\}, \\ &\Gamma(\varpi_k) = \{1, F_{2n\,1}, F_{2n\,k}, F_{k+1\,1}\} \ \textit{ if } \ 2 \leqslant k < n, \\ &\Gamma(\varpi_n) = \{1, F_{2n\,1}, F_{2n\,n}\}. \end{aligned}$$

Proof. The statements can be obtained by direct calculations. □

The dimension of $U(\lambda,\mu)$ is the product of n positive integers $(d_1+1)\dots(d_n+1)$, where

$$d_i = \min(\lambda_i, \mu_i) - \max(\lambda_{i+1}, \mu_{i+1}),$$

assuming that $\min(\lambda_1,\mu_1) = \lambda_1$ and $\lambda_{n+1} = \mu_{n+1} = 0$; see e.g. [15].

Consider the $\mathfrak{sl}_2$-triple $\{\frac{1}{2}F_{2n\,1}, F_{1\,1}, \frac{1}{2}F_{1\,2n}\}$. The corresponding subalgebra of $\mathfrak{g}$ acts on $U(\lambda,\mu)$, this representation is isomorphic to the tensor product $\mathcal{S}^{d_1}\mathbb{C}^2 \otimes \dots \otimes \mathcal{S}^{d_n}\mathbb{C}^2$. Moreover,

$$\xi_\mu \in V(\lambda)^+_\mu \cong U(\lambda,\mu)$$

is a highest weight vector of this representation and its $F_{1\,1}$-weight is equal to $d_1 + \dots + d_n$.

For a vector $Y = y_n^{a_n} \dots y_2^{a_2}$, where each y_i is either $F_{i\,1}$ or $F_{2n\,i}$, set

$$\iota_i = \begin{cases} 0 & \text{if } y_i = F_{i\,1}, \\ 1 & \text{if } y_i = F_{2n\,i}. \end{cases}$$

This defines a vector $\bar\iota = (\iota_2, \dots, \iota_n)$, which depends on Y. Set $\iota_1 = 0$ and $a_{n+1} = 0$.

We have

$$\lambda = (\lambda_1 - \lambda_2)\varpi_1 + \dots + (\lambda_{n-1} - \lambda_n)\varpi_{n-1} + \lambda_n \varpi_n.$$

Set $c_n = \lambda_n$ and $c_k = (\lambda_k - \lambda_{k+1})$ for $k < n$. Suppose that $\xi_\mu = pYv_\lambda \neq 0$ for some Y as above. It is not difficult to see then that

$$d_k = c_k - \iota_k a_k - (1 - \iota_{k+1}) a_{k+1} \tag{2·5}$$

for each $k \geqslant 2$. Informally speaking, each y_i in Y decreases c_k by 1 if $y_i \in \Gamma(\varpi_k)$. More formally, if $y_i \in \Gamma(\varpi_k)$, then $a_i \leqslant c_k$ and therefore $y_i^{a_i} \in \Gamma(\varpi_k)^{c_k} \subset \Gamma(c_k\varpi_k)$. Thus,

$Y \in \Gamma(\lambda)$. Note that Equation (2·5) defines the numbers $d_k = d_k(Y)$ for each vector Y as above.

The next step is to consider $\Gamma(\varpi_k{+}\varpi_j)$ with $k \neq j$.

Lemma 2.9. *Suppose that $j > k$ and $\lambda = \varpi_k + \varpi_j$. Then*

$$\mathcal{E}s(\lambda) \cap \mathcal{S}(\mathfrak{r})^{\mathfrak{h}_0} = \{1, F_{2n\,1}, F_{2n\,1}^2, F_{2n\,j}F_{j\,1}\}.$$

Proof. Set $\mu = \lambda|_{\mathfrak{h}_0}$. Then $\dim U(\lambda,\mu) = 4$. As a representation of $\mathfrak{sl}_2 = \langle F_{2n\,1}, F_{1\,1}, F_{1\,2n}\rangle_{\mathbb{C}}$, it decomposes as $\mathbb{C}^3 \oplus \mathbb{C}$. Since $F_{2n\,1} \in \Gamma(\varpi_i)$ for each i, we have $F_{2n\,1}, F_{2n\,1}^2 \in \mathcal{E}s(\lambda)$. It remains to show that $F_{2n\,j}F_{j\,1}$ is essential. In the case $k = j-1$, this follows form the inclusion (1·1) and Lemma 2.8. Therefore suppose that $k < j-1$. Then $\dim V(\lambda)^+ = 16$ and $\Gamma(\lambda)$ is the disjoint union of three subsets, $X = \{1, F_{2n\,1}, F_{2n\,k}, F_{2n\,j}, F_{k+1\,1}, F_{j+1\,1}\}$, the product $F_{2n\,1}X$, and the subset

$$\{F_{2n\,k}F_{2n\,j}, F_{2n\,k}F_{j+1\,1}, F_{k+1\,1}F_{2n\,j}, F_{k+1\,1}F_{j+1\,1}, x\},$$

where $pxv_\lambda \in V(\lambda)^+_\mu$ and the $F_{1\,1}$-weight of x is -2. Since $F_{2n\,1} \in X$, these two conditions on x imply that $x = F_{2n\,t}F_{t\,1}$ for some $t \leqslant n$.

First we show that $t \leqslant j$. If $j < n$, take $s > j$. Let us regard $V(\varpi_r)$ as $\bigwedge^r \mathbb{C}^n$ of the vector representation $\mathbb{C}^n$ with the standard basis $v_1, \ldots, v_n$. Then $\boldsymbol{v}_r = v_1 \wedge \ldots \wedge v_r$ is a highest weight vector of $V(\varpi_r)$. Set $u = F_{2n\,s}F_{s\,1}(\boldsymbol{v}_k \otimes \boldsymbol{v}_j)$. Then $u = \frac{1}{2}F_{n\,-n}(\boldsymbol{v}_k \otimes \boldsymbol{v}_j) + u'$, where

$$u' = (v_s \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_{s'} \wedge v_2 \wedge \ldots \wedge v_j) + (v_{s'} \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_s \wedge v_2 \wedge \ldots \wedge v_j).$$

Here $s' > n \geqslant s$ and $u' = \frac{1}{2}F_{s'\,s}\tilde{u}$ for

$$\tilde{u} = (v_s \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_s \wedge v_2 \wedge \ldots \wedge v_j).$$

Thereby $pu' = 0$ by (1·2), hence $pu = \frac{1}{2}F_{2n\,1}(\boldsymbol{v}_k \otimes \boldsymbol{v}_j)$ and $F_{s\,1}F_{2n\,s}$ is not essential for $\varpi_k + \varpi_j$. We have shown that $x \geqslant F_{j\,1}F_{2n\,j}$.

Assume that $F_{j\,1}F_{2n\,j}$ is not essential. Then $w = F_{j\,1}F_{2n\,j}(\boldsymbol{v}_k \otimes \boldsymbol{v}_j)$ lies in the linear span of smaller than $F_{j\,1}F_{2n\,j}$ essential monomials. Each such monomial is of the form m_0m_1, where m_1 has weight -2 w.r.t. F_{11} and $m_1 < F_{j\,1}F_{2n\,j}$. This is possible only for $F_{2n\,1}$, $F_{2n\,j}F_{k+1\,1}$, and $F_{j+1\,1}F_{k+1\,1}$.

The decomposition $V(\varpi_1) = \mathbb{C}v_1 \oplus V'(\varpi_1) \oplus \mathbb{C}v_{2n}$ leads to a $\mathfrak{g}_0$-invariant tri-grading on each $V(\varpi_r)$. In the tensor product $V(\varpi_k) \otimes V(\varpi_j)$, the vector $F_{2n\,j}F_{j\,1}(\boldsymbol{v}_k \otimes \boldsymbol{v}_j)$ has non-zero summands of degrees

$$(0, k-1, 1; 1, j-1, 0), \;\; (0, k, 0; 1, j-2, 1), \;\; (0, k, 0; 0, j, 0).$$

The monomials $F_{2n\,j}F_{k+1\,1}$ and $F_{j+1\,1}F_{k+1\,1}$ produce vectors of degrees

$$(0, k, 0; 1, j-2, 1), \;\; (0, k, 0; 0, j, 0), \;\; \text{and} \;\; (0, k, 0; 0, j, 0).$$

This implies that the summand of degree $(0, k-1, 1; 1, j-1, 0)$, which is equal to

$$w = (v_{2n} \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_1 \wedge \ldots \wedge v_j),$$

is written as $am_0 F_{2n\,1}(\boldsymbol{v}_k \otimes \boldsymbol{v}_j)$ for some $a \in \mathbb{C}$ and $m_0 \in \mathcal{U}(\mathfrak{n}_0^-)$. However, $F_{2n\,1}(\boldsymbol{v}_k \otimes \boldsymbol{v}_j) = 2(w + \tilde{w})$, where $\tilde{w} \neq 0$ is of degree $(1, k-1, 0; 0, j-1, 1)$. This contradiction finishes the proof. $\square$

Proposition 2.10. (i) *The defining inequalities for* $\Gamma(\lambda)$ *in terms of*

$$F_{2n\,1}^{b}(F_{2n\,n}F_{n\,1})^{b_n} \ldots (F_{2n\,2}F_{2\,1})^{b_2} y_n^{a_n} \ldots y_2^{a_2}$$

are:

$$0 \leqslant d_k, \quad \text{where the numbers } d_k \text{ are given by (2·5)}, \tag{2·6}$$

$$b_k \leqslant d_k, \tag{2·7}$$

$$b_k \leqslant d_1 + \sum_{i=2}^{k-1}(d_i - 2b_i) \ \text{ for each } k \text{ such that } 2 \leqslant k \leqslant n; \tag{2·8}$$

$$b + 2\sum_{k=2}^{n} b_k \leqslant \sum_{i=1}^{n} d_i. \tag{2·9}$$

(ii) *The semigroup* Γ *is generated by* $\Gamma(\varpi_t)$ *and* $\Gamma(\varpi_k + \varpi_j)$ *with* $1 \leqslant t, k, j \leqslant n$ *and* $j > k+1$.

Proof. (i) The inequalities (2·6) are equivalent to $U(\lambda, \mu) \neq 0$, where μ is the $\mathfrak{h}_0$-weight of $y_n^{a_n} \ldots y_2^{a_2} v_\lambda$. Each weight μ such that $U(\lambda, \mu) \neq 0$ defines the tuple $\bar{a} = (a_2, \ldots, a_n)$ uniquely. Let $\bar{a}$ be fixed.

Next we show that the number of tuples $(b, b_n, \ldots, b_2) \in \mathbb{Z}_+^n$ satisfying the inequalities (2·7)–(2·9) is equal to $\prod_{i=1}^{n}(d_i + 1) = \dim U(\lambda, \mu)$. We argue by induction on n. If $n = 1$, then there is just one inequality $b \leqslant d_1$. There are $d_1 + 1$ possibilities for b.

Suppose that $n = 2$. Then $b_2 \leqslant d_1, d_2$. Each admissible b_2 corresponds to the irreducible $\mathfrak{sl}_2$-submodule of $\mathcal{S}^{d_1}\mathbb{C}^2 \otimes \mathcal{S}^{d_2}\mathbb{C}^2$ of dimension $d_1 + d_2 + 1 - 2b_2$. If b_2 is fixed, then there are exactly $d_1 + d_2 - 2b_2 + 1$ possibilities for b. For $n = 2$, the number of tuples (b, b_2) is correct.

Suppose now that $n > 2$ and that for $n - 1$ there is a bijection between the tuples $\bar{b} = (b_2, \ldots, b_{n-1})$ satisfying the inequalities and the irreducible $\mathfrak{sl}_2$-submodules of

$$\mathcal{S}^{d_1}\mathbb{C}^2 \otimes \ldots \otimes \mathcal{S}^{d_{n-1}}\mathbb{C}^2$$

such that the module $V(\bar{b})$ corresponding to $\bar{b}$ is of dimension

$$\sum_{i=1}^{n-1} d_i + 1 - 2\sum_{i=2}^{n-1} b_i.$$

The irreducible submodules of $V(\bar{b}) \otimes \mathcal{S}^{d_n}\mathbb{C}^2$ can be enumerated by integers b_n such that

$$0 \leqslant b_n \leqslant \min(d_n, \dim V(\bar{b}) - 1).$$

We can arrange the submodules in such a way that the dimension decreases when b_n increases. Then b_n, or rather $(b_2, \ldots, b_{n-1}, b_n)$, corresponds to the summand of dimension

$$\sum_{i=1}^{n} d_i + 1 - 2\sum_{i=2}^{n} b_i.$$

This completes the inductive argument.

In the proof of part (ii) below, we show that each admissible tuple

$$(a_2, \ldots, a_n, b, b_2, \ldots, b_n)$$

defines a monomial of $\Gamma(\lambda)$. Hence by the dimension reasons, (i) holds.

(ii) For convenience, we will identify the monomials $m_1 \in \Gamma(\lambda)$ with the tuples of their exponents and use additive notation for $\Gamma(\lambda)$, so that $\Gamma(\lambda) + \Gamma(\lambda') \subset \Gamma(\lambda + \lambda')$; see (1·1).

Let $(\bar{b}, \bar{a})$ with $\bar{b} = (b, b_2, \ldots, b_n)$, $\bar{a} = (a_2, \ldots, a_n)$ be an admissible tuple. Recall that each y_i belongs to a unique $\Gamma(\varpi_s)$ with $s = s(i)$. Set $\tilde{\lambda} = \lambda - \sum\limits_{i=2}^{n} a_i \varpi_{s(i)}$. In view of (2·5), we have $\tilde{\lambda} = \sum\limits_{i=1}^{n-1} d_i \varpi_i$. The inequalities (2·6) guaranty that $\tilde{\lambda}$ is a dominant weight of $\mathfrak{g}$. If $\bar{b}$, identified with $(\bar{b}, \bar{0})$, lies in $\Gamma(\tilde{\lambda})$, then

$$(\bar{b}, \bar{a}) \in \Gamma(\tilde{\lambda}) + \sum_{i=2}^{n} a_i \Gamma(\varpi_{s(i)}) \subset \Gamma(\lambda).$$

Next we express $\bar{b}$ as a sum of tuples belonging to sets $\Gamma(\varpi_t)$ and $\Gamma(\varpi_k + \varpi_j)$ and show that indeed $\bar{b} \in \Gamma(\tilde{\lambda})$.

If all d_k are zero, then $\bar{b} = 0$ and there is nothing to prove. Suppose next that $d_k \neq 0$ only for $k = i$. Then $b_j = 0$ for all $j \geqslant 2$ and only $F_{2n\,1}^{b}$ with $b \leqslant d_i$ is left. Here $\bar{b} \in d_i\Gamma(\varpi_i)$. The proof continues by induction on $|\bar{b}| = b + b_2 + \ldots + b_n$.

Let $k < r$ be the smallest integers such that $d_r, d_k \neq 0$. Note that $b_2 = \ldots = b_{r-1} = 0$. If all b_i with $i \geqslant 2$ are equal to zero, then $\bar{b} = (b, 0, \ldots, 0)$, where $b \leqslant \sum\limits_{i=1}^{n} d_i$. Again, such $\bar{b}$ belongs to $\sum\limits_{i=1}^{n} d_i \Gamma(\varpi_i) \subset \Gamma(\tilde{\lambda})$. Therefore assume that $\bar{b} \neq (b, 0, \ldots, 0)$.

Let $j \geqslant r$ be the smallest integer such that $b_r \neq 0$. We divide our monomial by $F_{2n\,j}F_{j\,1}$, which is an element of $\Gamma(\varpi_k + \varpi_j)$ by Lemma 2.9. Note that in case $j = k+1$, we have $F_{2n\,j}F_{j\,1} \in \Gamma(\varpi_j)\Gamma(\varpi_k)$. The division corresponds to replacing $\bar{b}$ with $\bar{b}' = (b, b_2', \ldots, b_n')$, where $b_i' = b_i$ for $i \neq j$ and $b_j' = b_j - 1$. Accordingly, set $\lambda' = \tilde{\lambda} - (\varpi_k + \varpi_j)$. We have $\lambda' = \sum\limits_{i=1}^{n} d_i' \varpi_i$, where $d_i' = d_i$ for $i \neq k, j$ and $d_i' = d_i - 1$ for $i \in \{k, j\}$. The next task is to see that the inequalities (2·7)–(2·9) hold for $\bar{b}'$ and λ'.

Consider (2·7). For $i \neq k, j$, we have $b_i' = b_i \leqslant d_i = d_i'$. If $k = 1$, then there is no b_k. If $k \geqslant 2$, then $b_k' = b_k = 0$ and $b_k \leqslant d_k'$. Finally, $b_j' = b_j - 1 \leqslant d_j - 1 = d_j'$. These inequalities hold.

Consider (2·8). For $s < j$, we have $b_s = 0$. Clearly, the inequalities hold for all such s. For the index j, we have

$$b'_j \leqslant \left(\sum_{t=1}^{j-1} d_t\right) - 1 = \sum_{t=1}^{j-1} d'_t = d'_1 + \sum_{t=2}^{j-1}(d'_t - 2b'_t).$$

For $s > j$, the new right hand side $d'_1 + \sum\limits_{t=2}^{s-1}(d'_t - 2b'_t)$ is equal to the old one. Since $b'_s = b_s$ here, all the inequalities hold.

Finally, consider (2·9). We have

$$\sum_{i=1}^{n} d'_i - 2\sum_{t=2}^{n} b'_t = \sum_{i=1}^{n} d_i - 2\sum_{t=2}^{n} b_t.$$

Hence the inequality for b holds.

Summing up, $\bar{b}'$ belongs to $\Gamma(\lambda')$, because $|\bar{b}'| < |\bar{b}|$, and hence $\bar{b}$ belongs to $\Gamma(\tilde{\lambda})$. □

The perspective on $U(\lambda, \mu)$ taken in this section differs from the usual one. In order to obtain a basis, we have regarded $U(\lambda, \nu)$ as a direct sum of $\mathfrak{sl}_2$-modules instead of a tensor product. On the one side, this leads to a more complicated set of inequalities, on the other, we are getting one more basis.

Set $\tilde{p} = p_{\mathfrak{sl}_2} p$, where $p_{\mathfrak{sl}_2}$ is the extremal projector associated with $\mathfrak{sl}_2 = \langle F_{2n\,1}, F_{1\,1}, F_{1\,2n}\rangle_{\mathbb{C}}$ and p is the projector of $\mathfrak{g}_0$ as before. Let us restrict $V(\lambda)$ to $\mathfrak{g}_0 \oplus \mathfrak{sl}_2$.

Corollary 2.11. *The subspace of $V(\lambda)^+ \cap V(\lambda)^{F_{1\,2n}}$ has a basis*

$$\{\tilde{p} m_1 v_\lambda \mid m_1 \in \Gamma(\lambda) \textit{ is given by exponents } (0, b_2, \ldots, b_n, a_2, \ldots, a_n)\}.$$

The chain of subalgebras (2·3) can be used in order to extend the basis of Proposition 2.10 to a basis for $V(\lambda)$.

3. Relations to the Gelfand–Tsetlin and Littelmann bases

We start by recalling a construction of the celebrated basis of Gelfand and Tsetlin [8] for each finite-dimensional irreducible representation $L(\lambda)$ of $\mathfrak{gl}_n$ as defined in the Introduction. We refer the reader to the review paper [15] where several such constructions are discussed and we will follow the notation of that paper.

Consider the extremal projector p associated with the Lie algebra $\mathfrak{gl}_{n-1}$. Recall that the *Mickelsson–Zhelobenko algebra* $\mathrm{Z}(\mathfrak{gl}_n, \mathfrak{gl}_{n-1})$ is generated by the elements E_{nn}, pE_{in} and pE_{ni} with $i = 1, \ldots, n-1$; see [15, Sec. 2.3] for the definitions. The *lowering operators* are elements of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ which can be defined by the formulas

$$z_{nk} = pE_{nk}\,(h_k - h_{k+1}) \cdots (h_k - h_{n-1}), \tag{3·1}$$

where $h_k = E_{kk} - k + 1$. By the branching rule, the restriction of $L(\lambda)$ to the subalgebra $\mathfrak{gl}_{n-1}$ is isomorphic to the direct sum of irreducible $\mathfrak{gl}_{n-1}$-modules $L'(\mu)$,

$$L(\lambda)|_{\mathfrak{gl}_{n-1}} \simeq \underset{\mu}{\oplus}\, L'(\mu),$$

summed over the highest weights $\mu = (\mu_1, \dots, \mu_{n-1})$ satisfying the betweenness conditions

$$\lambda_i - \mu_i \in \mathbb{Z}_+ \quad \text{and} \quad \mu_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \dots, n-1. \tag{3·2}$$

The $\mathfrak{gl}_{n-1}$-submodule in $L(\lambda)$ isomorphic to $L'(\mu)$ is generated by the vector

$$\xi_\mu = z_{n1}^{\lambda_1 - \mu_1} \dots z_{n\, n-1}^{\lambda_{n-1} - \mu_{n-1}}\, \xi.$$

In the next lemma we suppose that the highest weights μ and μ' satisfy conditions (3·2) and we use the lexicographical ordering $\succ$ on such weights, where for complex numbers a and b we assume that $a \geqslant b$ if and only if $a - b \in \mathbb{Z}_+$.

Lemma 3.1. *For any given μ, in the module $L(\lambda)$ we have*

$$E_{n1}^{\lambda_1 - \mu_1} \dots E_{n\, n-1}^{\lambda_{n-1} - \mu_{n-1}}\, \xi = c\, \xi_\mu + \sum_{\mu' \succ \mu} u(\mu')\, \xi_{\mu'}$$

for a nonzero constant c and some elements $u(\mu') \in \mathrm{U}(\mathfrak{n}_0^-)$, where the sum is taken over the highest weights μ' satisfying conditions (3·2).

Proof. Starting from the rightmost generator which occurs in the product on the left hand side and proceeding to the left, we use the *inversion formula*

$$E_{nk} = pE_{nk} + \sum_{k < k_1 < \dots < k_s < n} E_{k_1 k} E_{k_2 k_1} \dots E_{k_s k_{s-1}}\, pE_{n k_s} \frac{1}{(h_{k_s} - h_k)(h_{k_s} - h_{k_1}) \cdots (h_{k_s} - h_{k_{s-1}})},$$

summed over $s = 1, 2, \dots$. Arguing by induction, observe that each generator E_{nl} with $l \leqslant k$ commutes with all factors $E_{k_1 k}, E_{k_2 k_1}, \dots, E_{k_s k_{s-1}}$ so that the proof is completed by using (3·1) and taking into account the fact that the lowering operators z_{nk} pairwise commute. $\square$

The vectors ξ_Λ of the Gelfand–Tsetlin basis of $L(\lambda)$ are parameterised by the patterns Λ defined in the Introduction. They are found by the formula

$$\xi_\Lambda = \overrightarrow{\prod_{k=2,\dots,n}} \Big(z_{k1}^{\lambda_{k1} - \lambda_{k-1\, 1}} \dots z_{k\, k-1}^{\lambda_{k\, k-1} - \lambda_{k-1\, k-1}} \Big)\, \xi. \tag{3·3}$$

Represent each pattern Λ associated with λ as the sequence of its rows:

$$\Lambda = (\bar\lambda_{n-1}, \dots, \bar\lambda_1), \qquad \bar\lambda_k = (\lambda_{k1}, \dots, \lambda_{kk}),$$

and consider the lexicographical ordering $\succ$ on the sequences by using the ordering on the highest weights introduced above. Recall the vectors π_Λ defined in Theorem A. We now obtain another proof of this theorem.

Proposition 3.2. *For each pattern Λ associated with λ, in the module $L(\lambda)$ we have*

$$\pi_\Lambda = \sum_{\Lambda' \succcurlyeq \Lambda} c_{\Lambda,\Lambda'}\,\xi_{\Lambda'}$$

for some constants $c_{\Lambda,\Lambda'}$, and $c_{\Lambda,\Lambda} \neq 0$. In particular, π_Λ is a basis of $L(\lambda)$.

Proof. Due to the inductive structure of the vectors (3·3), the proposition follows by a repeated application of Lemma 3.1. □

3.1. **Monomials in simple root vectors.** For any $\mathfrak{g}$, there is a way, that also involves branching, to produce a basis of $V(\lambda)$ by applying iterated negative simple root vectors to v_λ, see [11] and also [4, Sec. 11] for a connection with the FFLV-method. In type A, the construction is most transparent [10], [11, Sec. 5&10].

Set $f_k = E_{k+1\,k}$. The subspace $L(\lambda)^+$ is the linear span of vectors $pf_{n-1}^{a_{n-1}} \dots f_1^{a_1} v_\lambda$, where $a_{n-1} \geqslant a_{n-2} \geqslant \dots \geqslant a_1$. Set $a_0 = 0$. By the weight considerations,

$$\xi_\mu = pf_{n-1}^{a_{n-1}} \dots f_1^{a_1} v_\lambda \;\text{ with }\; a_k - a_{k-1} = \lambda_k - \mu_k,$$

up to a non-zero scalar. In view of the equality

$$[f_{n-1}, [f_{n-2}, \dots, [f_{k+1}, f_k] \dots]] = E_{n\,k},$$

we can conclude directly, without weight arguments, that

$$pf_{n-1}^{a_{n-1}} \dots f_1^{a_1} v_\lambda = pE_{n\,n-1}^{a_{n-1}-a_{n-2}} \dots E_{n\,1}^{a_1} v_\lambda\,.$$

A basis of $L(\lambda)$ is obtained inductively, omitting extremal projectors, so that the basis vectors have the form

$$\boldsymbol{f}(\Lambda) = f_1^{a_{n-1,1}} f_2^{a_{n-2,2}} f_1^{a_{n-2,1}} \dots f_{n-1}^{a_{1,n-1}} \dots f_1^{a_{1,1}} v_\lambda$$

and are naturally parameterised by the Gelfand–Tsetlin patterns, see [11, Corollary 5]. In the notation of (0·3),

$$a_{k,j} = \sum_{i=1}^{j} (\lambda_{n-k+1\,i} - \lambda_{n-k\,i}). \tag{3·4}$$

Let $m(\Lambda)$ be the leading term of $\boldsymbol{f}(\Lambda)$ in the monomial order used in Section 1.4. Then, $m(\Lambda)v_\lambda = \pi_\Lambda$. Hence

$$\boldsymbol{f}(\Lambda) \in \pi_\Lambda + \langle m v_\lambda \mid m < m(\Lambda) \rangle_{\mathbb{C}} = c_{\Lambda,\Lambda}\xi_\Lambda + \langle m v_\lambda \mid m < m(\Lambda) \rangle_{\mathbb{C}}$$

with a non-zero $c_{\Lambda,\Lambda} \in \mathbb{C}$. Therefore, the transition matrices between all three bases are triangular.

Remark 3.3. Let $\succ$ be the lexicographical order on $\mathbb{Z}^N$. Choose the enumeration of the basis vectors $\boldsymbol{f}(\Lambda)$ is such a way that the corresponding sequences

$$\bar{a} = \bar{a}(\Lambda) = (a_{n-1,1}, a_{n-2,2}, a_{n-2,1} \dots, a_{1,1}),$$

see (3·4), are decreasing. Then the transition matrix between $\{\boldsymbol{f}(\Lambda)\}$ and the canonical basis of $L(\lambda)$ is upper triangular and unipotent by [11, Prop. 10.3]. Refining the above considerations, one can show that

$$c_{\Lambda,\Lambda}\xi_\Lambda \in \boldsymbol{f}(\Lambda) + \langle \boldsymbol{f}(\Lambda') \mid \bar{a}(\Lambda') \succ \bar{a} \rangle_{\mathbb{C}}\,.$$

Therefore also the transition matrix between the canonical and the Gelfand–Tsetlin bases is triangular. We will give a different proof of this fact below.

3.2. **The PBW-parameterisation of the canonical basis.** The bases of Littelmann [11] arise as parameterisations of the canonical basis for $V(\lambda)$ constructed by Lusztig [12, 13]. A different approach to this problem leads to PBW-type bases.

Let $\omega_0 = s_{i_1} \dots s_{i_N}$ be a reduced decomposition of the longest element $\omega_0 \in W(\mathfrak{g}, \mathfrak{h})$ of the Weyl group. Define the sequence of positive roots $\beta_1, \dots, \beta_N$ by $\beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$, where α_r is the rth simple root. Then $\beta_t \neq \beta_k$ for $k \neq t$, see e.g. [4, Sec. 12]. Let f_k be the negative root vector corresponding to β_k. Make use of the *right opposite* lexicographical order on the monomials $f_1^{a_1} \dots f_N^{a_N}$, which means that $f_1^{a_1} \dots f_N^{a_N} < f_1^{a_1'} \dots f_N^{a_N'}$ if and only if there is k such that $1 \leqslant k \leqslant N$ and

$$a_k > a_k', \;\; a_r = a_r' \;\text{ for }\; r > k.$$

Use the same sequence of vectors for the elements of $\mathcal{U}(\mathfrak{n}^-)$. Then the elements of the canonical basis for $V(\lambda)$ are in bijection with $\mathcal{E}s(\lambda)$. Moreover, if the element $B(m)v_\lambda$ of the canonical basis corresponds to $m \in \mathcal{E}s(\lambda)$, then

$$B(m)v_\lambda \in mv_\lambda + \langle \tilde{m}v_\lambda \mid \tilde{m} < m \rangle_{\mathbb{C}}\,, \tag{3·5}$$

see e.g. [4, Sec. 12]. Note that we have omitted the "hight weighted function Ψ" of [4] on the monomials, because it becomes redundant once one fixes a finite-dimensional module $V(\lambda)$.

Example 3.4. Let $\mathfrak{g}$ be of type A_{n-1}. Choose the decomposition $\omega_0 = s_1 s_2 s_1 \dots s_{n-1} \dots s_2 s_1$. Then

$$f_1 \dots f_N = E_{2\,1}E_{3\,1}E_{3\,2} \dots E_{n\,1}E_{n\,2} \dots E_{n\,n-1}.$$

The right opposite lexicographical order satisfies the assumptions of Section 1.2 at each step of the reductions along the Gelfand–Tsetlin chain of subalgebras. Therefore, we get the basis of Theorem A, which is also the basis obtained in [18].

Corollary 3.5. *For each dominant λ, the transition matrix between the canonical and the Gelfand–Tsetlin bases of $L(\lambda)$ is triangular.*

Proof. If we use the right opposite lexicographical order as above, then (1·5) holds for $\mathfrak{g}_0 = \mathfrak{gl}_{n-1}$. In view of this and (3·5), we can conclude that $c_{\Lambda,\Lambda}\xi_\Lambda$ and $B(m)v_\lambda$, where $m = m(\Lambda)$ is given by Λ as in Theorem A, have the same leading term, namely, mv_λ. $\square$

Remark 3.6. In the case $n = 3$ the canonical basis is monomial [12, Example 3.4] so that this particular case of Corollary 3.5 follows by a simple calculation with the use of the Gelfand–Tsetlin formulas.

Outside type A, these PBW-type bases become less transparent, see e.g. [18].

Example 3.7. Let $\mathfrak{g}$ be of type C_n. Choose the decomposition

$$\omega_0 = s_n s_{n-1} s_n s_{n-1} \dots s_1 s_2 \dots s_{n-1} s_n s_{n-1} \dots s_2 s_1 \,.$$

Then $f_k \in \mathfrak{sp}_{2n-2}$ for $k \leqslant N - 2n + 1$ and

$$f_{N-2n+2} \dots f_N = F_{2n\,2} \dots F_{2n\,n} F_{2n\,1} F_{2n\,n+1} \dots F_{2n\,2n-1} \,.$$

It is not difficult to see that such a choice produces a branching semigroup related to $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ and that this semigroup is the same as in Section 2.2.

4. A Gelfand–Tsetlin-type basis for representations of $\mathfrak{sp}_{2n}$

We now aim to prove an analogue of Proposition 3.2 for the symplectic Lie algebra $\mathfrak{sp}_{2n}$. The vectors θ_Λ defined in Theorem B turn out to be related to a certain modification of the basis of [14]. In this section we will rely on the exposition in [16, Ch. 9] to produce this modification.

Given a type C pattern Λ associated with λ, as defined in the Introduction, set

$$l_{k\,i} = \lambda_{k\,i} - i - \frac{1}{2}, \qquad l'_{k\,i} = \lambda'_{k\,i} - i + \frac{1}{2}. \tag{4·1}$$

Theorem 4.1. *The $\mathfrak{sp}_{2n}$-module $V(\lambda)$ admits a basis ζ_Λ parameterised by the type* C *patterns Λ associated with λ such that the action of generators of $\mathfrak{sp}_{2n}$ in the basis is given by the formulas*

$$F_{k\,k}\,\zeta_\Lambda = \left(\sum_{i=1}^{k} \lambda_{k\,i} + \sum_{i=1}^{k-1} \lambda_{k-1\,i} - 2\sum_{i=1}^{k} \lambda'_{k\,i}\right)\zeta_\Lambda,$$

$$F_{k,-k}\,\zeta_\Lambda = \sum_{i=1}^{k} A_{k\,i}\,\zeta_{\Lambda-\delta'_{k\,i}}, \qquad F_{-k,k}\,\zeta_\Lambda = \sum_{i=1}^{k} B_{k\,i}\,\zeta_{\Lambda+\delta'_{k\,i}},$$

$$F_{k-1,-k}\,\zeta_\Lambda = -\sum_{i=1}^{k-1} C_{k\,i}\,\zeta_{\Lambda+\delta_{k-1\,i}} - \sum_{i=1}^{k}\sum_{j,m=1}^{k-1} D_{k\,i\,j\,m}\,\zeta_{\Lambda-\delta'_{k\,i}-\delta_{k-1\,j}-\delta'_{k-1\,m}},$$

where

$$A_{k\,i}=\prod_{a=1,\,a\neq i}^{k}\frac{1}{l'_{k\,a}-l'_{k\,i}},$$

$$B_{k\,i}=2\,A_{k\,i}\,\big(2\,l'_{k\,i}+1\big)\prod_{a=1}^{k}\big(l_{k\,a}-l'_{k\,i}\big)\prod_{a=1}^{k-1}\big(l_{k-1\,a}-l'_{k\,i}\big),$$

$$C_{k\,i}=\frac{1}{2\,l_{k-1\,i}+1}\prod_{a=1,\,a\neq i}^{k-1}\frac{1}{\big(l_{k-1\,i}-l_{k-1\,a}\big)\big(l_{k-1\,i}+l_{k-1\,a}+1\big)},$$

and

$$D_{k\,i\,j\,m}=A_{k\,i}A_{k-1\,m}C_{k\,j}\prod_{a=1,\,a\neq i}^{k}\big(l^2_{k-1\,j}-l'^2_{k\,a}\big)\prod_{a=1,\,a\neq m}^{k-1}\big(l^2_{k-1\,j}-l'^2_{k-1\,a}\big).$$

The arrays $\Lambda\pm\delta_{k\,i}$ and $\Lambda\pm\delta'_{k\,i}$ are obtained from Λ by replacing $\lambda_{k\,i}$ and $\lambda'_{k\,i}$ by $\lambda_{k\,i}\pm1$ and $\lambda'_{k\,i}\pm1$ respectively. The vector ζ_Λ is considered to be zero if the array Λ is not a pattern.

Proof. The proof is not essentially different from that of [16, Theorem 9.6.2], so we only point out some key steps and alternative choices made in the arguments.

Suppose that $\mu=(\mu_1,\dots,\mu_{n-1})$ is an $\mathfrak{sp}_{2n-2}$-highest weight. The multiplicity space $V(\lambda)^+_\mu$ is nonzero if and only if the components of λ and μ satisfy the inequalities

$$\lambda_i\geqslant\mu_{i+1},\quad i=1,\dots,n-2\qquad\text{and}\qquad\mu_i\geqslant\lambda_{i+1},\quad i=1,\dots,n-1. \tag{4·2}$$

When it is nonzero, the vector space $V(\lambda)^+_\mu$ carries an irreducible representation of the *twisted Yangian* $\mathrm{Y}(\mathfrak{sp}_2)$. By [16, Theorem 9.4.11], this representation is isomorphic to the tensor product,

$$V(\lambda)^+_\mu\cong L(\alpha_1,\beta_1)\otimes\ldots\otimes L(\alpha_n,\beta_n), \tag{4·3}$$

where

$$\alpha_i=\min\{\lambda_{i-1},\mu_{i-1}\}-i+\frac{1}{2},\qquad\beta_i=\max\{\lambda_i,\mu_i\}-i+\frac{1}{2},$$

assuming that $\lambda_0=\mu_0=0$ and $\max\{\lambda_n,\mu_n\}$ is understood as being equal to λ_n. Each factor $L(\alpha_i,\beta_i)$ is the highest weight $\mathfrak{gl}_2$-module which is extended to the *evaluation module* over the *Yangian* $\mathrm{Y}(\mathfrak{gl}_2)$. The coproduct on the Yangian allows one to equip the tensor product in (4·3) with a $\mathrm{Y}(\mathfrak{gl}_2)$-module structure. This module is then restricted to the subalgebra $\mathrm{Y}(\mathfrak{sp}_2)\subset\mathrm{Y}(\mathfrak{gl}_2)$.

The required modification of the construction relies on [16, Corollary 4.3.5] which implies an alternative isomorphism

$$V(\lambda)^+_\mu\cong L(-\beta_1,-\alpha_1)\otimes\ldots\otimes L(-\beta_n,-\alpha_n). \tag{4·4}$$

Although the tensor products in (4·3) and (4·4) are isomorphic as $\mathrm{Y}(\mathfrak{sp}_2)$-modules, they differ as $\mathrm{Y}(\mathfrak{gl}_2)$-modules. As we shall see below, the use of the alternative isomorphism leads to a different basis of the multiplicity space $V(\lambda)^+_\mu$.

The basis vectors of $V(\lambda)^+_\mu$ will be constructed with the use of the *Mickelsson–Zhelobenko algebra* $\mathrm{Z}(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n-2})$. The *lowering operators* are elements of $\mathrm{Z}(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n-2})$ defined by

$$z_{i,-n} = pF_{i,-n}(f_i - f_{i-1})\dots(f_i - f_{-n+1}), \qquad i = -n+1,\dots,n-1, \tag{4·5}$$

where p is the extremal projector for $\mathfrak{sp}_{2n-2}$, and we set

$$f_i = F_{ii} - i, \qquad f_{-i} = -F_{ii} + i,$$

for all $i = 1,\dots,n$. One more lowering operator $z_{n,-n} \in \mathrm{Z}(\mathfrak{sp}_{2n},\mathfrak{sp}_{2n-2})$ is defined by

$$z_{n,-n} = \sum_{n>i_1>\dots>i_s>-n} F_{n\,i_1}F_{i_1i_2}\cdots F_{i_s,-n}\,(f_n - f_{j_1})\cdots(f_n - f_{j_k}),$$

where $s = 0,1,\dots$ and $\{j_1,\dots,j_k\}$ is the complement to the subset $\{i_1,\dots,i_s\}$ in the set $\{-n+1,\dots,n-1\}$. We will also need an interpolation polynomial $Z_{n,-n}(u)$ with coefficients in the Mickelsson–Zhelobenko algebra given by

$$Z_{n,-n}(u) = F_{n,-n}\prod_{i=-n+1}^{n-1}(u+g_i) - \sum_{i=-n+1}^{n-1} z_{n\,i}\,z_{i,-n}\prod_{j=-n+1,\,j\neq i}^{n-1}\frac{u+g_j}{g_i-g_j}, \tag{4·6}$$

where $g_i = f_i + 1/2$ for all i and we set $z_{n\,i} = (-1)^{n-i}\,z_{-i,-n}$. This polynomial is even in u and has the properties

$$Z_{n,-n}(-g_i) = z_{n\,i}\,z_{i,-n}, \qquad i = -n+1,\dots,n-1, \tag{4·7}$$

and

$$Z_{n,-n}(-g_n) = z_{n,-n}. \tag{4·8}$$

Recall that the dimension of the multiplicity space $V(\lambda)^+_\mu$ equals the number of n-tuples of integers $\nu = (\nu_1,\dots,\nu_n)$ satisfying the *betweenness conditions* (2·2). Let us set

$$\gamma_i = \nu_i - i + \frac{1}{2}, \qquad i = 1,\dots,n. \tag{4·9}$$

The highest vector of the $\mathrm{Y}(\mathfrak{sp}_2)$-module $V(\lambda)^+_\mu$ is given by the formula (it coincides with the vector in [16, (9.69)] up to a sign):

$$\xi_\mu = \prod_{i=1}^{n-1}\Big(z_{n,-i}^{\max\{\lambda_i,\mu_i\}-\lambda_i}\, z_{-i,-n}^{\max\{\lambda_i,\mu_i\}-\mu_i}\Big)\,\xi, \tag{4·10}$$

so that following the proof of [16, Theorem 9.5.1] and using the isomorphism (4·4) instead of (4·3), we find that the vectors

$$\prod_{i=1}^{n} Z_{n,-n}(\gamma_i+1)\dots Z_{n,-n}(\alpha_i-1)\,Z_{n,-n}(\alpha_i)\,\xi_\mu \tag{4·11}$$

with ν satisfying the betweenness conditions form a basis of $V(\lambda)^+_\mu$. By repeating the argument of that proof, we can conclude that the vectors

$$\xi_\nu = \prod_{i=1}^{n-1} z_{n,-i}^{\mu_i-\nu_{i+1}} z_{-i,-n}^{\lambda_i-\nu_{i+1}} \cdot Z_{n,-n}(\gamma_1+1)\dots Z_{n,-n}(\alpha_1)\,\xi \tag{4·12}$$

parameterised by the n-tuples ν satisfying the betweenness conditions form a basis of the multiplicity space $V(\lambda)^+_\mu$.

Taking into account the decomposition (0·1) and applying the same argument to the subalgebras of the chain (2·3), we obtain that the vectors

$$\xi_\Lambda = \prod_{k=1,\dots,n}^{\longrightarrow} \Big(\prod_{i=1}^{k-1} z_{k,-i}^{\lambda_{k-1\,i}-\lambda'_{k\,i+1}} z_{-i,-k}^{\lambda_{k\,i}-\lambda'_{k\,i+1}} \cdot Z_{k,-k}(\lambda'_{k\,1}+1/2)\dots Z_{k,-k}(-1/2)\Big)\,\xi$$

parameterised by all patterns Λ associated with λ form a basis of the representation $V(\lambda)$ of $\mathfrak{sp}_{2n}$. The same calculations as in the proof of [16, Theorem 9.6.2] allow one to get the formulas for the action of the generators of the Lie algebra $\mathfrak{sp}_{2n}$ in the basis ξ_Λ and then write them in terms of the normalised basis vectors

$$\zeta_\Lambda = \prod_{k=2}^{n} \prod_{1\leqslant i<j\leqslant k} \frac{1}{(-l'_{k\,i}-l'_{k\,j}-1)!}\,\xi_\Lambda$$

thus completing the proof. □

Remark 4.2. When written for the basis vectors ξ_Λ, the matrix elements of the generators of $\mathfrak{sp}_{2n}$ provided by Theorem 4.1 and those of [16, Theorem 9.6.2] exhibit the following symmetry: the formal replacements $\Lambda \mapsto -\Lambda$ together with $l_{k\,i} \mapsto -l_{k\,i}$ and $l'_{k\,i} \mapsto -l'_{k\,i}$ transform the matrix elements from one case to the other.

5. Connection between the monomial and Gelfand–Tsetlin-type bases

We will demonstrate that the transition matrix between the basis θ_Λ of the $\mathfrak{sp}_{2n}$-module $V(\lambda)$ provided by Theorem B and the basis ζ_Λ of Theorem 4.1 is triangular.

Using the notation from the previous section, for each n-tuple ν satisfying the betweenness conditions, introduce the vector $\eta_\nu \in V(\lambda)^+_\mu$ by

$$\eta_\nu = \prod_{i=1}^{n-1} z_{n,-i}^{\mu_i-\nu_{i+1}} z_{-i,-n}^{\lambda_i-\nu_{i+1}}\, F_{n,-n}^{-\nu_1}\,\xi. \tag{5·1}$$

By a result of Zhelobenko [19, Theorem 6.1], the vectors η_ν form a basis of $V(\lambda)^+_\mu$. This fact will also follow from a relationship between the vectors ξ_ν and η_ν as described in the next lemma. We will consider the lexicographical orderings $\succ$ on the set of n-tuples ν and on the set of $(n-1)$-tuples μ.

Lemma 5.1. *For any ν we have the relation*

$$\eta_\nu = \sum_{\nu' \succcurlyeq \nu} c_{\nu,\nu'}\, \xi_{\nu'}$$

for some constants $c_{\nu,\nu'}$, and $c_{\nu,\nu} \neq 0$. In particular, the vectors η_ν form a basis of $V(\lambda)^+_\mu$.

Proof. Since $F_{n,-n}$ commutes with the lowering operators $z_{n\,j}$, the vector (5·1) can be written as $\eta_\nu = F_{n,-n}^{-\nu_1}\xi_{\nu^0}$, where ν^0 is the n-tuple obtained from ν by replacing ν_1 with 0. On the other hand, by the formulas of Theorem 4.1 for any ν we have

$$F_{n,-n}\,\xi_\nu = \sum_{i=1}^{n} \prod_{a=1,\, a\neq i}^{n} \frac{1}{\gamma_i^2 - \gamma_a^2}\, \xi_{\nu-\delta_i}.$$

A repeated application of this formula allows us to write $F_{n,-n}^{-\nu_1}\xi_{\nu^0}$ as a linear combination of the basis vectors ξ_ν which clearly has the required form. $\square$

Lemma 5.1 implies that the vectors

$$\eta_\Lambda = \prod_{k=1,\dots,n}^{\longrightarrow} \left(F_{k,-k}^{-\lambda'_{k\,1}} \prod_{i=1}^{k-1} z_{k,-i}^{\lambda_{k-1\,i}-\lambda'_{k\,i+1}}\, z_{-i,-k}^{\lambda_{k\,i}-\lambda'_{k\,i+1}} \right) \xi, \tag{5·2}$$

parameterised by all type C patterns Λ associated with λ form a basis of the representation $V(\lambda)$.

Since the weight μ will now be varied, we will denote the vector (5·1) by $\eta_{\nu\mu}$. The following lemma is essentially a particular case of [19, Theorem 7] or [20, Lemma 2].

Lemma 5.2. *For any given pair (ν,μ) satisfying the betweenness conditions, in the module $V(\lambda)$ we have*

$$F_{n,-n}^{-\nu_1} \prod_{i=1}^{n-1} F_{n,-i}^{\mu_i-\nu_{i+1}} F_{-i,-n}^{\lambda_i-\nu_{i+1}}\, \xi = c\,\eta_{\nu\mu} + \sum_{\nu',\,\mu'} u(\nu',\mu')\,\eta_{\nu'\mu'} \tag{5·3}$$

for a nonzero constant c and some elements $u(\nu',\mu') \in \mathrm{U}(\mathfrak{n}_0^-)$, where the sum is taken over the pairs (ν',μ') satisfying the betweenness conditions, and $u(\nu',\mu') = 0$ unless $\mu' \succ \mu$, or $\mu' = \mu$ and $\nu' \succ \nu$.

Proof. Write the product on the left hand side in the order

$$F_{n,-n+1}^{\mu_{n-1}-\nu_n} \dots F_{n,-1}^{\mu_1-\nu_2}\, F_{n,-n}^{-\nu_1}\, F_{-1,-n}^{\lambda_1-\nu_2} \dots F_{-n+1,-n}^{\lambda_{n-1}-\nu_n}\,\xi.$$

Taking into account that $F_{n,-k} = F_{k,-n}$ for positive values of k, start from the rightmost generator and proceed to the left by using the *inversion formula* [16, Lemma 9.2.2] to replace $F_{i,-n}$ with $i = -n+1,\dots,n-1$ by the expression:

$$F_{i,-n} = pF_{i,-n} + \sum_{i>i_1>\dots>i_s>-n} F_{i\,i_1}F_{i_1 i_2}\dots F_{i_{s-1} i_s}\, pF_{i_s,-n}\frac{1}{(f_{i_s}-f_i)(f_{i_s}-f_{i_1})\dots(f_{i_s}-f_{i_{s-1}})},$$

summed over $s = 1, 2, \ldots$. Apply relation (4·5) to write the right hand side of the inversion formula in terms of the lowering operators $z_{k,-n}$. We will use the following property of these operators: $z_{i,-n}$ and $z_{j,-n}$ commute for $i + j \neq 0$; see [16, Proposition 9.2.5]. Let $\tilde{\mathfrak{n}}_0^-$ denote the subalgebra of $\mathfrak{n}_0^-$ spanned by the elements $F_{j\,i}$ with $1 \leqslant i < j \leqslant n-1$. The same argument as in the proof of Lemma 3.1 shows that

$$F_{n,-n}^{-\nu_1} F_{-1,-n}^{\lambda_1-\nu_2} \ldots F_{-n+1,-n}^{\lambda_{n-1}-\nu_n} \xi = d\, \eta_{\nu\tilde{\nu}} + \sum_{\sigma \succ \nu} u(\sigma)\, \eta_{\sigma\tilde{\nu}}$$

for a nonzero constant d and some elements $u(\sigma) \in \mathcal{U}(\tilde{\mathfrak{n}}_0^-)$, where $\tilde{\nu} = (\nu_2, \ldots, \nu_n)$. Now we will be applying the inversion formula for positive values of i and note that each term with $i_s < 0$ in the sum on the right hand side contains a generator $F_{i_k\, i_{k+1}}$ with $i_k > 0 > i_{k+1}$. However, such a generator commutes with all elements $F_{i,-n}$ for $i > 0$. Therefore, all these terms with $i_s < 0$ will only contribute to the sum on the right hand side of the expansion (5·3) within the summands of the form $u(\nu', \mu')\, \eta_{\nu'\mu'}$ with $\mu' \succ \mu$.

On the other hand, for any element $u \in \mathcal{U}(\tilde{\mathfrak{n}}_0^-)$ we have the relation

$$F_{i,-n} u = u F_{i,-n} + \sum_{j=i+1}^{n-1} F_{j,-n} u_j$$

for certain elements $u_j \in \mathcal{U}(\tilde{\mathfrak{n}}_0^-)$. Hence, considering the terms in the inversion formula with the property $i_s > 0$, we may conclude that nonzero summands on the right hand side of (5·3) of the form $u(\nu', \mu)\, \eta_{\nu'\mu}$ must have the property $\nu' \succcurlyeq \nu$ and $u(\nu, \mu)$ is a nonzero constant. □

Consider the vectors $\xi_\Lambda \in V(\lambda)$ introduced in Section 4. They are parameterised by the type C patterns Λ defined in the Introduction. Represent each pattern Λ associated with λ as the sequence of the rows:

$$\Lambda = (\bar{\lambda}_{n-1}, \bar{\lambda}'_n, \bar{\lambda}_{n-2}, \bar{\lambda}'_{n-1}, \ldots, \bar{\lambda}'_1),$$

where we set

$$\bar{\lambda}_k = (\lambda_{k\,1}, \ldots, \lambda_{k\,k}) \qquad \text{and} \qquad \bar{\lambda}'_k = (\lambda'_{k\,1}, \ldots, \lambda'_{k\,k}).$$

Introduce the lexicographical ordering $\succ$ on the sequences Λ by using the lexicographical orderings on the vectors $\bar{\lambda}_k$ and $\bar{\lambda}'_k$. Recall the vectors θ_Λ defined in Theorem B. We can now obtain another proof of the theorem.

Proposition 5.3. *For each type* C *pattern Λ associated with λ, in the module $V(\lambda)$ we have*

$$\theta_\Lambda = \sum_{\Lambda' \succcurlyeq \Lambda} c_{\Lambda,\Lambda'}\, \xi_{\Lambda'}$$

for some constants $c_{\Lambda,\Lambda'}$, and $c_{\Lambda,\Lambda} \neq 0$. In particular, θ_Λ is a basis of $V(\lambda)$.

Proof. We will use an induction on n. Consider the part of the product defining the vector θ_Λ which corresponds to the value $k = n$. By applying Lemma 5.2 and using the induction hypothesis, we can write θ_Λ as a linear combination of the basis vectors η_M defined in (5·2) so that it contains the vector η_Λ with a nonzero coefficient, while the remaining vectors occurring in the linear combination have the property $M \succ \Lambda$. It remains to expand the vectors η_M as linear combinations of basis vectors $\xi_{\Lambda'}$ by using Lemma 5.1 which yields the expansion of θ_Λ with the required properties. □

Remark 5.4. The inversion formula can be used also for rewriting the basis of Proposition 2.10 in terms of the lowering operators. Therefore the subspace $V(\lambda)^+$ has a basis

$$\{F_{2n\,1}^{b} z_{2n\,n}^{b_n+\iota_n a_n} z_{n\,1}^{b_n+(1-\iota_n)a_n} \dots z_{2n\,2}^{b_2+\iota_2 a_2} z_{2\,1}^{b_2+(1-\iota_2)a_2} v_\lambda \mid a_2, \dots a_n, b, b_2, \dots b_n \text{ satisfy } (2{\cdot}6) - (2{\cdot}9)\},$$

where $\iota_k \in \{0, 1\}$.

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