

Classical \mathcal{W} -algebras in types A, B, C, D and G

A. I. Molev and E. Ragoucy

Abstract

We produce explicit generators of the classical \mathcal{W} -algebras associated with the principal nilpotents in the simple Lie algebras of all classical types and in the exceptional Lie algebra of type G_2 . The generators are given by determinant formulas in the context of the Poisson vertex algebras. We also show that the images of the \mathcal{W} -algebra generators under the Chevalley-type isomorphism coincide with the elements defined via the corresponding Miura transformations.

School of Mathematics and Statistics
University of Sydney, NSW 2006, Australia
alexander.molev@sydney.edu.au

Laboratoire de Physique Théorique LAPTh, CNRS and Université de Savoie
BP 110, 74941 Annecy-le-Vieux Cedex, France
eric.ragoucy@lapth.cnrs.fr

1 Introduction

Given a finite-dimensional simple Lie algebra \mathfrak{g} over \mathbb{C} and a nilpotent element $f \in \mathfrak{g}$, the corresponding *classical \mathcal{W} -algebra* $\mathcal{W}(\mathfrak{g}, f)$ can be defined as a Poisson algebra of functions on an infinite-dimensional manifold. The algebras $\mathcal{W}(\mathfrak{g}, f)$ were defined by Drinfeld and Sokolov [9] in the case where f is the principal nilpotent and were used to introduce equations of the KdV type for arbitrary simple Lie algebras.

Originally introduced in physics to investigate Toda systems [19] or minimal representations of conformal field theories [3], [24], the \mathcal{W} -algebras were firstly studied in the context of constrained Wess–Zumino–Witten models; see [4], [10], and references therein for the physics literature.

In a recent work by De Sole, Kac and Valeri (see [7] and review [8]) the construction of Drinfeld and Sokolov (generalized to an arbitrary nilpotent element f) was described in the framework of Poisson vertex algebras. This description was then applied to construct integrable hierarchies of bi-Hamiltonian equations.

We follow the approach of [7] to produce explicit generators of the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ for all simple Lie algebras \mathfrak{g} of types A, B, C, D and G for the case where f is the principal nilpotent element of \mathfrak{g} . The generators are given as certain determinants of matrices formed by elements of differential algebras associated with \mathfrak{g} . These formulas were suggested by the recent work [2], where the rectangular affine \mathcal{W} -algebras were explicitly produced in type A .

We show that the images of the generators of the algebra $\mathcal{W}(\mathfrak{g}, f)$ under a Chevalley-type projection coincide with the elements defined in [9] via the Miura transformation. The projection yields an isomorphism of Chevalley type to an algebra of polynomials defined as the intersection of the kernels of screening operators. This provides a direct connection between two presentations of the classical \mathcal{W} -algebras: one in the context of the Poisson vertex algebras and the other as the Harish-Chandra image of the Feigin–Frenkel center (i.e., the center of the affine vertex algebra at the critical level); see [12, Theorem 8.1.5]. Explicit generators of the center were constructed in [5], [6] and [20] for the Lie algebras \mathfrak{g} of all classical types. We make a connection between the Harish-Chandra images of these elements (*loc. cit.* and [21]) and the generators of $\mathcal{W}(\mathfrak{g}, f)$ constructed via the determinant formulas.

The Chevalley-type isomorphism in the case of the Lie algebra of type G_2 allows us to use the determinant formula for the generators to produce explicit elements of $\mathcal{W}(\mathfrak{g}, f)$ via a Miura transformation.

This work was completed during the first author’s visit to the *Laboratoire d’Annecy-le-Vieux de Physique Théorique*. He would like to thank the Laboratory for the warm hospitality.

2 Definitions and preliminaries

We will follow [7] to introduce the principal classical \mathcal{W} -algebras in the context of Poisson vertex algebras.

2.1 Differential algebras and λ -brackets

Let \mathfrak{g} be a vector space over \mathbb{C} and let X_1, \dots, X_d be a basis of \mathfrak{g} . Following [7], consider the *differential algebra* $\mathcal{V} = \mathcal{V}(\mathfrak{g})$ which is defined as the *algebra of differential polynomials* in the variables X_1, \dots, X_d ,

$$\mathcal{V} = \mathbb{C}[X_1^{(r)}, \dots, X_d^{(r)} \mid r = 0, 1, 2, \dots] \quad \text{with} \quad X_i^{(0)} = X_i,$$

equipped with the derivation ∂ defined by $\partial(X_i^{(r)}) = X_i^{(r+1)}$ for all $i = 1, \dots, d$ and $r \geq 0$.

Suppose now that \mathfrak{g} is a simple (or reductive) Lie algebra and fix a symmetric invariant bilinear form $(\cdot | \cdot)$ on \mathfrak{g} . Introduce the λ -*bracket* on \mathcal{V} as a linear map

$$\mathcal{V} \otimes \mathcal{V} \rightarrow \mathbb{C}[\lambda] \otimes \mathcal{V}, \quad a \otimes b \mapsto \{a_\lambda b\}.$$

By definition, it is given by

$$\{X_\lambda Y\} = [X, Y] + (X|Y)\lambda \quad \text{for} \quad X, Y \in \mathfrak{g}, \quad (2.1)$$

and extended to \mathcal{V} by sesquilinearity ($a, b \in \mathcal{V}$):

$$\{\partial a_\lambda b\} = -\lambda \{a_\lambda b\}, \quad \{a_\lambda \partial b\} = (\lambda + \partial) \{a_\lambda b\},$$

skewsymmetry

$$\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\},$$

and the Leibniz rule ($a, b, c \in \mathcal{V}$):

$$\{a_\lambda bc\} = \{a_\lambda b\}c + \{a_\lambda c\}b.$$

The λ -bracket defines the *affine Poisson vertex algebra* structure on \mathcal{V} [7].

Now choose a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Set $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{h}$ and define the projection map $\pi_{\mathfrak{p}} : \mathfrak{g} \rightarrow \mathfrak{p}$ with the kernel \mathfrak{n}_+ . From now on we will assume that $f \in \mathfrak{n}_-$ is a principal nilpotent element of \mathfrak{g} . We regard $\mathcal{V}(\mathfrak{p})$ as a differential subalgebra of \mathcal{V} and define the differential algebra homomorphism

$$\rho : \mathcal{V} \rightarrow \mathcal{V}(\mathfrak{p})$$

by setting

$$\rho(X) = \pi_{\mathfrak{p}}(X) + (f|X), \quad X \in \mathfrak{g}. \quad (2.2)$$

The *classical* \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ is defined by

$$\mathcal{W}(\mathfrak{g}, f) = \{P \in \mathcal{V}(\mathfrak{p}) \mid \rho\{X_\lambda P\} = 0 \text{ for all } X \in \mathfrak{n}_+\}. \quad (2.3)$$

By [7, Lemma 3.2], the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ is a differential subalgebra of $\mathcal{V}(\mathfrak{p})$; moreover, it is a Poisson vertex algebra equipped with the λ -bracket

$$\{a_\lambda b\}_\rho = \rho\{a_\lambda b\}, \quad a, b \in \mathcal{W}(\mathfrak{g}, f).$$

We will need a sufficient condition for a family of elements of the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ to be its set of generators as a differential algebra (Proposition 2.1). Include the principal nilpotent element f into an \mathfrak{sl}_2 -triple $\{e, f, h\}$ with the standard commutation relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f,$$

and set $x = h/2$. The subspace \mathfrak{p} is assumed to be compatible with the $\text{ad } x$ -eigenspace decomposition. In particular, \mathfrak{p} contains the centralizer \mathfrak{g}^f of the element f in \mathfrak{g} . Let v_1, \dots, v_n be a basis of \mathfrak{g}^f consisting of $\text{ad } x$ -eigenvectors. If b is an $\text{ad } x$ -eigenvector, we let $\delta_x(b)$ denote its eigenvalue. The following is a particular case of [7, Corollary 3.19].

Proposition 2.1. *Suppose that w_1, \dots, w_n is an arbitrary collection of elements of $\mathcal{W}(\mathfrak{g}, f)$ of the form $w_j = v_j + g_j$, where g_j is the sum of products of $\text{ad } x$ -eigenvectors $b_i \in \mathfrak{p}$,*

$$g_j = \sum b_1^{(m_1)} \dots b_s^{(m_s)}, \quad \sum_{i=1}^s (1 - \delta_x(b_i) + m_i) = 1 - \delta_x(v_j),$$

with $s + m_1 + \dots + m_s > 1$. Then w_1, \dots, w_n are generators of the differential algebra $\mathcal{W}(\mathfrak{g}, f)$. \square

2.2 Chevalley-type theorem

Keeping the notation of the previous section, introduce the standard Chevalley generators e_i, h_i, f_i with $i = 1, \dots, n$ of the simple Lie algebra \mathfrak{g} of rank n . The generators h_i form a basis of the Cartan subalgebra \mathfrak{h} of \mathfrak{g} , while the e_i and f_i generate the respective nilpotent subalgebras \mathfrak{n}_+ and \mathfrak{n}_- . Let $A = [a_{ij}]$ be the Cartan matrix of \mathfrak{g} so that the defining relations of \mathfrak{g} take the form

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_i, & [h_i, h_j] &= 0, \\ [h_i, e_j] &= a_{ij} e_j, & [h_i, f_j] &= -a_{ij} f_j, \end{aligned}$$

together with the Serre relations; see e.g. [16]. There exists a diagonal matrix $D = \text{diag}[\epsilon_1, \dots, \epsilon_n]$ with positive rational entries such that the matrix $B = D^{-1}A$ is symmetric. Following [16, Ch. 2], normalize the symmetric invariant bilinear form on \mathfrak{g} so that

$$(e_i | f_j) = \delta_{ij} \epsilon_i.$$

We let Δ denote the root system of \mathfrak{g} . For each root $\alpha \in \Delta$ choose a nonzero root vector e_α so that the set $\{e_\alpha \mid \alpha \in \Delta\} \cup \{h_i \mid i = 1, \dots, n\}$ is a basis of \mathfrak{g} . The subset of positive roots will be denoted by Δ^+ . The respective sets of elements e_α and $e_{-\alpha}$ with $\alpha \in \Delta^+$ form bases of \mathfrak{n}_+ and \mathfrak{n}_- . We also let $\alpha_1, \dots, \alpha_n$ denote the simple roots so that

$$e_i = e_{\alpha_i}, \quad f_i = e_{-\alpha_i}, \quad i = 1, \dots, n.$$

The elements of the differential algebra $\mathcal{V}(\mathfrak{g})$ corresponding to the generators of \mathfrak{g} will be denoted by

$$e_i^{(r)} = \partial^r(e_i), \quad f_i^{(r)} = \partial^r(f_i), \quad h_i^{(r)} = \partial^r(h_i).$$

Let

$$\phi : \mathcal{V}(\mathfrak{p}) \rightarrow \mathcal{V}(\mathfrak{h}) \tag{2.4}$$

denote the homomorphism of differential algebras defined on the generators as the projection $\mathfrak{p} \rightarrow \mathfrak{h}$ with the kernel \mathfrak{n}_- . For each $i = 1, \dots, n$ introduce the *screening operator*

$$V_i : \mathcal{V}(\mathfrak{h}) \rightarrow \mathcal{V}(\mathfrak{h})$$

defined by the formula

$$V_i = \sum_{r=0}^{\infty} V_{ir} \sum_{j=1}^n a_{ji} \frac{\partial}{\partial h_j^{(r)}}, \tag{2.5}$$

where the coefficients V_{ir} are elements of $\mathcal{V}(\mathfrak{h})$ found by the relation

$$\sum_{r=0}^{\infty} \frac{V_{ir} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{h_i^{(m-1)} z^m}{\epsilon_i m!} \right). \tag{2.6}$$

The following statement is well-known; it can be regarded as an analogue of the classical Chevalley theorem providing a relationship between two presentations of the classical \mathcal{W} -algebra; cf. [7], [9] and [12, Ch. 8].

Proposition 2.2. *The restriction of the homomorphism ϕ to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ yields an isomorphism*

$$\phi : \mathcal{W}(\mathfrak{g}, f) \rightarrow \widetilde{\mathcal{W}}(\mathfrak{g}, f), \tag{2.7}$$

where $\widetilde{\mathcal{W}}(\mathfrak{g}, f)$ is the subalgebra of $\mathcal{V}(\mathfrak{h})$ which consists of the elements annihilated by all screening operators V_i ,

$$\widetilde{\mathcal{W}}(\mathfrak{g}, f) = \bigcap_{i=1}^n \ker V_i.$$

Proof. We start by showing that for any $P \in \mathcal{W}(\mathfrak{g}, f)$ its image $\phi(P)$ is annihilated by each operator V_i . It will be convenient to work with an equivalent affine version of the differential algebra $\mathcal{V}(\mathfrak{g})$; see [17, Sec. 2.7]. Recall that the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined as the central extension

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K, \quad (2.8)$$

where $\mathfrak{g}[t, t^{-1}]$ is the Lie algebra of Laurent polynomials in t with coefficients in \mathfrak{g} . For any $r \in \mathbb{Z}$ and $X \in \mathfrak{g}$ we set $X[r] = X t^r$. The commutation relations of the Lie algebra $\widehat{\mathfrak{g}}$ have the form

$$[X[r], Y[s]] = [X, Y][r + s] + r \delta_{r, -s} (X|Y) K, \quad X, Y \in \mathfrak{g},$$

and the element K is central in $\widehat{\mathfrak{g}}$. Consider the quotient $S(\widehat{\mathfrak{g}})/I$ of the symmetric algebra $S(\widehat{\mathfrak{g}})$ by its ideal I generated by the subspace $\mathfrak{g}[t]$ and the element $K - 1$. This quotient can be identified with the symmetric algebra $S(t^{-1}\mathfrak{g}[t^{-1}])$, as a vector space. We will identify the differential algebras $\mathcal{V}(\mathfrak{g}) \cong S(t^{-1}\mathfrak{g}[t^{-1}])$ via the isomorphism

$$X^{(r)} \mapsto r! X[-r - 1], \quad X \in \mathfrak{g}, \quad r \geq 0, \quad (2.9)$$

so that the derivation ∂ will correspond to the derivation $-d/dt$. Similarly, we will identify the differential algebras $\mathcal{V}(\mathfrak{p}) \cong S(t^{-1}\mathfrak{p}[t^{-1}])$.

By definition (2.3), if an element $P \in \mathcal{V}(\mathfrak{p})$ belongs to the subalgebra $\mathcal{W}(\mathfrak{g}, f)$, then $\rho\{e_{i\lambda} P\} = 0$ for all $i = 1, \dots, n$. Now observe that, regarding P as an element of the $\mathfrak{g}[t]$ -module $S(\widehat{\mathfrak{g}})/I \cong S(t^{-1}\mathfrak{g}[t^{-1}])$, we can write

$$\{e_{i\lambda} P\} = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} e_i[r] P.$$

We have the following relations in $\widehat{\mathfrak{g}}$,

$$[e_i[r], f_i[-s - 1]] = h_i[r - s - 1] + r \delta_{r, s+1} \epsilon_i K,$$

$$[e_i[r], h_j[-s - 1]] = -a_{ji} e_i[r - s - 1].$$

Moreover, for each positive root $\alpha \neq \alpha_i$ we also have

$$[e_i[r], e_{-\alpha}[-s - 1]] = c_i(\alpha) e_{-\alpha+\alpha_i}[r - s - 1]$$

for a certain constant $c_i(\alpha)$, if $\alpha - \alpha_i$ is a root; otherwise the commutator is zero. Hence, recalling the definition (2.2) of the homomorphism ρ , we can conclude that the condition that P belongs to the subalgebra $\mathcal{W}(\mathfrak{g}, f)$ implies the relations

$$\widehat{e}_i[r] P = 0 \quad \text{for all } i = 1, \dots, n \quad \text{and } r \geq 0, \quad (2.10)$$

where $\widehat{e}_i[r]$ is the operator on $S(t^{-1}\mathfrak{p}[t^{-1}])$ given by

$$\begin{aligned} \widehat{e}_i[r] &= \sum_{s=r}^{\infty} h_i[r-s-1] \frac{\partial}{\partial f_i[-s-1]} + \epsilon_i r \frac{\partial}{\partial f_i[-r]} - \sum_{j=1}^n a_{ji} \frac{\partial}{\partial h_j[-r-1]} \\ &+ \sum_{\alpha \in \Delta^+, \alpha \neq \alpha_i} \sum_{s=r}^{\infty} c_i(\alpha) e_{-\alpha+\alpha_i}[r-s-1] \frac{\partial}{\partial e_{-\alpha}[-s-1]}, \end{aligned}$$

and $e_{-\alpha+\alpha_i}$ is understood as being equal to zero, if $\alpha - \alpha_i$ is not a root. Denote the generating function in z introduced in (2.6) by $V_i(z)$ and replace $h_i^{(m-1)}/(m-1)!$ with $h_i[-m]$ for $m \geq 1$ in accordance with (2.9). We have the relation for its derivative,

$$V_i'(z) = V_i(z) \left(- \sum_{m=1}^{\infty} \frac{h_i[-m] z^{m-1}}{\epsilon_i} \right).$$

Taking the coefficient of z^{p-1} with $p \geq 1$ we get the relations

$$\epsilon_i p \frac{V_{ip}}{p!} + \sum_{r=0}^{p-1} \frac{V_{ir}}{r!} h_i[r-p] = 0. \quad (2.11)$$

By (2.10) the element P has the property

$$\sum_{r=0}^{\infty} \frac{V_{ir}}{r!} \widehat{e}_i[r] P = 0. \quad (2.12)$$

Note that by (2.11) all the differentiations $\partial/\partial f_i[-s-1]$ with $s \geq 0$ will cancel in the expansion of the left hand side of (2.12). Moreover, the elements of the form $e_{-\alpha+\alpha_i}[r-s-1]$ occurring in the expansion of $\widehat{e}_i[r]$ will vanish under the projection (2.4). Therefore, (2.12) implies that the image $\phi(P)$ with respect to this projection satisfies the relation

$$\sum_{r=0}^{\infty} \frac{V_{ir}}{r!} \sum_{j=1}^n a_{ji} \frac{\partial}{\partial h_j[-r-1]} \phi(P) = 0$$

which is equivalent to

$$\sum_{r=0}^{\infty} V_{ir} \sum_{j=1}^n a_{ji} \frac{\partial}{\partial h_j^{(r)}} \phi(P) = 0,$$

that is, $V_i \phi(P) = 0$, as claimed. This shows that the restriction of the homomorphism ϕ to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ provides the homomorphism (2.7).

It is clear from the definition that ϕ is a differential algebra homomorphism. It is known that $\mathcal{W}(\mathfrak{g}, f)$ is an algebra of differential polynomials in n variables (a proof of this claim for an arbitrary nilpotent element f is given in [7, Theorem 3.14]). Therefore, the proof of the proposition can be completed by verifying that the images of the generators of $\mathcal{W}(\mathfrak{g}, f)$ under the homomorphism ϕ are differential algebra generators of $\widetilde{\mathcal{W}}(\mathfrak{g}, f)$. We will omit this verification for the general case. In types A, B, C, D and G this will follow easily from the explicit construction of generators of $\mathcal{W}(\mathfrak{g}, f)$ given in the next sections. \square

3 Generators of $\mathcal{W}(\mathfrak{gl}_n, f)$

We will consider square matrices of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-11} & a_{n-12} & a_{n-13} & \dots & \dots & a_{n-1n} \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & a_{nn} \end{bmatrix} \quad (3.1)$$

such that the entries a_{ij} belong to a certain ring. It is well-known and can be easily verified that even if the ring is noncommutative, the column-determinant and row-determinant of the matrix A coincide, provided that the entries $a_{12}, a_{23}, \dots, a_{n-1n}$ belong to the center of the ring. We will assume that this condition holds, and define the (noncommutative) determinant of A by setting

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot a_{\sigma(1)1} \dots a_{\sigma(n)n} = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn} \sigma \cdot a_{1\sigma(1)} \dots a_{n\sigma(n)}. \quad (3.2)$$

A particular case which we will often use below is where the ring contains the identity and the matrix A has the form

$$A = \begin{bmatrix} a_{11} & 1 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 1 & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-11} & a_{n-12} & a_{n-13} & \dots & \dots & 1 \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & a_{nn} \end{bmatrix}. \quad (3.3)$$

One easily derives the following explicit formula for the determinant of this matrix:

$$\det A = (-1)^{n-1} a_{n1} + \sum_{1 \leq i_1 < \dots < i_k < n} (-1)^{n-k-1} a_{i_1 1} a_{i_2 i_1+1} a_{i_3 i_2+1} \dots a_{n i_k+1}, \quad (3.4)$$

where k runs over the set of values $1, \dots, n-1$.

Returning to more general matrices (3.1), denote by D_i (resp., \overline{D}_i) the determinant of the $i \times i$ submatrix of A corresponding to the first (resp., last) i rows and columns. We suppose that $D_0 = \overline{D}_0 = 1$.

Lemma 3.1. *Fix $p \in \{0, 1, \dots, n\}$. Then for the determinant of the matrix (3.1) we have*

$$\det A = D_p \overline{D}_{n-p} + \sum_{j=1}^p \sum_{i=p+1}^n (-1)^{j+i} D_{j-1} \tilde{a}_{ij} \overline{D}_{n-i},$$

where

$$\tilde{a}_{ij} = a_{ij} a_{j,j+1} a_{j+1,j+2} \cdots a_{i-1,i} \quad \text{for } i > j. \quad (3.5)$$

Proof. The formula follows easily from the definition of the determinant (3.2). In the applications which we consider below, the central elements $a_{12}, a_{23}, \dots, a_{n-1,n}$ turn out to be invertible. Then the lemma is reduced to the particular case of matrices (3.3) and it is immediate from the explicit formula (3.4). \square

3.1 Determinant-type generators

Take \mathfrak{g} to be the general linear Lie algebra \mathfrak{gl}_n with its standard basis E_{ij} , $i, j = 1, \dots, n$. The elements E_{11}, \dots, E_{nn} span a Cartan subalgebra of \mathfrak{gl}_n which we will denote by \mathfrak{h} . The respective subsets of basis elements E_{ij} with $i < j$ and $i > j$ span the nilpotent subalgebras \mathfrak{n}_+ and \mathfrak{n}_- . The subalgebra $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{h}$ is then spanned by the elements E_{ij} with $i \geq j$. Take the principal nilpotent element f in the form

$$f = E_{21} + E_{32} + \cdots + E_{nn-1}.$$

For the \mathfrak{sl}_2 -triple $\{e, f, h\}$ take

$$e = \sum_{i=1}^{n-1} i(n-i) E_{i,i+1} \quad \text{and} \quad h = \sum_{i=1}^n (n-2i+1) E_{ii}.$$

We will be working with the algebra of differential operators $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$, where the commutation relations are given by

$$\partial E_{ij}^{(r)} - E_{ij}^{(r)} \partial = E_{ij}^{(r+1)}.$$

In other words, ∂ will be regarded as a generator of this algebra rather than the derivation on $\mathcal{V}(\mathfrak{p})$. For any element $g \in \mathcal{V}(\mathfrak{p})$ and any nonnegative integer r the element $g^{(r)} = \partial^r(g)$ coincides with the constant term of the differential operator $\partial^r g$ so that

$$g^{(r)} = \partial^r g \mathbf{1}, \quad (3.6)$$

assuming that $\partial \mathbf{1} = 0$.

The invariant symmetric bilinear form on \mathfrak{gl}_n is defined by

$$(X|Y) = \text{tr } XY, \quad X, Y \in \mathfrak{gl}_n, \quad (3.7)$$

where X and Y are understood as $n \times n$ matrices over \mathbb{C} .

Consider the determinant (3.2) of the matrix with entries in $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

$$\det \begin{bmatrix} \partial + E_{11} & 1 & 0 & 0 & \dots & 0 \\ E_{21} & \partial + E_{22} & 1 & 0 & \dots & 0 \\ E_{31} & E_{32} & \partial + E_{33} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ E_{n-11} & E_{n-12} & E_{n-13} & \dots & \dots & 1 \\ E_{n1} & E_{n2} & E_{n3} & \dots & \dots & \partial + E_{nn} \end{bmatrix} = \partial^n + w_1 \partial^{n-1} + \dots + w_n. \quad (3.8)$$

Theorem 3.2. *All elements w_1, \dots, w_n belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{gl}_n, f)$. Moreover, the elements $w_1^{(r)}, \dots, w_n^{(r)}$ with $r = 0, 1, \dots$ are algebraically independent and generate the algebra $\mathcal{W}(\mathfrak{gl}_n, f)$.*

Proof. Denote the determinant on the left hand side of (3.8) by D_n . For each $1 \leq k < n$ we will identify the Lie algebra \mathfrak{gl}_k with the natural subalgebra of \mathfrak{gl}_n so that the determinant D_k will also be regarded as an element of the algebra $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$. We also set $D_0 = 1$. We will be proving the first part of the theorem by induction on n . The induction base is trivial and for any $1 \leq i < n - 1$ the induction hypothesis implies

$$\rho\{E_{i+1} \lambda D_k\} = 0 \quad (3.9)$$

for all $k = i + 1, \dots, n$. Moreover, relation (3.9) clearly holds for the values $k = 1, \dots, i - 1$ as well, while

$$\rho\{E_{i+1} \lambda D_i\} = -\rho(D_{i-1}^+ E_{i+1}) = -D_{i-1}^+,$$

where for each polynomial $P = P(\partial) \in \mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$ we denote by P^+ the polynomial $P(\partial + \lambda)$ regarded as an element of $\mathbb{C}[\lambda] \otimes \mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$. By Lemma 3.1 (with $p = n - 1$) we have the relation

$$\begin{aligned} D_n &= D_{n-1}(\partial + E_{nn}) - D_{n-2}E_{nn-1} + D_{n-3}E_{nn-2} \\ &\quad + \dots + (-1)^{n-2}D_1E_{n2} + (-1)^{n-1}D_0E_{n1}. \end{aligned} \quad (3.10)$$

Hence, using the properties of the λ -bracket we get

$$\rho\{E_{i+1} \lambda D_n\} = (-1)^{n-i+1} D_{i-1}^+ E_{ni+1} + (-1)^{n-i} D_{i-1}^+ E_{ni+1} = 0$$

for all $i = 1, \dots, n - 2$. Furthermore,

$$\begin{aligned} \rho\{E_{n-1n} \lambda D_n\} &= D_{n-1}^+ - D_{n-2}^+(\partial + E_{nn}) - D_{n-2}^+(E_{n-1n-1} - E_{nn} + \lambda) \\ &\quad + D_{n-3}^+ E_{n-1n-2} + \dots + (-1)^{n-2} D_1^+ E_{n-12} + (-1)^{n-1} D_0^+ E_{n-11} \end{aligned}$$

which is zero due to relation (3.10) applied to the determinant D_{n-1}^+ instead of D_n . Since $\rho\{E_{ii+1} \lambda D_n\} = 0$ for all $i = 1, \dots, n-1$, we may conclude that $\rho\{X \lambda D_n\} = 0$ for all $X \in \mathfrak{n}_+$ so that all elements w_1, \dots, w_n belong to the subalgebra $\mathcal{W}(\mathfrak{gl}_n, f)$ of $\mathcal{V}(\mathfrak{p})$.

To prove the second part of the theorem we apply Proposition 2.1. Note that the powers of the $n \times n$ matrix f form a basis of the centralizer \mathfrak{gl}_n^f so that for $j = 1, \dots, n$ we can take v_j to be equal, up to a sign, to f^{j-1} . Then the condition of Proposition 2.1 holds for the elements w_1, \dots, w_n thus implying that they are generators of the differential algebra $\mathcal{W}(\mathfrak{gl}_n, f)$.

Finally, the images of the elements w_k under the homomorphism (2.7) are the elements $\tilde{w}_k \in \mathcal{V}(\mathfrak{h})$ found from the relation

$$(\partial + E_{11}) \dots (\partial + E_{nn}) = \partial^n + \tilde{w}_1 \partial^{n-1} + \dots + \tilde{w}_n.$$

In the notation of Sec. 2.2 we have

$$h_j^{(r)} = E_{jj}^{(r)} - E_{j+1j+1}^{(r)}, \quad j = 1, \dots, n-1.$$

The Cartan matrix is of the size $(n-1) \times (n-1)$,

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

so that $a_{ii} = 2$ for $i = 1, \dots, n-1$ and $a_{ii+1} = a_{i+1i} = -1$ for $i = 1, \dots, n-2$, while all other entries are zero. Hence, regarding $\mathcal{V}(\mathfrak{h})$ as the algebra of polynomials in the variables $E_{ii}^{(r)}$ with $i = 1, \dots, n$ and $r = 0, 1, \dots$, we get

$$\sum_{j=1}^{n-1} a_{ji} \frac{\partial}{\partial h_j^{(r)}} = \frac{\partial}{\partial E_{ii}^{(r)}} - \frac{\partial}{\partial E_{i+1i+1}^{(r)}}.$$

Therefore, the screening operators (2.5) take the form

$$V_i = \sum_{r=0}^{\infty} V_{ir} \left(\frac{\partial}{\partial E_{ii}^{(r)}} - \frac{\partial}{\partial E_{i+1i+1}^{(r)}} \right), \quad i = 1, \dots, n-1,$$

where the coefficients V_{ir} are found by the relation

$$\sum_{r=0}^{\infty} \frac{V_{ir} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{E_{ii}^{(m-1)} - E_{i+1i+1}^{(m-1)}}{m!} z^m \right).$$

The differential algebra $\widetilde{\mathcal{W}}(\mathfrak{gl}_n, f)$ consists of the polynomials in the variables $E_{ii}^{(r)}$, which are annihilated by all operators V_i . It is easy to verify directly that the elements $\widetilde{w}_1, \dots, \widetilde{w}_n$ belong to $\widetilde{\mathcal{W}}(\mathfrak{gl}_n, f)$; cf. [21]. Moreover, the elements $\widetilde{w}_1^{(r)}, \dots, \widetilde{w}_n^{(r)}$ with r running over nonnegative integers are algebraically independent generators of the algebra $\widetilde{\mathcal{W}}(\mathfrak{gl}_n, f)$; see [12, Ch. 8]. Hence, the generators $w_1^{(r)}, \dots, w_n^{(r)}$ of $\mathcal{W}(\mathfrak{gl}_n, f)$ are also algebraically independent. \square

The injective homomorphism $\mathcal{W}(\mathfrak{gl}_n, f) \hookrightarrow \mathcal{V}(\mathfrak{h})$ taking w_i to \widetilde{w}_i constructed in the proof of Theorem 3.2 can be shown to respect the λ -brackets. This fact is known as the *Kupershmidt–Wilson theorem*, and the embedding is called the *Miura transformation*; see e.g. [1], [12, Ch. 8], [14] and [18]. Note also that the arguments used in the proof imply that the homomorphism (2.7) is bijective in the case $\mathfrak{g} = \mathfrak{gl}_n$; see the proof of Proposition 2.2.

This embedding can be used to calculate λ -brackets in the classical \mathcal{W} -algebra inside $\mathcal{V}(\mathfrak{h})$, as we illustrate below; cf. [22]. The λ -bracket on $\mathcal{V}(\mathfrak{h})$ is determined by the relations

$$\{E_{ii} \lambda E_{jj}\} = \delta_{ij} \lambda.$$

Let us set $x_i = \partial + E_{ii}$ for $i = 1, \dots, n$ and let u be a variable commuting with the x_i . Setting $C = E_{11} + \dots + E_{nn}$, we find

$$\{C_\lambda(u + x_1) \dots (u + x_n)\} = (u + x_1^+) \dots (u + x_n^+) - (u + x_1) \dots (u + x_n),$$

where $x_i^+ = \partial + \lambda + E_{ii}$. Similarly, consider the quadratic element

$$P = -\frac{1}{2} \sum_{i=1}^n E_{ii}^2 - \sum_{i=1}^n (n-i) E_{ii}.$$

Then

$$\{P_\lambda(u + x_1) \dots (u + x_n)\} = u(u + x_1^+) \dots (u + x_n^+) - (\partial + n\lambda + u)(u + x_1) \dots (u + x_n),$$

which follows by a straightforward calculation based on the properties of the λ -brackets. Defining the coefficients \widetilde{e}_m of the polynomial in u , by

$$(u + x_1) \dots (u + x_n) = \sum_{m=0}^n \widetilde{e}_m u^{n-m}$$

we get

$$\{P_\lambda \widetilde{e}_m\} = \widetilde{e}_{m+1}^+ - \widetilde{e}_{m+1} - (\partial + n\lambda) \widetilde{e}_m,$$

where \widetilde{e}_{m+1}^+ is obtained from \widetilde{e}_{m+1} by replacing ∂ with $\partial + \lambda$. Note that the \widetilde{e}_m coincide with the images of the respective differential operators e_m under the isomorphism (2.7); see (3.14) below.

3.2 MacMahon theorem

We now return to an arbitrary matrix A of the form (3.3) with entries in a certain ring with the identity. Let t be a variable commuting with elements of the ring. Define elements e_m of the ring by the expansion

$$\det(1 + tA) = \sum_{m=0}^n e_m t^m \quad (3.11)$$

and denote this polynomial in t by $e(t)$.

Lemma 3.3. *The elements e_m are found by*

$$e_m = \sum_{s=1}^m \sum_{i_k \geq j_k \text{ and } i_k < j_{k+1}} (-1)^{m-s} a_{i_1 j_1} \dots a_{i_s j_s},$$

where the second sum is taken over the indices i_1, \dots, i_s and j_1, \dots, j_s such that $i_k \geq j_k$ for $k = 1, \dots, s$ and $i_k < j_{k+1}$ for $k = 1, \dots, s-1$ satisfying the condition

$$\sum_{k=1}^s (j_k - i_k) = m - s. \quad (3.12)$$

Proof. For any $1 \leq k \leq n$ we will use the notation $e_m^{\{k\}}$ to indicate the elements e_m associated with the submatrix of A corresponding to the first k rows and columns. The obvious analogue of (3.10) for the determinant $\det(1 + tA)$ gives the recurrence relation

$$\begin{aligned} e_m^{\{n\}} &= e_m^{\{n-1\}} + e_{m-1}^{\{n-1\}} a_{nn} - e_{m-2}^{\{n-2\}} a_{nn-1} \\ &\quad + \dots + (-1)^{m-2} e_1^{\{n-m+1\}} a_{nn-m+2} + (-1)^{m-1} a_{nn-m+1} \end{aligned}$$

which implies the desired formula. \square

For an arbitrary matrix A of the form (3.3) over a ring, we also define the family of elements h_m of the ring by setting $h_0 = 1$ and

$$h_m = \sum_{s=1}^{\infty} \sum_{i_k \geq j_k \text{ and } i_k \geq j_{k+1}} a_{i_1 j_1} \dots a_{i_s j_s}, \quad m \geq 1,$$

where the second sum is taken over the indices i_1, \dots, i_s and j_1, \dots, j_s such that $i_k \geq j_k$ for $k = 1, \dots, s$ and $i_k \geq j_{k+1}$ for $k = 1, \dots, s-1$ satisfying (3.12). Combine these elements into the formal series

$$h(t) = \sum_{m=0}^{\infty} h_m t^m.$$

The following identity is a version of the MacMahon Master Theorem for noncommutative matrices of the form (3.3).

Proposition 3.4. *We have the identity*

$$h(t) e(-t) = 1.$$

Proof. We need to verify that for any $m \geq 1$ we have the relation

$$h_m - h_{m-1} e_1 + h_{m-2} e_2 + \cdots + (-1)^n h_{m-n} e_n = 0, \quad (3.13)$$

where we assume that $h_k = 0$ for $k < 0$. The expansion of the left hand side as a linear combination of monomials in the entries of the matrix A will contain monomials of the form

$$a_{i_1 j_1} \cdots a_{i_p j_p} a_{i_{p+1} j_{p+1}} \cdots a_{i_{p+r} j_{p+r}}$$

such that $i_k \geq j_{k+1}$ for $k = 1, \dots, p$ and $i_k < j_{k+1}$ for $k = p+1, \dots, p+r-1$. Such a monomial will occur twice in the expansion with the opposite signs. Indeed, it will occur in the expansion of $h_a e_b$ for two pairs of indices (a, b) . In the first pair,

$$a = \sum_{k=1}^p (j_k - i_k) + p, \quad b = \sum_{k=p+1}^{p+r} (j_k - i_k) + r$$

so that the sign of the monomial in the expansion is $(-1)^r$. In the second pair,

$$a = \sum_{k=1}^{p+1} (j_k - i_k) + p + 1, \quad b = \sum_{k=p+2}^{p+r} (j_k - i_k) + r - 1,$$

and the sign of the monomial in the expansion is $(-1)^{r-1}$. \square

Remark 3.5. One can regard $e(t)$ and $h(t)$ as specializations of the generating series of the noncommutative elementary and complete symmetric functions, respectively; see [15, Sec. 3]. Explicit formulas for the specializations of the power sums symmetric functions and ribbon Schur functions are also easy to write down in terms of the entries of A . \square

3.3 Permanent-type generators

Now we specialize the matrix A given in (3.3) and take

$$a_{ij} = \delta_{ij} \partial + E_{ij}, \quad i \geq j,$$

so that the entries belong to the algebra $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$. Consider the elements e_m and h_m associated with this specialization of the matrix A and write them as differential operators,

$$e_m = e_{m0} + e_{m1} \partial + \cdots + e_{mm} \partial^m, \quad (3.14)$$

$$h_m = h_{m0} + h_{m1} \partial + \cdots + h_{mm} \partial^m. \quad (3.15)$$

In particular, the constant terms are found by the formulas

$$e_{m0} = \sum_{s=1}^m \sum_{i_k \geq j_k \text{ and } i_k < j_{k+1}} (-1)^{m-s} (\delta_{i_1 j_1} \partial + E_{i_1 j_1}) \dots (\delta_{i_s j_s} \partial + E_{i_s j_s}) 1,$$

where the second sum is taken over the indices i_1, \dots, i_s and j_1, \dots, j_s such that $i_k \geq j_k$ for $k = 1, \dots, s$ and $i_k < j_{k+1}$ for $k = 1, \dots, s-1$ satisfying (3.12); and

$$h_{m0} = \sum_{s=1}^{\infty} \sum_{i_k \geq j_k \text{ and } i_k \geq j_{k+1}} (\delta_{i_1 j_1} \partial + E_{i_1 j_1}) \dots (\delta_{i_s j_s} \partial + E_{i_s j_s}) 1,$$

where the second sum is taken over the indices i_1, \dots, i_s and j_1, \dots, j_s such that $i_k \geq j_k$ for $k = 1, \dots, s$ and $i_k \geq j_{k+1}$ for $k = 1, \dots, s-1$ satisfying (3.12).

Proposition 3.6. *We have*

$$e_{mi} = \binom{n-m+i}{i} w_{m-i} \quad \text{for all } 0 \leq i \leq m \leq n, \quad (3.16)$$

where the elements w_1, \dots, w_n are introduced in (3.8).

Proof. Consider the determinant $D_n = D_n(\partial)$ defined in (3.8). We have

$$\det(1 + tA) = t^n D_n(\partial + t^{-1}).$$

Hence

$$\sum_{m=0}^n \sum_{i=0}^m e_{mi} \partial^i t^m = \sum_{k=0}^n w_k t^k (1 + t\partial)^{n-k}$$

which implies (3.16). \square

In particular, $e_{m0} = w_m$ and so the family $e_{10}^{(r)}, \dots, e_{n0}^{(r)}$ with $r = 0, 1, \dots$ is algebraically independent and generates the algebra $\mathcal{W}(\mathfrak{gl}_n, f)$.

Corollary 3.7. *All elements h_{mi} belong to $\mathcal{W}(\mathfrak{gl}_n, f)$. Moreover, the family $h_{10}^{(r)}, \dots, h_{n0}^{(r)}$ with $r = 0, 1, \dots$ is algebraically independent and generates the algebra $\mathcal{W}(\mathfrak{gl}_n, f)$.*

Proof. By Proposition 3.6, each e_m is a linear combination of differential operators whose coefficients belong to $\mathcal{W}(\mathfrak{gl}_n, f)$. Furthermore, Proposition 3.4 implies that each h_m is a differential operator with coefficients in $\mathcal{W}(\mathfrak{gl}_n, f)$. This proves the first part of the corollary.

Furthermore, it is easy to verify (see also [15, Sec. 4.1]) that the recurrence relation (3.13) is equivalent to

$$h_m = \sum_{i_1 + \dots + i_k = m} (-1)^{m-k} e_{i_1} \dots e_{i_k},$$

summed over k -tuples of positive integers (i_1, \dots, i_k) with $k = 1, \dots, m$. Together with (3.16) this implies an explicit expression for h_{m0} in terms of the generators $e_{10}^{(r)}, \dots, e_{n0}^{(r)}$,

$$h_{m0} = (-1)^{m-1} e_{m0} + \sum \text{const} \cdot e_{j_1 0}^{(r_1)} \dots e_{j_k 0}^{(r_k)},$$

summed over k -tuples of positive integers (j_1, \dots, j_k) and k -tuples of nonnegative integers (r_1, \dots, r_k) such that $j_1 + \dots + j_k + r_1 + \dots + r_k = m$, with certain coefficients, where for the summands with $k = 1$ we have $r_1 \geq 1$. This shows that the elements $h_{10}^{(r)}, \dots, h_{n0}^{(r)}$ with $r = 0, 1, \dots$ are algebraically independent and generate the algebra $\mathcal{W}(\mathfrak{gl}_n, f)$. \square

4 Generators of $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$

Given a positive integer N , we will use the involution on the set $\{1, \dots, N\}$ defined by $i \mapsto i' = N - i + 1$. The Lie subalgebra of \mathfrak{gl}_N spanned by the elements

$$F_{ij} = E_{ij} - E_{j'i'}, \quad i, j = 1, \dots, N, \quad (4.1)$$

is the *orthogonal Lie algebra* \mathfrak{o}_N . The simple Lie algebras of type B_n correspond to the odd values $N = 2n + 1$, while the simple Lie algebras of type D_n correspond to the even values $N = 2n$. In both cases, we have the commutation relations

$$[F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kl} - \delta_{ki'} F_{j'l} + \delta_{j'l} F_{ki'}, \quad i, j, k, l \in \{1, \dots, N\}. \quad (4.2)$$

Note also the symmetry relation

$$F_{ij} + F_{j'i'} = 0, \quad i, j \in \{1, \dots, N\}. \quad (4.3)$$

The elements F_{11}, \dots, F_{nn} span a Cartan subalgebra of \mathfrak{o}_N which we will denote by \mathfrak{h} . The respective subsets of elements F_{ij} with $i < j$ and $i > j$ span the nilpotent subalgebras \mathfrak{n}_+ and \mathfrak{n}_- . The subalgebra $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{h}$ is then spanned by the elements F_{ij} with $i \geq j$.

As with the case of \mathfrak{gl}_n , we will be working with the algebra of differential operators $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$, where the commutation relations are given by

$$\partial F_{ij}^{(r)} - F_{ij}^{(r)} \partial = F_{ij}^{(r+1)}. \quad (4.4)$$

For any element $g \in \mathcal{V}(\mathfrak{p})$ and any nonnegative integer r the element $g^{(r)}$ coincides with the constant term of the differential operator $\partial^r g$ as in (3.6).

In this section we will work with the odd orthogonal Lie algebras \mathfrak{o}_{2n+1} . Take the principal nilpotent element $f \in \mathfrak{o}_{2n+1}$ in the form

$$f = F_{21} + F_{32} + \dots + F_{n+1n}. \quad (4.5)$$

The \mathfrak{sl}_2 -triple is now formed by the elements $\{e, f, h\}$ with

$$e = \sum_{i=1}^n i(2n - i + 1) F_{ii+1} \quad \text{and} \quad h = 2 \sum_{i=1}^n (n - i + 1) F_{ii}.$$

The invariant symmetric bilinear form on \mathfrak{o}_{2n+1} is defined by

$$(X|Y) = \frac{1}{2} \operatorname{tr} XY, \quad X, Y \in \mathfrak{o}_{2n+1}, \quad (4.6)$$

where X and Y are understood as matrices over \mathbb{C} which are skew-symmetric with respect to the antidiagonal.

Consider the determinant (3.2) of the matrix with entries in $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

$$\det \begin{bmatrix} \partial + F_{11} & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ F_{21} & \partial + F_{22} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots \\ F_{n1} & F_{n2} & \dots & \partial + F_{nn} & 1 & 0 & \dots & 0 & 0 \\ F_{n+11} & F_{n+12} & \dots & F_{n+1n} & \partial & -1 & \dots & 0 & 0 \\ F_{n'1} & F_{n'2} & \dots & 0 & F_{n'n+1} & \partial + F_{n'n'} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ F_{2'1} & 0 & \dots & \dots & F_{2'n+1} & \dots & \dots & \partial + F_{2'2'} & -1 \\ 0 & F_{1'2} & \dots & \dots & F_{1'n+1} & \dots & \dots & F_{1'2'} & \partial + F_{1'1'} \end{bmatrix}$$

so that the (i, j) entry of the matrix is $\delta_{ij} \partial + F_{ij}$ for $i \geq j$, the $(i, i+1)$ entry is 1 for $i \leq n$ and -1 for $i > n$, while the remaining entries are zero. The determinant has the form

$$\partial^{2n+1} + w_2 \partial^{2n-1} + w_3 \partial^{2n-2} + \dots + w_{2n+1}, \quad w_i \in \mathcal{V}(\mathfrak{p}). \quad (4.7)$$

Theorem 4.1. *All elements w_2, \dots, w_{2n+1} belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$. Moreover, the elements $w_2^{(r)}, w_4^{(r)}, \dots, w_{2n}^{(r)}$ with $r = 0, 1, \dots$ are algebraically independent and generate the algebra $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$.*

Proof. The argument is similar to the proof of Theorem 3.2. Denote the determinant by D and let D_i (resp., \overline{D}_i) denote the $i \times i$ minor corresponding to the first (resp., last) i rows and columns. We suppose that $D_0 = \overline{D}_0 = 1$. Lemma 3.1 implies the expansion

$$D = D_n \partial \overline{D}_n + \sum_{j,k=1}^{n+1} (-1)^{n-j+1} D_{j-1} F_{k'j} \overline{D}_{k-1}.$$

To prove the first part of the theorem, note that the elements F_{ij} with $1 \leq i, j \leq n$ span a subalgebra of \mathfrak{o}_{2n+1} isomorphic to \mathfrak{gl}_n . Hence, due to Theorem 3.2, if $1 \leq i \leq n-1$ then

$$\rho\{F_{i+1\lambda} D_k\} = 0 \quad \text{and} \quad \rho\{F_{i+1\lambda} \overline{D}_k\} = 0 \quad (4.8)$$

for all $1 \leq k \leq n$ with $k \neq i$. Moreover,

$$\rho\{F_{i+1} \lambda D_i\} = -D_{i-1}^+ \quad \text{and} \quad \rho\{F_{i+1} \lambda \bar{D}_i\} = \bar{D}_{i-1},$$

where, as before, $P^+ = P(\partial + \lambda)$ for any polynomial $P = P(\partial) \in \mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$. Hence, for any $k \in \{1, \dots, n+1\}$ and $k \neq i, i+1$ we have

$$\rho\{F_{i+1} \lambda D_i F_{k'} \bar{D}_{k-1} - D_{i-1} F_{k'} \bar{D}_{k-1}\} = 0.$$

Similarly, for any $j \in \{1, \dots, n+1\}$ and $j \neq i, i+1$ we have

$$\rho\{F_{i+1} \lambda D_{j-1} F_{j'} \bar{D}_{i-1} + D_{j-1} F_{(i+1)'} \bar{D}_i\} = 0$$

and

$$\rho\{F_{i+1} \lambda D_{i-1} F_{(i+1)'} \bar{D}_i - D_i F_{i'} \bar{D}_{i-1}\} = 0.$$

These relations imply $\rho\{F_{i+1} \lambda D\} = 0$. Finally, performing similar calculations we get

$$\begin{aligned} \rho\{F_{n+1} \lambda D\} &= D_n^+ (\partial + \lambda) \bar{D}_{n-1} + \sum_{k=1}^{n-1} D_n^+ F_{k'} \bar{D}_{k-1} - D_n^+ (F_{nn} + \lambda) \bar{D}_{n-1} \\ &\quad - D_{n-1}^+ \partial \bar{D}_n + \sum_{j=1}^{n-1} (-1)^{n-j+1} D_{j-1}^+ F_{nj} \bar{D}_n - D_{n-1}^+ (F_{nn} + \lambda) \bar{D}_n. \end{aligned}$$

Applying Lemma 3.1 to the determinants D_n^+ and \bar{D}_n , we get the relations

$$D_n^+ = D_{n-1}^+ (\partial + \lambda + F_{nn}) + \sum_{j=1}^{n-1} (-1)^{n-j} D_{j-1}^+ F_{nj} \quad (4.9)$$

and

$$\bar{D}_n = (\partial + F_{n'n'}) \bar{D}_{n-1} + \sum_{k=1}^{n-1} F_{k'n'} \bar{D}_{k-1} \quad (4.10)$$

which imply that $\rho\{F_{n+1} \lambda D\} = 0$. This shows that all elements w_2, \dots, w_{2n+1} belong to the subalgebra $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$ of $\mathcal{V}(\mathfrak{p})$.

Now use Proposition 2.1. The odd powers of the matrix f form a basis of the centralizer \mathfrak{o}_{2n+1}^f so that for $j = 1, \dots, n$ we can take v_j to be equal, up to a sign, to f^{2j-1} . Then the condition of Proposition 2.1 will hold for the family of elements w_2, w_4, \dots, w_{2n} , thus implying that they are generators of the differential algebra $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$.

The images of the elements w_k under the homomorphism (2.7) are the elements $\tilde{w}_k \in \mathcal{V}(\mathfrak{h})$ found from the relation

$$\begin{aligned} &(\partial + F_{11}) \dots (\partial + F_{nn}) \partial (\partial + F_{n'n'}) \dots (\partial + F_{1'1'}) \\ &= \partial^{2n+1} + \tilde{w}_2 \partial^{2n-1} + \tilde{w}_3 \partial^{2n-2} + \dots + \tilde{w}_{2n+1}. \end{aligned}$$

In the notation of Sec. 2.2 we have

$$h_j^{(r)} = F_{jj}^{(r)} - F_{j+1j+1}^{(r)}, \quad j = 1, \dots, n-1, \quad \text{and} \quad h_n^{(r)} = 2F_{nn}^{(r)}.$$

The Cartan matrix is of the size $n \times n$,

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -2 & 2 \end{bmatrix}$$

so that $a_{ii} = 2$ for $i = 1, \dots, n$ and $a_{ii+1} = a_{i+1i} = a_{n-1n} = -1$ for $i = 1, \dots, n-2$, while $a_{nn-1} = -2$ and all other entries are zero. The entries of the diagonal matrix $D = \text{diag}[\epsilon_1, \dots, \epsilon_n]$ are found by

$$\epsilon_1 = \dots = \epsilon_{n-1} = 1 \quad \text{and} \quad \epsilon_n = 2.$$

Hence, regarding $\mathcal{V}(\mathfrak{h})$ as the algebra of polynomials in the variables $F_{ii}^{(r)}$ with $i = 1, \dots, n$ and $r = 0, 1, \dots$, we get

$$\sum_{j=1}^n a_{ji} \frac{\partial}{\partial h_j^{(r)}} = \frac{\partial}{\partial F_{ii}^{(r)}} - \frac{\partial}{\partial F_{i+1i+1}^{(r)}}, \quad i = 1, \dots, n-1,$$

and

$$\sum_{j=1}^n a_{jn} \frac{\partial}{\partial h_j^{(r)}} = \frac{\partial}{\partial F_{nn}^{(r)}}.$$

Therefore, the screening operators (2.5) take the form

$$V_i = \sum_{r=0}^{\infty} V_{ir} \left(\frac{\partial}{\partial F_{ii}^{(r)}} - \frac{\partial}{\partial F_{i+1i+1}^{(r)}} \right), \quad i = 1, \dots, n-1,$$

and

$$V_n = \sum_{r=0}^{\infty} V_{nr} \frac{\partial}{\partial F_{nn}^{(r)}},$$

where the coefficients V_{ir} are found by the relations

$$\sum_{r=0}^{\infty} \frac{V_{ir} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{F_{ii}^{(m-1)} - F_{i+1i+1}^{(m-1)}}{m!} z^m \right), \quad i = 1, \dots, n-1,$$

and

$$\sum_{r=0}^{\infty} \frac{V_{nr} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{F_{nn}^{(m-1)}}{m!} z^m \right).$$

The differential algebra $\widetilde{\mathcal{W}}(\mathfrak{o}_{2n+1}, f)$ consists of the polynomials in the variables $F_{ii}^{(r)}$, which are annihilated by all operators V_i . It is easy to verify directly that the elements \widetilde{w}_k belong to $\widetilde{\mathcal{W}}(\mathfrak{o}_{2n+1}, f)$; cf. [21, Sec. 4.2]. Furthermore, the elements $\widetilde{w}_2^{(r)}, \widetilde{w}_4^{(r)}, \dots, \widetilde{w}_{2n}^{(r)}$ with r running over nonnegative integers are algebraically independent generators of the algebra $\widetilde{\mathcal{W}}(\mathfrak{o}_{2n+1}, f)$; see [12, Ch. 8]. Hence, the generators $w_2^{(r)}, w_4^{(r)}, \dots, w_{2n}^{(r)}$ of $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$ are also algebraically independent. \square

The injective homomorphism $\mathcal{W}(\mathfrak{o}_{2n+1}, f) \hookrightarrow \mathcal{V}(\mathfrak{h})$ taking w_i to \widetilde{w}_i is known as the *Miura transformation* in type B [9]; see also [12, Ch. 8]. Note that by the arguments used in the proof of Theorem 4.1, the homomorphism (2.7) is bijective in the case $\mathfrak{g} = \mathfrak{o}_{2n+1}$.

Remark 4.2. There is an alternative way to prove the first part of Theorem 4.1 based on the idea of *folding* which was already used in the work [11]; see also [22, Ch. 2]. To outline the argument, let θ denote the involutive automorphism of the differential algebra $\mathcal{V} = \mathcal{V}(\mathfrak{gl}_{2n+1})$ defined by

$$\theta : E_{ij}^{(r)} \mapsto -E_{j'i'}^{(r)}.$$

We have the direct sum decomposition

$$\mathfrak{gl}_{2n+1} = \mathfrak{o}_{2n+1} \oplus \mathfrak{gl}_{2n+1}^-,$$

where \mathfrak{o}_{2n+1} is identified with the fixed point subalgebra under θ , while

$$\mathfrak{gl}_{2n+1}^- = \{X \in \mathfrak{gl}_{2n+1} \mid \theta(X) = -X\}$$

is the subspace of anti-invariants. With the identification $\mathcal{V} = \mathbb{S}(t^{-1}\mathfrak{gl}_{2n+1}[t^{-1}])$ used in Sec. 2.2, for the subalgebra of θ -invariants in \mathcal{V} we have

$$\mathcal{V}^\theta = \mathbb{S}\left(t^{-1}\mathfrak{o}_{2n+1}[t^{-1}] \oplus \mathbb{S}^2\left(t^{-1}\mathfrak{gl}_{2n+1}^-[t^{-1}]\right)\right).$$

This subalgebra is stable under the λ -bracket so that its restriction defines a λ -bracket on \mathcal{V}^θ . Let \mathbb{J} be the ideal of \mathcal{V}^θ generated by the subspace $\mathbb{S}^2\left(t^{-1}\mathfrak{gl}_{2n+1}^-[t^{-1}]\right)$. For any element $P \in \mathcal{V}^\theta$ we have $\{P_\lambda \mathbb{J}\} \subset \mathbb{J}$. The quotient space $\mathcal{V}^\theta/\mathbb{J}$ is naturally identified with the differential algebra $\mathcal{V}(\mathfrak{o}_{2n+1})$. This quotient is equipped with a λ -bracket induced from that of \mathcal{V}^θ . The resulting bracket on $\mathcal{V}(\mathfrak{o}_{2n+1})$ is then obtained as the *folding* of the λ -bracket on $\mathcal{V}(\mathfrak{gl}_{2n+1})$. It coincides with the λ -bracket defined in (2.1).

For any element $P \in \mathcal{V}$ denote by $\text{pr}(P)$ its projection to the subspace \mathcal{V}^θ along the second summand in the decomposition

$$\mathcal{V} = \mathcal{V}^\theta \oplus t^{-1}\mathfrak{gl}_{2n+1}^-[t^{-1}]\mathcal{V}^\theta.$$

Furthermore, let $\tilde{P} \in \mathcal{V}(\mathfrak{o}_{2n+1})$ be the image of $\text{pr}(P)$ under the natural epimorphism

$$\mathcal{V}^\theta \rightarrow \mathcal{V}^\theta/\mathcal{J} \cong \mathcal{V}(\mathfrak{o}_{2n+1}).$$

Set

$$\tilde{f} = E_{21} + E_{32} + \cdots + E_{n+1n} - E_{n+2n+1} - \cdots - E_{2n+12n} \in \mathfrak{gl}_{2n+1}.$$

Theorem 3.2 remains valid with the modified determinant (of the size $(2n+1) \times (2n+1)$) obtained from (3.8) by changing the sign of the entries 1 in the last n columns. Note that \tilde{f} coincides with the element f defined in (4.5). The corresponding coefficients of the differential operator will belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{gl}_{2n+1}, \tilde{f})$. Moreover, the images of these coefficients under the map $P \mapsto \tilde{P}$ belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$. Upon an appropriate re-normalizing of the form (3.7) to match (4.6), we recover the corresponding coefficients of the differential operator (4.7) thus proving that they belong to $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$.

The same argument can be applied to the case of the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$; cf. Sec. 5 below. Some modifications are required for the case of $\mathcal{W}(\mathfrak{g}_2, f)$ as we need to deal with a third order automorphism of \mathfrak{o}_8 ; see Sec. 7 for the details. Note, however, that such a folding argument is not applicable for the reduction in type D , i.e., for the reduction from \mathfrak{gl}_{2n} to \mathfrak{o}_{2n} . The reason, which was already pointed out in [9], is the fact that the principal nilpotent element $f \in \mathfrak{o}_{2n}$ cannot be lifted to a principal nilpotent of the Lie algebra \mathfrak{gl}_{2n} containing \mathfrak{o}_{2n} . \square

Now we use Proposition 3.4 to produce another family of generators of $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$; cf. Corollary 3.7. Observe that the determinant yielding the differential operator (4.7) has the form (3.1). As in (3.5), for any indices $2n+1 \geq i \geq j \geq 1$ set

$$\tilde{a}_{ij} = \begin{cases} \delta_{ij} \partial + F_{ij} & \text{if } n+1 \geq i \geq j, \\ (-1)^{i-n-1} (\delta_{ij} \partial + F_{ij}) & \text{if } i \geq n+1 \geq j, \\ (-1)^{i-j} (\delta_{ij} \partial + F_{ij}) & \text{if } i \geq j \geq n+1. \end{cases} \quad (4.11)$$

By analogy with Sec. 3.3 introduce elements $e_{m0}, h_{m0} \in \mathcal{V}(\mathfrak{p})$ as constant terms of the differential operators,

$$e_{m0} = \sum_{s=1}^m \sum_{i_k \geq j_k \text{ and } i_k < j_{k+1}} (-1)^{m-s} \tilde{a}_{i_1 j_1} \cdots \tilde{a}_{i_s j_s} 1, \quad (4.12)$$

where the second sum is taken over the indices i_1, \dots, i_s and j_1, \dots, j_s such that $i_k \geq j_k$ for $k = 1, \dots, s$ and $i_k < j_{k+1}$ for $k = 1, \dots, s-1$ satisfying (3.12); and

$$h_{m0} = \sum_{s=1}^{\infty} \sum_{i_k \geq j_k \text{ and } i_k \geq j_{k+1}} \tilde{a}_{i_1 j_1} \cdots \tilde{a}_{i_s j_s} 1, \quad (4.13)$$

where the second sum is taken over the indices i_1, \dots, i_s and j_1, \dots, j_s such that $i_k \geq j_k$ for $k = 1, \dots, s$ and $i_k \geq j_{k+1}$ for $k = 1, \dots, s-1$ satisfying (3.12). Proposition 3.6 implies $e_{m0} = w_m$ for all $m = 2, \dots, 2n+1$. Hence, by Theorem 4.1 the family $e_{20}^{(r)}, e_{40}^{(r)}, \dots, e_{2n0}^{(r)}$ with $r = 0, 1, \dots$ is algebraically independent and generates the algebra $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$.

Corollary 4.3. *All elements h_{m0} belong to the algebra $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$. Moreover, the family $h_{20}^{(r)}, h_{40}^{(r)}, \dots, h_{2n0}^{(r)}$ with $r = 0, 1, \dots$ is algebraically independent and generates $\mathcal{W}(\mathfrak{o}_{2n+1}, f)$.*

Proof. The proof is essentially the same as that of Corollary 3.7; the only difference is that the elements e_{2k+10} (resp., h_{2k+10}) with $k = 0, 1, \dots$ are expressible in terms of the elements e_{2k0} (resp., h_{2k0}). \square

5 Generators of $\mathcal{W}(\mathfrak{sp}_{2n}, f)$

As in Sec. 4, for any positive integer n , we use the involution on the set $\{1, \dots, 2n\}$ defined by $i \mapsto i' = 2n - i + 1$. The Lie subalgebra of \mathfrak{gl}_{2n} spanned by the elements

$$F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{j'i'}, \quad i, j = 1, \dots, 2n, \quad (5.1)$$

is the *symplectic Lie algebra* \mathfrak{sp}_{2n} , where we set $\varepsilon_i = 1$ for $i = 1, \dots, n$ and $\varepsilon_i = -1$ for $i = n+1, \dots, 2n$. This is a simple Lie algebra of type C_n . We have the commutation relations

$$[F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} - \varepsilon_i \varepsilon_j (\delta_{ki'} F_{j'l} - \delta_{j'l} F_{ki'}) \quad (5.2)$$

for all $i, j, k, l \in \{1, \dots, 2n\}$. Note also the symmetry relation

$$F_{ij} + \varepsilon_i \varepsilon_j F_{j'i'} = 0, \quad i, j \in \{1, \dots, 2n\}.$$

The elements F_{11}, \dots, F_{nn} span a Cartan subalgebra of \mathfrak{sp}_{2n} which we will denote by \mathfrak{h} . The respective subsets of elements F_{ij} with $i < j$ and $i > j$ span the nilpotent subalgebras \mathfrak{n}_+ and \mathfrak{n}_- . The subalgebra $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{h}$ is then spanned by the elements F_{ij} with $i \geq j$.

As before, we will work with the algebra of differential operators $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$, where the commutation relations are given by

$$\partial F_{ij}^{(r)} - F_{ij}^{(r)} \partial = F_{ij}^{(r+1)}.$$

For any element $g \in \mathcal{V}(\mathfrak{p})$ and any nonnegative integer r the element $g^{(r)}$ coincides with the constant term of the differential operator $\partial^r g$ as in (3.6).

Take the principal nilpotent element $f \in \mathfrak{sp}_{2n}$ in the form

$$f = F_{21} + F_{32} + \dots + F_{nn-1} + \frac{1}{2} F_{n'n}. \quad (5.3)$$

The \mathfrak{sl}_2 -triple is formed by the elements $\{e, f, h\}$ with

$$e = \sum_{i=1}^{n-1} i(2n-i) F_{i,i+1} + \frac{n^2}{2} F_{nn'} \quad \text{and} \quad h = \sum_{i=1}^n (2n-2i+1) F_{ii}.$$

The invariant symmetric bilinear form on \mathfrak{sp}_{2n} is defined by

$$(X|Y) = \frac{1}{2} \operatorname{tr} XY, \quad X, Y \in \mathfrak{sp}_{2n}, \quad (5.4)$$

where X and Y are understood as $2n \times 2n$ symplectic matrices over \mathbb{C} .

Consider the determinant (3.2) of the matrix with entries in $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

$$\det \begin{bmatrix} \partial + F_{11} & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ F_{21} & \partial + F_{22} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots \\ F_{n1} & F_{n2} & \dots & \partial + F_{nn} & 1 & 0 & \dots & 0 & 0 \\ F_{n'1} & F_{n'2} & \dots & F_{n'n} & \partial + F_{n'n'} & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ F_{2'1} & F_{2'2} & \dots & F_{2'n} & F_{2'n'} & \dots & \dots & \partial + F_{2'2'} & -1 \\ F_{1'1} & F_{1'2} & \dots & F_{1'n} & F_{1'n'} & \dots & \dots & F_{1'2'} & \partial + F_{1'1'} \end{bmatrix}$$

so that the (i, j) entry of the matrix is $\delta_{ij} \partial + F_{ij}$ for $i \geq j$, the $(i, i+1)$ entry is 1 for $i \leq n$ and -1 for $i > n$, while the remaining entries are zero. The determinant has the form

$$\partial^{2n} + w_2 \partial^{2n-2} + w_3 \partial^{2n-3} + \dots + w_{2n}, \quad w_i \in \mathcal{V}(\mathfrak{p}). \quad (5.5)$$

Theorem 5.1. *All elements w_2, \dots, w_{2n} belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$. Moreover, the elements $w_2^{(r)}, w_4^{(r)}, \dots, w_{2n}^{(r)}$ with $r = 0, 1, \dots$ are algebraically independent and generate the algebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$.*

Proof. Denote the determinant by D and let D_i (resp., \overline{D}_i) denote the $i \times i$ minor corresponding to the first (resp., last) i rows and columns. We suppose that $D_0 = \overline{D}_0 = 1$. Lemma 3.1 implies the expansion

$$D = D_n \overline{D}_n + \sum_{j,k=1}^n (-1)^{n-j+1} D_{j-1} F_{k'j} \overline{D}_{k-1}.$$

If $1 \leq i \leq n-1$ then the relation $\rho\{F_{i,i+1} \lambda D\} = 0$ follows by the same calculation as in the proof of Theorem 4.1. Furthermore,

$$\begin{aligned} \rho\{F_{n'n'} \lambda D\} &= -2D_{n-1}^+ \overline{D}_n + 2D_n^+ \overline{D}_{n-1} + 2 \sum_{k=1}^{n-1} D_{n-1}^+ F_{k'n'} \overline{D}_{k-1} \\ &\quad - 2D_{n-1}^+ (2F_{nn} + \lambda) \overline{D}_{n-1} + 2 \sum_{j=1}^{n-1} (-1)^{n-j+1} D_{j-1}^+ F_{nj} \overline{D}_{n-1}. \end{aligned}$$

This is zero since the relations (4.9) and (4.10) are valid for the case of \mathfrak{sp}_{2n} as well. Thus, all elements w_2, \dots, w_{2n} belong to the subalgebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$ of $\mathcal{V}(\mathfrak{p})$.

The odd powers of the matrix f form a basis of the centralizer \mathfrak{sp}_{2n}^f . For $j = 1, \dots, n$ take v_j to be equal, up to a sign, to f^{2j-1} . The condition of Proposition 2.1 will hold for the family of elements w_2, w_4, \dots, w_{2n} , thus implying that they are generators of the differential algebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$.

The images of the elements w_k under the homomorphism (2.7) are the elements $\tilde{w}_k \in \mathcal{V}(\mathfrak{h})$ found from the relation

$$\begin{aligned} (\partial + F_{11}) \dots (\partial + F_{nn}) (\partial + F_{n'n'}) \dots (\partial + F_{1'1'}) \\ = \partial^{2n} + \tilde{w}_2 \partial^{2n-2} + \tilde{w}_3 \partial^{2n-3} + \dots + \tilde{w}_{2n}. \end{aligned}$$

In the notation of Sec. 2.2 we have

$$h_j^{(r)} = F_{jj}^{(r)} - F_{j+1j+1}^{(r)}, \quad j = 1, \dots, n-1, \quad \text{and} \quad h_n^{(r)} = F_{nn}^{(r)}.$$

The Cartan matrix is of the size $n \times n$,

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -2 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix}$$

so that $a_{ii} = 2$ for $i = 1, \dots, n$ and $a_{ii+1} = a_{i+1i} = a_{nn-1} = -1$ for $i = 1, \dots, n-2$, while $a_{n-1n} = -2$ and all other entries are zero. The entries of the diagonal matrix $D = \text{diag}[\epsilon_1, \dots, \epsilon_n]$ are found by

$$\epsilon_1 = \dots = \epsilon_{n-1} = 1 \quad \text{and} \quad \epsilon_n = 1/2.$$

Hence, regarding $\mathcal{V}(\mathfrak{h})$ as the algebra of polynomials in the variables $F_{ii}^{(r)}$ with $i = 1, \dots, n$ and $r = 0, 1, \dots$, we get

$$\sum_{j=1}^n a_{ji} \frac{\partial}{\partial h_j^{(r)}} = \frac{\partial}{\partial F_{ii}^{(r)}} - \frac{\partial}{\partial F_{i+1i+1}^{(r)}}, \quad i = 1, \dots, n-1,$$

and

$$\sum_{j=1}^n a_{jn} \frac{\partial}{\partial h_j^{(r)}} = 2 \frac{\partial}{\partial F_{nn}^{(r)}}.$$

Therefore, the screening operators (2.5) take the form

$$V_i = \sum_{r=0}^{\infty} V_{ir} \left(\frac{\partial}{\partial F_{ii}^{(r)}} - \frac{\partial}{\partial F_{i+1 i+1}^{(r)}} \right), \quad i = 1, \dots, n-1,$$

and

$$V_n = 2 \sum_{r=0}^{\infty} V_{nr} \frac{\partial}{\partial F_{nn}^{(r)}},$$

where the coefficients V_{ir} are found by the relations

$$\sum_{r=0}^{\infty} \frac{V_{ir} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{F_{ii}^{(m-1)} - F_{i+1 i+1}^{(m-1)}}{m!} z^m \right), \quad i = 1, \dots, n-1,$$

and

$$\sum_{r=0}^{\infty} \frac{V_{nr} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{2 F_{nn}^{(m-1)}}{m!} z^m \right).$$

The differential algebra $\widetilde{\mathcal{W}}(\mathfrak{sp}_{2n}, f)$ consists of the polynomials in the variables $F_{ii}^{(r)}$, which are annihilated by all operators V_i . One easily verifies directly that the elements \widetilde{w}_k belong to $\widetilde{\mathcal{W}}(\mathfrak{sp}_{2n}, f)$; cf. [21, Sec. 4.2]. Moreover, the elements $\widetilde{w}_2^{(r)}, \widetilde{w}_4^{(r)}, \dots, \widetilde{w}_{2n}^{(r)}$ with r running over nonnegative integers are algebraically independent generators of the algebra $\widetilde{\mathcal{W}}(\mathfrak{sp}_{2n}, f)$; see [12, Ch. 8]. Hence, the generators $w_2^{(r)}, w_4^{(r)}, \dots, w_{2n}^{(r)}$ of $\mathcal{W}(\mathfrak{sp}_{2n}, f)$ are also algebraically independent. \square

The injective homomorphism $\mathcal{W}(\mathfrak{sp}_{2n}, f) \hookrightarrow \mathcal{V}(\mathfrak{h})$ taking w_i to \widetilde{w}_i is known as the *Miura transformation* in type C [9]; see also [12, Ch. 8]. Note that by the arguments used in the proof of Theorem 5.1, the homomorphism (2.7) is bijective in the case $\mathfrak{g} = \mathfrak{sp}_{2n}$. An alternative proof of the first part of the theorem can be obtained with the use of the folding procedure; see Remark 4.2.

Another family of generators of the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$ can be constructed by using Proposition 3.4; cf. Corollaries 3.7 and 4.3. Extend the definition (4.11) to the symplectic case by restricting the range of the indices to $2n \geq i \geq j \geq 1$. Furthermore, define elements $e_{m0}, h_{m0} \in \mathcal{V}(\mathfrak{p})$ by the formulas (4.12) and (4.13). Proposition 3.6 implies $e_{m0} = w_m$ for $m = 2, 3, \dots, 2n$. Hence, by Theorem 5.1 the family $e_{20}^{(r)}, e_{40}^{(r)}, \dots, e_{2n0}^{(r)}$ with $r = 0, 1, \dots$ is algebraically independent and generates the algebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$.

Corollary 5.2. *All elements h_{m0} belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{sp}_{2n}, f)$. Moreover, the family $h_{20}^{(r)}, h_{40}^{(r)}, \dots, h_{2n0}^{(r)}$ with $r = 0, 1, \dots$ is algebraically independent and generates $\mathcal{W}(\mathfrak{sp}_{2n}, f)$. \square*

6 Generators of $\mathcal{W}(\mathfrak{o}_{2n}, f)$

We keep the notation for the generators of the Lie algebra \mathfrak{o}_{2n} introduced in the beginning of Sec. 4 by taking $N = 2n$. In particular, we have the relations (4.2) and (4.3). We will work with the algebra of pseudo-differential operators $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}((\partial^{-1}))$, where the relations are given by (4.4) and

$$\partial^{-1} F_{ij}^{(r)} = \sum_{s=0}^{\infty} (-1)^s F_{ij}^{(r+s)} \partial^{-s-1}.$$

Take the principal nilpotent element $f \in \mathfrak{o}_{2n}$ in the form

$$f = F_{21} + F_{32} + \cdots + F_{nn-1} + F_{n'n-1}. \quad (6.1)$$

The \mathfrak{sl}_2 -triple is formed by the elements $\{e, f, h\}$ with

$$e = \sum_{i=1}^{n-2} i(2n-i-1) F_{ii+1} + \frac{n^2-n}{2} (F_{n-1n} + F_{n-1n'}) \quad \text{and} \quad h = 2 \sum_{i=1}^{n-1} (n-i) F_{ii}.$$

The invariant symmetric bilinear form on \mathfrak{o}_{2n} is defined by

$$(X|Y) = \frac{1}{2} \text{tr } XY, \quad X, Y \in \mathfrak{o}_{2n}, \quad (6.2)$$

where X and Y are understood as matrices over \mathbb{C} which are skew-symmetric with respect to the antidiagonal.

Consider the following $(2n+1) \times (2n+1)$ matrix with entries in $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}((\partial^{-1}))$,

$$\begin{bmatrix} \partial + F_{11} & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ F_{21} & \partial + F_{22} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{n1} - F_{n'1} & F_{n2} - F_{n'2} & \cdots & \partial + F_{nn} & 0 & -2\partial & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \partial^{-1} & 0 & \cdots & 0 & 0 \\ F_{n'1} & F_{n'2} & \cdots & 0 & 0 & \partial + F_{n'n'} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ F_{2'1} & 0 & \cdots & \cdots & 0 & F_{2'n'} - F_{2'n} & \cdots & \partial + F_{2'2'} & -1 \\ 0 & F_{1'2} & \cdots & \cdots & 0 & F_{1'n'} - F_{1'n} & \cdots & F_{1'2'} & \partial + F_{1'1'} \end{bmatrix}$$

so that the first $n-1$ rows, the last $n-1$ columns and the lower-left $n \times n$ submatrix respectively coincide with those of the matrix introduced for \mathfrak{o}_{2n+1} in Sec. 4. All entries in the row and column $n+1$ are zero, except for the $(n+1, n+1)$ entry which equals ∂^{-1} . The (n, j) entries are $F_{nj} - F_{n'j}$ for $j = 1, \dots, n-1$, the (n, n) entry is $\partial + F_{nn}$ and the $(n, n+2)$ entry is -2∂ , while the remaining entries in row n are zero. Finally, the

remaining nonzero entries in column $n + 2$ are $F_{k'n'} - F_{k'n}$ for $k = 1, 2, \dots, n - 1$ which occur in the respective rows $2n - k + 2$, and $\partial + F_{n'n'}$ which occurs in row $n + 2$.

One easily verifies that the column-determinant and row-determinant of this matrix coincide, so that the determinant (3.2) is well-defined and we denote it by D . Applying the simultaneous column expansion along the first n columns and using Lemma 3.1, we derive that it can be written in the form

$$D = D_n \partial^{-1} \overline{D}_n + 2 \sum_{j,k=1}^n (-1)^{n-j} D_{j-1} F_{k'j} \overline{D}_{k-1}, \quad (6.3)$$

where D_i (resp., \overline{D}_i) denotes the $i \times i$ minor corresponding to the first (resp., last) i rows and columns. We suppose that $D_0 = \overline{D}_0 = 1$. Write

$$\begin{aligned} D_n &= \partial^n + y_1 \partial^{n-1} + y_2 \partial^{n-2} + \dots + y_n, \\ \overline{D}_n &= \partial^n + \partial^{n-1} \bar{y}_1 + \partial^{n-2} \bar{y}_2 + \dots + \bar{y}_n, \end{aligned}$$

for certain uniquely determined elements $y_i, \bar{y}_i \in \mathcal{V}(\mathfrak{p})$.

Lemma 6.1. *We have $\bar{y}_i = (-1)^i y_i$ for all $i = 1, \dots, n$.*

Proof. Replace ∂ by $-\partial$ in the minor \overline{D}_n and multiply each row by -1 . The lemma can then be equivalently stated as the identity

$$(-1)^n \overline{D}_n \Big|_{\partial \mapsto -\partial} = \partial^n + \partial^{n-1} y_1 + \partial^{n-2} y_2 + \dots + y_n.$$

The left hand side is the determinant

$$\det \begin{bmatrix} \partial + F_{nn} & 1 & 0 & 0 & \dots & 0 \\ F_{nn-1} - F_{n'n-1} & \partial + F_{n-1n-1} & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ F_{n2} - F_{n'2} & F_{n-12} & \dots & \dots & \partial + F_{22} & 1 \\ F_{n1} - F_{n'1} & F_{n-11} & \dots & \dots & F_{21} & \partial + F_{11} \end{bmatrix}.$$

The claim now follows from the observation that this determinant coincides with the image of D_n under the anti-automorphism of the algebra $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$ which is the identity on the generators F_{ij} and ∂ . \square

Lemma 6.1 implies that the pseudo-differential operator D can be written as

$$D = \partial^{2n-1} + w_2 \partial^{2n-3} + w_3 \partial^{2n-4} + \dots + w_{2n-1} + (-1)^n y_n \partial^{-1} y_n, \quad w_i \in \mathcal{V}(\mathfrak{p}). \quad (6.4)$$

Theorem 6.2. *The elements $w_2, w_3, \dots, w_{2n-1}$ and y_n belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{o}_{2n}, f)$. Moreover, the elements $w_2^{(r)}, w_4^{(r)}, \dots, w_{2n-2}^{(r)}, y_n^{(r)}$ with $r = 0, 1, \dots$ are algebraically independent and generate the algebra $\mathcal{W}(\mathfrak{o}_{2n}, f)$.*

Proof. The relation $\rho\{F_{i+1} \lambda D\} = 0$ for $1 \leq i \leq n-1$ follows by the same calculations as in the proof of Theorem 4.1. Furthermore, let $\sigma = (n n')$ be the permutation of the set of indices $\{1, \dots, 2n\}$ which swaps n and $n' = n+1$ and leaves all other indices fixed. The mapping

$$\varsigma : F_{ij} \mapsto F_{\sigma(i)\sigma(j)} \quad (6.5)$$

defines an involutive automorphism of the Lie algebra \mathfrak{o}_{2n} . It also extends to an involutive automorphism of the Poisson vertex algebra $\mathcal{V}(\mathfrak{o}_{2n})$. We claim that all coefficients of the pseudo-differential operator D are ς -invariant. Indeed, let us apply the following operations on rows and columns of the given matrix. Replace row $n+2$ by the sum of rows n and $n+2$. Then replace column n by the sum of columns n and $n+2$. Finally, multiply row n and column $n+2$ by -1 . As a result, we get the image of the matrix with respect to the involution (6.5). On the other hand, the determinant D remains unchanged. This proves the relation $\rho\{F_{n-1n'} \lambda D\} = 0$. This shows that all coefficients of the operator D belong to $\mathcal{W}(\mathfrak{o}_{2n}, f)$.

Now consider the minor D_n . It can be written in the form

$$\begin{aligned} D_n = & D_{n-1}(\partial + F_{nn}) - D_{n-2}(F_{nn-1} - F_{n'n-1}) + D_{n-3}(F_{nn-2} - F_{n'n-2}) \\ & + \dots + (-1)^{n-2} D_1(F_{n2} - F_{n'2}) + (-1)^{n-1} D_0(F_{n1} - F_{n'1}). \end{aligned} \quad (6.6)$$

The same calculations as in the proof of Theorem 3.2, show that $\rho\{F_{i+1} \lambda D_n\} = 0$ for $i = 1, \dots, n-1$. Furthermore,

$$\begin{aligned} \rho\{F_{n-1n'} \lambda D_n\} = & -D_{n-1}^+ - D_{n-2}^+(\partial + F_{nn}) + D_{n-2}^+(F_{n-1n-1} + F_{nn} + \lambda) \\ & - D_{n-3}^+ F_{n-1n-2} + \dots + (-1)^{n-1} D_1^+ F_{n-12} + (-1)^n D_0^+ F_{n-11}. \end{aligned}$$

Applying relation (3.10) to the determinant D_{n-1}^+ we get

$$\rho\{F_{n-1n'} \lambda D_n\} = -2 D_{n-2}^+ \partial.$$

This implies $\rho\{F_{n-1n'} \lambda y_n\} = 0$ so that the constant term y_n of the differential operator D_n belongs to $\mathcal{W}(\mathfrak{o}_{2n}, f)$.

To use Proposition 2.1, note that the odd powers f, f^3, \dots, f^{2n-3} of the matrix f together with the element $F_{n1} - F_{n'1}$ form a basis of the centralizer \mathfrak{o}_{2n}^f . Take v_j to be equal, up to a sign, to f^{2j-1} for $j = 1, \dots, n-1$ and set $v_n = F_{n1} - F_{n'1}$. The condition of Proposition 2.1 holds for the family $w_2, w_4, \dots, w_{2n-2}, y_n$, thus implying that they are generators of the differential algebra $\mathcal{W}(\mathfrak{o}_{2n}, f)$.

The images of the coefficients of the operator D under the homomorphism (2.7) are the elements $\tilde{w}_k, \tilde{y}_n \in \mathcal{V}(\mathfrak{h})$ found from the relation

$$\begin{aligned} & (\partial + F_{11}) \dots (\partial + F_{nn}) \partial^{-1} (\partial + F_{n'n'}) \dots (\partial + F_{1'1'}) \\ & = \partial^{2n-1} + \tilde{w}_2 \partial^{2n-3} + \tilde{w}_3 \partial^{2n-4} + \dots + \tilde{w}_{2n-1} + (-1)^n \tilde{y}_n \partial^{-1} \tilde{y}_n. \end{aligned}$$

In particular,

$$\tilde{y}_n = (\partial + F_{11}) \dots (\partial + F_{nn}) 1.$$

In the notation of Sec. 2.2 we have

$$h_j^{(r)} = F_{jj}^{(r)} - F_{j+1j+1}^{(r)}, \quad j = 1, \dots, n-1, \quad \text{and} \quad h_n^{(r)} = F_{n-1n-1}^{(r)} + F_{nn}^{(r)}.$$

The Cartan matrix is of the size $n \times n$,

$$A = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{bmatrix}$$

so that $a_{ii} = 2$ for $i = 1, \dots, n$ and $a_{ii+1} = a_{i+1i} = a_{n-2n} = a_{nn-2} = -1$ for $i = 1, \dots, n-2$ and all other entries are zero. Then the diagonal matrix D (see Sec. 2.2) is the identity matrix so that $\epsilon_1 = \dots = \epsilon_n = 1$. Hence, regarding $\mathcal{V}(\mathfrak{h})$ as the algebra of polynomials in the variables $F_{ii}^{(r)}$ with $i = 1, \dots, n$ and $r = 0, 1, \dots$, we get

$$\sum_{j=1}^n a_{ji} \frac{\partial}{\partial h_j^{(r)}} = \frac{\partial}{\partial F_{ii}^{(r)}} - \frac{\partial}{\partial F_{i+1i+1}^{(r)}}, \quad i = 1, \dots, n-1,$$

and

$$\sum_{j=1}^n a_{jn} \frac{\partial}{\partial h_j^{(r)}} = \frac{\partial}{\partial F_{n-1n-1}^{(r)}} + \frac{\partial}{\partial F_{nn}^{(r)}}.$$

Therefore, the screening operators (2.5) take the form

$$V_i = \sum_{r=0}^{\infty} V_{ir} \left(\frac{\partial}{\partial F_{ii}^{(r)}} - \frac{\partial}{\partial F_{i+1i+1}^{(r)}} \right), \quad i = 1, \dots, n-1,$$

and

$$V_n = \sum_{r=0}^{\infty} V_{nr} \left(\frac{\partial}{\partial F_{n-1n-1}^{(r)}} + \frac{\partial}{\partial F_{nn}^{(r)}} \right),$$

where the coefficients V_{ir} are found by the relations

$$\sum_{r=0}^{\infty} \frac{V_{ir} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{F_{ii}^{(m-1)} - F_{i+1i+1}^{(m-1)}}{m!} z^m \right), \quad i = 1, \dots, n-1,$$

and

$$\sum_{r=0}^{\infty} \frac{V_{nr} z^r}{r!} = \exp \left(- \sum_{m=1}^{\infty} \frac{F_{n-1n-1}^{(m-1)} + F_{nn}^{(m-1)}}{m!} z^m \right).$$

The differential algebra $\widetilde{\mathcal{W}}(\mathfrak{o}_{2n}, f)$ consists of the polynomials in the variables $F_{ii}^{(r)}$, which are annihilated by all operators V_i . It is easy to verify directly that the elements \widetilde{w}_k and \widetilde{y}_n belong to $\widetilde{\mathcal{W}}(\mathfrak{o}_{2n}, f)$; cf. [21, Sec. 4.2]. The elements $\widetilde{w}_2^{(r)}, \widetilde{w}_4^{(r)}, \dots, \widetilde{w}_{2n-2}^{(r)}, \widetilde{y}_n^{(r)}$ with r running over nonnegative integers are known to be algebraically independent generators of the algebra $\widetilde{\mathcal{W}}(\mathfrak{o}_{2n}, f)$; see [12, Ch. 8]. Hence, the generators $w_2^{(r)}, w_4^{(r)}, \dots, w_{2n-2}^{(r)}, y_n^{(r)}$ of $\mathcal{W}(\mathfrak{o}_{2n}, f)$ are also algebraically independent. \square

By the arguments used in the proof of Theorem 6.2, the homomorphism (2.7) is bijective in the case $\mathfrak{g} = \mathfrak{o}_{2n}$.

Another family of generators of $\mathcal{W}(\mathfrak{o}_{2n}, f)$ analogous to those described in Corollaries 3.7, 4.3 and 5.2, can be obtained from an appropriate analogue of Proposition 3.4. This is yet another version of the MacMahon Master Theorem involving the noncommutative elementary and complete symmetric functions associated with the $(2n+1) \times (2n+1)$ generators matrix introduced in the beginning of this section.

Using the determinant $D = D(\partial)$ define the elements $e_m \in \mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$ as the coefficients of the formal power series in t ,

$$\sum_{m=0}^{\infty} e_m t^m = t^{2n-1} D(\partial + t^{-1}).$$

Denote this series by $e(t)$. Furthermore, define the elements $h_m \in \mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$ as the coefficients of the series

$$h(t) = \sum_{m=0}^{\infty} h_m t^m, \quad h(t) = e(-t)^{-1}.$$

Write the expansions (3.14) and (3.15). By Theorem 6.2, all coefficients of the differential operators e_m and h_m belong to $\mathcal{W}(\mathfrak{o}_{2n}, f)$. Explicit formulas for these coefficients in terms of the generators F_{ij} and ∂ take the form similar to those of Sec. 3.2 but are more complicated. For this reason we will not reproduce them. On the other hand, their images under the isomorphism (2.7) are easy to describe. Set $a_{ii} = \partial + F_{ii}$. We have

$$\phi : e(t) \mapsto (1 + t a_{11}) \dots (1 + t a_{nn}) (1 + t \partial)^{-1} (1 + t a_{n'n'}) \dots (1 + t a_{1'1'})$$

and hence

$$\phi : h(t) \mapsto (1 - t a_{1'1'})^{-1} \dots (1 - t a_{n'n'})^{-1} (1 - t \partial) (1 - t a_{nn})^{-1} \dots (1 - t a_{11})^{-1}.$$

In particular, the constant terms e_{m0} of the differential operators e_m coincide with the elements w_m for $m = 2, \dots, 2n - 1$. Observe that since $a_{nn} + a_{n'n'} = 2\partial$, we have the relation

$$(1 - t a_{n'n'})^{-1} (1 - t \partial) (1 - t a_{nn})^{-1} = \frac{1}{2} \left((1 - t a_{nn})^{-1} + (1 - t a_{n'n'})^{-1} \right).$$

Therefore, the images of the elements h_m can be written explicitly as

$$\phi(h_m) = \frac{1}{2} \sum_{k_1 + \dots + k_1 = m} a_{1'1'}^{k_1'} \dots a_{n'n'}^{k_n'} a_{nn}^{k_n} \dots a_{11}^{k_1},$$

summed over nonnegative integers k_i such that either $k_n = 0$ or $k_{n'} = 0$.

The generators $\phi(w_i)$ and $\phi(y_n)$ of the differential algebra $\widetilde{\mathcal{W}}(\mathfrak{o}_{2n}, f)$ were introduced in [9] in relation with the *Miura transformation* in type D . The coefficients of the differential operators $\phi(h_m)$ were found in [21] as the Harish-Chandra images of generators of the center of the affine vertex algebra at the critical level in type D (in a slightly different notation). The relation $e(-t)h(t) = 1$ thus makes a connection between the two families of generators. Moreover, we obtain the following.

Corollary 6.3. *The family $h_{20}^{(r)}, h_{40}^{(r)}, \dots, h_{2n-20}^{(r)}, y_n^{(r)}$ with $r = 0, 1, \dots$ is algebraically independent and generates $\mathcal{W}(\mathfrak{o}_{2n}, f)$. \square*

7 Generators of $\mathcal{W}(\mathfrak{g}_2, f)$

We denote by \mathfrak{g}_2 the simple Lie algebra of type G_2 with the Cartan matrix

$$A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

This Lie algebra is 14-dimensional. An explicit basis and the multiplication table is given in [13, Lec. 22]. A simple combinatorial construction of \mathfrak{g}_2 is given in [23] which we will follow below. We let α and β denote the simple roots, and the set of positive roots is

$$\alpha, \quad \beta, \quad \alpha + \beta, \quad \alpha + 2\beta, \quad \alpha + 3\beta, \quad 2\alpha + 3\beta.$$

For each positive root γ we let X_γ and Y_γ denote the root vectors associated with γ and $-\gamma$, respectively. The Cartan subalgebra \mathfrak{h} of \mathfrak{g}_2 is two-dimensional and we let H_α and H_β denote its basis elements. The full list of commutators between the basis elements of \mathfrak{g}_2 is given in [23].¹ In particular,

$$[X_\alpha, Y_\alpha] = H_\alpha, \quad [X_\beta, Y_\beta] = H_\beta$$

¹The preprint version of [23] provides such relations for the basis elements which differ from the journal version by signs.

and

$$\begin{aligned} [H_\alpha, X_\alpha] &= -2X_\alpha, & [H_\alpha, X_\beta] &= X_\beta, \\ [H_\beta, X_\beta] &= -2X_\beta, & [H_\beta, X_\alpha] &= 3X_\alpha. \end{aligned}$$

The commutation relations can also be recovered from the identification of the basis elements of \mathfrak{g}_2 as elements of \mathfrak{o}_7 or \mathfrak{o}_8 under the embeddings $\mathfrak{g}_2 \subset \mathfrak{o}_7$ or $\mathfrak{g}_2 \subset \mathfrak{o}_8$ which we use below.

The respective subsets of elements X_γ and Y_γ with γ running over the set of positive roots span the nilpotent subalgebras \mathfrak{n}_+ and \mathfrak{n}_- . As before, we set $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{h}$. We will be working with the algebra of differential operators $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$, where

$$\partial X^{(r)} - X^{(r)} \partial = X^{(r+1)}, \quad X \in \mathfrak{p}.$$

Take the principal nilpotent element $f \in \mathfrak{g}_2$ in the form

$$f = Y_\alpha + Y_\beta. \quad (7.1)$$

The \mathfrak{sl}_2 -triple is formed by the elements $\{e, f, h\}$ with

$$e = -10X_\alpha - 6X_\beta \quad \text{and} \quad h = -10H_\alpha - 6H_\beta.$$

The product $D^{-1}A$ is a symmetric matrix for $D = \text{diag}[1, 3]$ so that $\epsilon_1 = 1$ and $\epsilon_2 = 3$. The invariant symmetric bilinear form on \mathfrak{g}_2 is then uniquely determined by the conditions

$$(X_\alpha | Y_\alpha) = -1 \quad \text{and} \quad (X_\beta | Y_\beta) = -3. \quad (7.2)$$

Consider the determinant (3.2) of the 7×7 matrix with entries in $\mathcal{V}(\mathfrak{p}) \otimes \mathbb{C}[\partial]$,

$$\det \begin{bmatrix} \partial + \tilde{F}_{11} & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} Y_\beta & \partial + \tilde{F}_{22} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} Y_{\alpha+\beta} & Y_\alpha & \partial + \tilde{F}_{33} & 1 & 0 & 0 & 0 \\ \frac{4}{9} Y_{\alpha+2\beta} & -\frac{2}{3} Y_{\alpha+\beta} & \frac{2}{3} Y_\beta & \partial & -1 & 0 & 0 \\ -\frac{4}{9} Y_{\alpha+3\beta} & \frac{4}{9} Y_{\alpha+2\beta} & 0 & -\frac{2}{3} Y_\beta & \partial - \tilde{F}_{33} & -1 & 0 \\ \frac{4}{9} Y_{2\alpha+3\beta} & 0 & -\frac{4}{9} Y_{\alpha+2\beta} & \frac{2}{3} Y_{\alpha+\beta} & -Y_\alpha & \partial - \tilde{F}_{22} & -1 \\ 0 & -\frac{4}{9} Y_{2\alpha+3\beta} & \frac{4}{9} Y_{\alpha+3\beta} & -\frac{4}{9} Y_{\alpha+2\beta} & -\frac{1}{3} Y_{\alpha+\beta} & -\frac{1}{3} Y_\beta & \partial - \tilde{F}_{11} \end{bmatrix},$$

where

$$\tilde{F}_{11} = -H_\alpha - \frac{2}{3} H_\beta, \quad \tilde{F}_{22} = -H_\alpha - \frac{1}{3} H_\beta \quad \text{and} \quad \tilde{F}_{33} = -\frac{1}{3} H_\beta.$$

The determinant has the form

$$D = \partial^7 + w_2 \partial^5 + w_3 \partial^4 + w_4 \partial^3 + w_5 \partial^2 + w_6 \partial + w_7, \quad w_i \in \mathcal{V}(\mathfrak{p}). \quad (7.3)$$

Theorem 7.1. *The elements w_2, \dots, w_7 belong to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}_2, f)$. Moreover, the elements $w_2^{(r)}, w_6^{(r)}$ with $r = 0, 1, \dots$ are algebraically independent and generate the algebra $\mathcal{W}(\mathfrak{g}_2, f)$.*

Proof. Consider the Lie algebra \mathfrak{o}_8 and its diagram automorphism of order 3. We can regard \mathfrak{g}_2 as its fixed point subalgebra. We will use a folding procedure to construct the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}_2, f)$ from its counterpart $\mathcal{W}(\mathfrak{o}_8, f)$; cf. [11]. More precisely, using the notation of Sec. 6 for \mathfrak{o}_8 , identify \mathfrak{g}_2 with a subalgebra of \mathfrak{o}_8 by setting

$$X_\alpha = -F_{23}, \quad X_\beta = -F_{12} - F_{34} - F_{34'}, \quad Y_\alpha = F_{32}, \quad Y_\beta = F_{21} + F_{43} + F_{4'3},$$

where $i' = 9 - i$. The remaining basis elements of \mathfrak{g}_2 are then uniquely recovered, and for the positive root vectors we have

$$\begin{aligned} X_{\alpha+\beta} &= -F_{13} + F_{24} + F_{24'}, & X_{\alpha+2\beta} &= -F_{14} - F_{14'} - F_{23'}, \\ X_{\alpha+3\beta} &= F_{13'}, & X_{2\alpha+3\beta} &= -F_{12'}. \end{aligned}$$

The mapping $\psi : F_{ij} \mapsto -F_{ji}$ defines an involutive automorphism of \mathfrak{o}_8 . It preserves the subalgebra \mathfrak{g}_2 and the restriction to this subalgebra yields an involutive automorphism of the latter such that

$$\psi : X_\gamma \mapsto Y_\gamma, \quad Y_\gamma \mapsto X_\gamma,$$

for all positive roots γ . Moreover, $\psi(H_\alpha) = -H_\alpha$ and $\psi(H_\beta) = -H_\beta$. So we thus obtain expressions of the remaining basis vectors of \mathfrak{g}_2 in terms of the generators of \mathfrak{o}_8 ,

$$H_\alpha = -F_{22} + F_{33}, \quad H_\beta = -F_{11} + F_{22} - 2F_{33}$$

and

$$\begin{aligned} Y_{\alpha+\beta} &= F_{31} - F_{42} - F_{4'2}, & Y_{\alpha+2\beta} &= F_{41} + F_{4'1} + F_{3'2}, \\ Y_{\alpha+3\beta} &= -F_{3'1}, & Y_{2\alpha+3\beta} &= F_{2'1}. \end{aligned}$$

Observe that the subalgebra $\mathfrak{g}_2 \subset \mathfrak{o}_8$ is contained in the subalgebra $\mathfrak{o}_7 \subset \mathfrak{o}_8$ defined as the span of the elements

$$F_{ij}^\circ = F_{ij}, \quad F_{ij'}^\circ = \frac{1}{2} F_{ij'}, \quad F_{i'j}^\circ = 2 F_{i'j}, \quad 1 \leq i, j \leq 3,$$

and

$$F_{4i}^\circ = F_{4i} + F_{4'i}, \quad F_{i4}^\circ = \frac{1}{2} (F_{i4} + F_{i4'}), \quad 1 \leq i \leq 3.$$

In particular, we can identify $f \in \mathfrak{g}_2$ with the principal nilpotent elements

$$f = F_{21}^\circ + F_{32}^\circ + F_{43}^\circ \in \mathfrak{o}_7 \quad \text{and} \quad f = F_{21} + F_{32} + F_{43} + F_{4'3} \in \mathfrak{o}_8.$$

Now we apply the folding procedure as a reduction from the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{o}_8, f)$ to $\mathcal{W}(\mathfrak{g}_2, f)$; cf. Remark 4.2. Let ϑ denote the diagram automorphism of the differential algebra $\mathcal{V} = \mathcal{V}(\mathfrak{o}_8)$ defined by

$$\vartheta : F_{23}^{(r)} \mapsto F_{23}^{(r)}, \quad F_{12}^{(r)} \mapsto F_{34}^{(r)}, \quad F_{34}^{(r)} \mapsto F_{34'}^{(r)}, \quad F_{34'}^{(r)} \mapsto F_{12}^{(r)}$$

and

$$\vartheta : F_{32}^{(r)} \mapsto F_{32}^{(r)}, \quad F_{21}^{(r)} \mapsto F_{43}^{(r)}, \quad F_{43}^{(r)} \mapsto F_{4'3}^{(r)}, \quad F_{4'3}^{(r)} \mapsto F_{21}^{(r)}.$$

Note that $\vartheta^3 = \text{id}$ and we have the direct sum decomposition

$$\mathfrak{o}_8 = \mathfrak{g}_2 \oplus \mathfrak{o}_8^{(1)} \oplus \mathfrak{o}_8^{(2)},$$

where $\mathfrak{g}_2 = \mathfrak{o}_8^{(0)}$ is identified with the fixed point subalgebra under ϑ , while

$$\mathfrak{o}_8^{(k)} = \{X \in \mathfrak{o}_8 \mid \vartheta(X) = \omega^k X\}, \quad k = 0, 1, 2, \quad \omega = e^{2\pi i/3},$$

is the eigenspace of ϑ corresponding to the eigenvalue ω^k . As in Sec. 2.2, we will regard $\mathcal{V} = \mathcal{V}(\mathfrak{o}_8)$ as the symmetric algebra $S(t^{-1}\mathfrak{o}_8[t^{-1}])$. For the subalgebra of ϑ -invariants in \mathcal{V} we have

$$\mathcal{V}^\vartheta = S(t^{-1}\mathfrak{g}_2[t^{-1}] \oplus S^3(t^{-1}\mathfrak{o}_8^{(1)}[t^{-1}]) \oplus S^3(t^{-1}\mathfrak{o}_8^{(2)}[t^{-1}]) \oplus t^{-1}\mathfrak{o}_8^{(1)}[t^{-1}]t^{-1}\mathfrak{o}_8^{(2)}[t^{-1}]).$$

The λ -bracket on \mathcal{V} defines a λ -bracket on \mathcal{V}^ϑ by restriction. Let J be the ideal of \mathcal{V}^ϑ generated by the subspace

$$S^3(t^{-1}\mathfrak{o}_8^{(1)}[t^{-1}]) \oplus S^3(t^{-1}\mathfrak{o}_8^{(2)}[t^{-1}]) \oplus t^{-1}\mathfrak{o}_8^{(1)}[t^{-1}]t^{-1}\mathfrak{o}_8^{(2)}[t^{-1}].$$

For any element $P \in \mathcal{V}^\vartheta$ we have $\{P_\lambda J\} \subset J$. The quotient space \mathcal{V}^ϑ/J is naturally identified with the differential algebra $\mathcal{V}(\mathfrak{g}_2)$. This quotient is equipped with a λ -bracket induced from that of \mathcal{V}^ϑ . The resulting bracket on $\mathcal{V}(\mathfrak{g}_2)$ is then obtained as the *folding* of the λ -bracket on $\mathcal{V}(\mathfrak{o}_8)$. It coincides with the λ -bracket defined in (2.1) for $\mathfrak{g} = \mathfrak{g}_2$.

For any element $P \in \mathcal{V}$ denote by $\text{pr}(P)$ its projection to the subspace \mathcal{V}^ϑ in the decomposition

$$\mathcal{V} = \mathcal{V}^\vartheta \oplus t^{-1}\mathfrak{o}_8^{(1)}[t^{-1}]\mathcal{V}^\vartheta \oplus t^{-1}\mathfrak{o}_8^{(2)}[t^{-1}]\mathcal{V}^\vartheta.$$

Furthermore, let $\tilde{P} \in \mathcal{V}(\mathfrak{g}_2)$ be the image of $\text{pr}(P)$ under the natural epimorphism

$$\mathcal{V}^\vartheta \rightarrow \mathcal{V}^\vartheta/J \cong \mathcal{V}(\mathfrak{g}_2).$$

Now, if P belongs to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{o}_8, f)$, then its image \tilde{P} belongs to the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}_2, f)$. To prove the first part of the theorem, we will now verify that the coefficients of the differential operator D in (7.3) are obtained as the respective

images of the coefficients of the pseudo-differential operator (6.3). Note that if $P \in \mathfrak{o}_8$, then we can write

$$3P = P^{(0)} + P^{(1)} + P^{(2)},$$

where

$$\begin{aligned} P^{(0)} &= P + \vartheta(P) + \vartheta^2(P) \in \mathfrak{g}_2, \\ P^{(1)} &= P + \omega^2 \vartheta(P) + \omega \vartheta^2(P) \in \mathfrak{o}_8^{(1)}, \\ P^{(2)} &= P + \omega \vartheta(P) + \omega^2 \vartheta^2(P) \in \mathfrak{o}_8^{(2)}. \end{aligned}$$

Hence, for the image \tilde{P} we have $3\tilde{P} = P^{(0)}$. This gives the following formulas for the images of the generators of \mathfrak{o}_8 :

$$\begin{aligned} \tilde{F}_{32} &= Y_\alpha, & \tilde{F}_{21} &= \tilde{F}_{43} = \tilde{F}_{4'3} = \frac{1}{3} Y_\beta, & \tilde{F}_{31} &= -\tilde{F}_{42} = -\tilde{F}_{4'2} = \frac{1}{3} Y_{\alpha+\beta}, \\ \tilde{F}_{41} &= \tilde{F}_{4'1} = \frac{2}{9} Y_{\alpha+2\beta}, & \tilde{F}_{3'1} &= -\frac{2}{9} Y_{\alpha+3\beta}, & \tilde{F}_{2'1} &= \frac{2}{9} Y_{2\alpha+3\beta}. \end{aligned}$$

Furthermore,

$$\tilde{F}_{11} = -H_\alpha - \frac{2}{3} H_\beta, \quad \tilde{F}_{22} = -H_\alpha - \frac{1}{3} H_\beta, \quad \tilde{F}_{33} = -\frac{1}{3} H_\beta \quad \text{and} \quad \tilde{F}_{44} = 0.$$

Thus, the 9×9 matrix introduced in the beginning of Sec. 6 (with $n = 4$) reduces to a matrix with entries in $\mathcal{V}(\mathfrak{p})$, where $\mathfrak{p} = \mathfrak{n}_- \oplus \mathfrak{h}$ is the subspace of \mathfrak{g}_2 . The determinant of this matrix remains unchanged if the $(4, 6)$ entry -2∂ is replaced by $-\partial$ and each entry of the lower left 4×4 submatrix is multiplied by 2. Moreover, the determinant will also be unchanged if we replace row 6 by the sum of rows 4 and 6. Finally, the simultaneous expansion along the rows 5 and 6 reduces the determinant to the given form. This shows that all elements w_i belong to $\mathcal{W}(\mathfrak{g}_2, f)$.

The centralizer \mathfrak{g}_2^f is spanned by the elements f and $Y_{2\alpha+3\beta}$. The condition of Proposition 2.1 holds for the elements w_2, w_6 , thus implying that they are generators of the differential algebra $\mathcal{W}(\mathfrak{g}_2, f)$.

The images of the elements w_2, \dots, w_7 under the homomorphism (2.7) are the elements $\tilde{w}_2, \dots, \tilde{w}_7 \in \mathcal{V}(\mathfrak{h})$ found from the relation

$$\begin{aligned} (\partial + \tilde{F}_{11})(\partial + \tilde{F}_{22})(\partial + \tilde{F}_{33}) \partial (\partial - \tilde{F}_{33})(\partial - \tilde{F}_{22})(\partial - \tilde{F}_{11}) \\ = \partial^7 + \tilde{w}_2 \partial^5 + \tilde{w}_3 \partial^4 + \tilde{w}_4 \partial^3 + \tilde{w}_5 \partial^2 + \tilde{w}_6 \partial + \tilde{w}_7. \end{aligned}$$

In the notation of Sec. 2.2 we have

$$h_1^{(r)} = -H_\alpha^{(r)} \quad \text{and} \quad h_2^{(r)} = -H_\beta^{(r)}.$$

The differential algebra $\widetilde{\mathcal{W}}(\mathfrak{g}_2, f)$ consists of the polynomials in the variables $H_\alpha^{(r)}$ and $H_\beta^{(r)}$, which are annihilated by the screening operators V_1 and V_2 defined in (2.5). Observe that

the definition of the elements \tilde{w}_i coincides with that for the classical \mathcal{W} -algebra $\mathcal{W}(\mathfrak{o}_7, f)$ with the additional condition $\tilde{F}_{22} + \tilde{F}_{33} = \tilde{F}_{11}$; see Sec. 4. If this condition is ignored, then the elements $\tilde{w}_2^{(r)}, \tilde{w}_4^{(r)}$ and $\tilde{w}_6^{(r)}$ with r running over nonnegative integers are algebraically independent generators of the algebra $\widetilde{\mathcal{W}}(\mathfrak{o}_7, f)$; see [12, Ch. 8]. A direct calculation shows that the condition $\tilde{F}_{22} + \tilde{F}_{33} = \tilde{F}_{11}$ implies

$$\tilde{w}_4 = \frac{1}{4} \tilde{w}_2^2 + 3\tilde{w}_2^{(2)}$$

and that the family $\tilde{w}_2^{(r)}, \tilde{w}_6^{(r)}$ with $r \geq 0$ is algebraically independent. \square

The injective homomorphism $\mathcal{W}(\mathfrak{g}_2, f) \hookrightarrow \mathcal{V}(\mathfrak{h})$ taking w_i to \tilde{w}_i can be regarded as the *Miura transformation* associated with the Lie algebra \mathfrak{g}_2 ; see [12, Ch. 8].

References

- [1] M. Adler, *On a trace functional for formal pseudo differential operators and the symplectic structure of the Korteweg–de Vries type equations*, Invent. Math. **50** (1978/79), 219–248.
- [2] T. Arakawa and A. Molev, *Explicit generators in rectangular affine \mathcal{W} -algebras of type A*, arXiv:1403.1017.
- [3] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, *Extensions of the Virasoro algebra constructed from Kac–Moody algebras using higher order Casimir invariants*, Nucl. Phys. B **304** (1988), 348–370.
- [4] P. Bouwknegt and K. Schoutens, *\mathcal{W} symmetry in conformal field theory*, Phys. Rep. **223** (1993), 183–276.
- [5] A. V. Chervov and A. I. Molev, *On higher order Sugawara operators*, Int. Math. Res. Not. (2009), 1612–1635.
- [6] A. Chervov and D. Talalaev, *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, arXiv:hep-th/0604128.
- [7] A. De Sole, V. G. Kac and D. Valeri, *Classical \mathcal{W} -algebras and generalized Drinfeld–Sokolov bi-Hamiltonian systems within the theory of Poisson vertex algebras*, Comm. Math. Phys. **323** (2013), 663–711.
- [8] A. De Sole, *On classical finite and affine \mathcal{W} -algebras*, arXiv:1306.6592.
- [9] V. G. Drinfeld and V. V. Sokolov, *Lie algebras and equations of Korteweg–de Vries type*, J. Sov. Math. **30** (1985), 1975–2036.

- [10] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, *On Hamiltonian reductions of the Wess–Zumino–Novikov–Witten theories*, Phys. Rep. **222** (1992), 1–64.
- [11] L. Frappat, E. Ragoucy and P. Sorba, *Folding the W algebras*, Nucl. Phys. B **404** (1993), 805–836.
- [12] E. Frenkel, *Langlands correspondence for loop groups*, Cambridge Studies in Advanced Mathematics, 103. Cambridge University Press, Cambridge, 2007.
- [13] W. Fulton and J. Harris, *Representation theory. A first course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991.
- [14] I. M. Gelfand and L. A. Dickey, *A family of Hamiltonian structures connected with integrable nonlinear differential equations*, in “Collected papers. Vol. I.”, pp. 625–646, Springer-Verlag, Berlin, 1987.
- [15] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), 218–348.
- [16] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd edition, Cambridge University Press, Cambridge, 1990.
- [17] V. Kac, *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, Providence, RI, 1997.
- [18] B. A. Kupershmidt and G. Wilson, *Modifying Lax equations and the second Hamiltonian structure*, Invent. Math. **62** (1981), 403–436.
- [19] A. N. Leznov and M. V. Saveliev, *Two-dimensional exactly and completely integrable dynamical systems*, Comm. Math. Phys. **89** (1983), 59–75.
- [20] A. I. Molev, *Feigin–Frenkel center in types B , C and D* , Invent. Math. **191** (2013), 1–34.
- [21] A. I. Molev and E. E. Mukhin, *Yangian characters and classical \mathcal{W} -algebras*, arXiv:1212.4032.
- [22] D. Valeri, *Classical \mathcal{W} -algebras*, PhD Thesis, Sapienza Università di Roma, 2012.
- [23] N. J. Wildberger, *A combinatorial construction of G_2* , J. Lie Theory **13** (2003), 155–165.
- [24] A. B. Zamolodchikov, *Infinite additional symmetries in two-dimensional conformal quantum field theory*, Theor. Math. Phys. **65** (1985), 1205–1213.