The p-Dirichlet-to-Neumann Operator with applications to elliptic and parabolic problems☆

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Abstract

In this paper, we investigate the Dirichlet-to-Neumann operator associated with second order quasi-linear operators of p-Laplace type for $1 < p < \infty$, which acts on the boundary of a bounded Lipschitz domain in $\mathbb{R}^d$ for $d \geq 2$. We establish well-posedness and Hölder-continuity with uniform estimates of weak solutions of some elliptic boundary-value problems involving the Dirichlet-to-Neumann operator. By employing these regularity results of weak solutions of elliptic problems, we show that the semigroup generated by the negative Dirichlet-to-Neumann operator on L^q enjoys an $L^q - C^{0,\alpha}$-smoothing effect and the negative Dirichlet-to-Neumann operator on the set of continuous functions on the boundary of the domain generates a strongly continuous and order-preserving semigroup. Moreover, we establish convergence in large time with decay rates of all trajectories of the semigroup, and in the singular case $(1+\varepsilon) \vee \frac{2d}{d+2} \leq p < 2$ for some $\varepsilon > 0$, we give upper estimates of the finite time of extinction.

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1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$, $d \geq 2$ and $1 < p < \infty$. Then for every boundary value $\varphi \in W^{1-1/p,p}(\partial\Omega)$, there is a unique weak solution $u \in W^{1,p}(\Omega)$ of the p-Dirichlet problem

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ \quad\;\; u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We denote this unique weak solution u by $P\varphi$. If $P\varphi$ is smooth enough then we define

$$\Lambda\varphi := |\nabla P\varphi|^{p-2} \tfrac{\partial P\varphi}{\partial\nu}.$$

Here ν denotes the outward pointing unit normal vector on $\partial\Omega$ and $|\nabla P\varphi|^{p-2} \frac{\partial P\varphi}{\partial\nu}$ the p-normal derivative of $P\varphi$ associated with the p-Laplace operator $\Delta_p P\varphi := \mathrm{div}(|\nabla P\varphi|^{p-2}\nabla P\varphi)$.

☆Dedicated to Prof. José Mazón on the occasion of his 60th birthday.

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The aim of this article is to investigate the mapping $\varphi \mapsto \Lambda\varphi$ and to establish Hölder-continuity of solutions of some elliptic and parabolic problems associated with this map. The mapping Λ is called the *Dirichlet-to-Neumann operator* associated with the p-Laplace operator and is also referred as the interior capacity operator (cf. [19, §II.5.1]) or the Neumann operator (cf. [36, p. 41]). This operator appears in a natural way, for instance, in inverse problems associated with the p-Laplace operator (cf. [15] for $p = 2$ and [35, 12, 13] for $p \neq 2$), in the mathematical notion of p-capacity (see [19]) or in the celebrated Signorini problem (for instance, cf. [20, 21, 25]). The Dirichlet-to-Neumann operator Λ is a *nonlocal* operator; in the sense that for a given boundary function φ, the value of $\Lambda\varphi$ on a relatively open neighbourhood of some $x \in \partial\Omega$ does not only depend on the value of φ on the same neighbourhood of x, but on the value of φ at every point of $\partial\Omega$ (see Remark 3.2).

Elliptic and parabolic problems associated with the Dirichlet-to-Neumann operator Λ attracted in the past (see, for instance, [19, 36] or [29, 23, 22]) and currently (see, for instance, [6, 17, 37] concerning the linear case $p = 2$ and see, for instance, [35, 12, 13] for the general case $1 < p < \infty$) much interests. However, it seems that already rather much is known about the linear case $p = 2$ while only partial results exist concerning the general (nonlinear) case $1 < p < \infty$.

To the best of our knowledge, first results on evolution equations in $L^q(\partial\Omega)$, $1 \leq q \leq \infty$, associated with the nonlinear Dirichlet-to-Neumann operator Λ go back to the papers [20] and [25]. Results about existence and uniqueness of entropy solution of elliptic and parabolic equations involving the operator Λ in $L^1(\partial\Omega)$ have been first announced in [2] and later established in the thesis [3] (see also [4, 1]).

The results in this article complement the existing literature by establishing well-posedness of some elliptic and parabolic problems involving the Dirichlet-to-Neumann operator Λ associated with general second order quasi-linear operators A (defined in (19) in Section 3) of Leray-Lions type (cf. [27]), by proving Hölder-continuity of weak solutions of elliptic equations involving Λ (for the definition of weak solution see Section 4), by showing that the part Λ_c of Λ in $C(\partial\Omega)$ is m-completely accretive, by establishing the large time asymptotic behaviour with decay rates of all trajectories of the semigroup generated by the negative Dirichlet-to-Neumann operator on $L^q(\partial\Omega)$ for $1 \leq q \leq \infty$ and on $C(\partial\Omega)$, and by giving upper estimates of the finite time of extinction of the trajectories of the semigroup in the singular case $(1+\varepsilon) \vee \frac{2d}{d+2} \leq p < 2$ for some $\varepsilon > 0$. The p-Laplace operator $A = \Delta_p$ is one important prototype operator of the class of Leray-Lions operators. Hence, we apply our results obtained in this paper to the Dirichlet-to-Neumann operator Λ associated with the p-Laplace operator $A = \Delta_p$ and present them in this section.

Our first main result of this article reads as follows:

Theorem 1.1. *The following assertions hold.*

1. *For every $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$ there is a unique weak solution $\varphi \in W_m^{1-1/p,p}(\partial\Omega)$ of the elliptic problem*

$$\Lambda\varphi = \psi \qquad \text{on } \partial\Omega, \tag{2}$$

$$\int_{\partial\Omega} \varphi \,\mathrm{d}\mathcal{H} = 0. \tag{3}$$

Moreover, the mapping $\psi \mapsto \varphi$ from $W_m^{-(1-1/p),p'}(\partial\Omega)$ to $W_m^{1-1/p,p}(\partial\Omega)$ is continuous.

2. *Let $q = \frac{d-1}{p-1-\varepsilon}$ for some $\varepsilon \in (0,1)$ if $p \le d$ and $q = 1$ if $p > d$. Further, let $\psi \in L^q(\partial\Omega)$ satisfying (3). Then there are $\alpha \in (0,1)$ and $c_\alpha \ge 0$ such that every weak solution $\varphi \in W^{1-1/p,p}(\partial\Omega)$ of equation (2) belongs to $C^{0,\alpha}(\partial\Omega)$ and satisfies*

$$\|\varphi\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left(\|\psi\|_{L^q(\partial\Omega)}^{\frac{1}{p-1}} + \|P\varphi\|_{L^p(\Omega)} \right) + c_\alpha. \tag{4}$$

Here, we denote by $W_m^{1-1/p,p}(\partial\Omega)$ the subspace of all $\varphi \in W^{1-1/p,p}(\partial\Omega)$ satisfying the so-called *compatibility condition* (3) equipped with the induced norm and by $W_m^{-(1-1/p),p'}(\partial\Omega)$ the dual space of $W_m^{1-1/p,p}(\partial\Omega)$.

The statements of Theorem 1.1 follow as a special case of Theorem 4.2 (stated in Section 4). To establish well-posedness of problem (2)-(3), we employ the classical theory of monotone operators (cf. [29]). The Hölder-continuity of weak solutions of equation (2) is based on a recent regularity result [32] of weak solutions of nonlinear elliptic Neumann boundary-value problems on a bounded domain with Lipschitz boundary.

Our second main result is concerned with the well-posedness of initial value problem

$$\tfrac{d\varphi}{dt} + \Lambda\varphi = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{5}$$

$$\varphi(0) = \varphi_0 \tag{6}$$

and to investigate the regularisation effect of mild solutions of (5) depending on the regularity of the initial value φ_0. It is worth noting that the parabolic equation (5) is, in fact, equivalent to the *elliptic-parabolic boundary-value problem*

$$\begin{aligned} -\Delta_p u &= 0 && \text{on } \Omega \times (0,\infty),\\ \tfrac{du}{dt} + |\nabla u|^{p-2}\tfrac{\partial u}{\partial \nu} &= 0 && \text{on } \partial\Omega \times (0,\infty). \end{aligned}$$

Note, in this paper, we focus our attention only on the homogeneous equation (5), although our results (see Proposition 3.13) imply existence and uniqueness of the inhomogeneous equation

$$\tfrac{d\varphi}{dt} + \Lambda\varphi = \psi \qquad \text{on } \partial\Omega \times (0,T)$$

for every given $\psi \in L^2(0,T;L^2(\partial\Omega))$ and $\varphi_0 \in L^2(\partial\Omega)$ (cf. [14]). For $1 \le q \le \infty$, we denote by Λ_q the part of Λ in $L^q(\partial\Omega)$, $D(\Lambda_q)$ the domain of Λ_q, $\overline{\Lambda_q}$ the closure of Λ_q in $L^q(\partial\Omega)$, (see Section 3.2), and $Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ the linear continuous right-inverse of the trace operator $\mathcal{T}r$ on $W^{1,p}(\Omega)$; see Section 2 for further details. We now state our second main result.

Theorem 1.2. *The negative Dirichlet-to-Neumann operator $-\Lambda_2$ in $L^2(\partial\Omega)$ generates a strongly continuous, order preserving semigroup $\{e^{-t\,\Lambda_2}\}$ of contractions on $L^2(\partial\Omega)$ satisfying:*

1. *The semigroup $\{e^{-t\,\Lambda_2}\}$ can be extrapolated to a strongly continuous, order preserving semigroup of contractions on $L^q(\partial\Omega)$ for $1 \le q < \infty$, on the closure $\overline{D(\Lambda_\infty)}$ in $L^\infty(\partial\Omega)$ for $q = \infty$, and on the closure $\overline{D(\Lambda_c)}$ in $C(\partial\Omega)$, where $\overline{D(\Lambda_c)} = C(\partial\Omega)$ if Ω has a $C^{1,\beta}$ boundary from some $\beta \in (0,1)$. In particular, the extrapolated semigroup of $\{e^{-t\,\Lambda_2}\}$ coincides with the semigroup $\{e^{-t\,\overline{\Lambda_q}}\}$ generated by $-\overline{\Lambda_q}$ on $L^q(\partial\Omega)$ if $1 \le q < 2$ and with the semigroup $\{e^{-t\,\Lambda_q}\}$ generated by $-\Lambda_q$ on $L^q(\partial\Omega)$ (resp., on $\overline{D(\Lambda_\infty)}$) if $2 \le q \le \infty$.*

2. *For every* $\varphi \in L^q(\partial\Omega)$ $(1 \le q \le \infty)$*, one has the conservation of mass equality*

$$\int_{\partial\Omega} e^{-tB}\varphi \,\mathrm{d}\mathcal{H} = \int_{\partial\Omega} \varphi \,\mathrm{d}\mathcal{H} \qquad \text{for every } t \ge 0, \tag{7}$$

where $B = \overline{\Lambda_q}$ *if* $1 \le q < 2$ *and* $B = \Lambda_q$ *if* $2 \le q \le \infty$.

3. *For every* $\varphi \in L^q(\partial\Omega)$ $(2 \le q \le \infty)$ *or* $\varphi \in \overline{D(\Lambda_c)}$*, the mild solution* $t \mapsto e^{-t\,\Lambda_q}\varphi$ *of* (5)-(6) *in* $L^q(\partial\Omega)$ *coincides with the unique strong solution of* (5)-(6) *in* $L^2(\partial\Omega)$ *and has regularity*

$$e^{-\cdot\,\Lambda_q}\varphi \in C([0,\infty); L^q(\partial\Omega)) \cap C((0,\infty); W^{1-1/p,p}(\partial\Omega)) \cap W^{1,\infty}([\delta,\infty); L^2(\partial\Omega))$$

for every $\delta > 0$*,* $e^{-t\,\Lambda_q}\varphi$ *is right-hand side differentiable in* $L^2(\partial\Omega)$ *and satisfies*

$$\int_{\partial\Omega} \tfrac{d}{dt_+} e^{-t\,\Lambda_q}\varphi\,\xi \,\mathrm{d}\mathcal{H} + \int_\Omega |\nabla P(e^{-t\,\Lambda_q}\varphi)|^{p-2}\nabla P(e^{-t\,\Lambda_q}\varphi)\nabla Z\xi \,\mathrm{d}x = 0$$

for every $\xi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$*. The function*

$$t \mapsto \mathcal{E}(e^{-t\,\Lambda_q}\varphi) := \tfrac{1}{p}\int_\Omega |\nabla P(e^{-t\,\Lambda_q}\varphi)|^p \,\mathrm{d}x \tag{8}$$

is convex, decreasing, Lipschitz continuous on $[\delta,\infty)$ *for every* $\delta > 0$*, and*

$$\tfrac{d}{dt}\mathcal{E}(e^{-t\,\Lambda_q}\varphi) = -\|\tfrac{d}{dt}(t)e^{-t\,\Lambda_q}\varphi\|^2_{L^2(\partial\Omega)} \tag{9}$$

for a.e. $t > 0$.

4. *For* $B = \overline{\Lambda_q}$ *if* $1 < q < 2$ *and* $B = \Lambda_q$ *if* $2 \le q < \infty$*, we have that for every* $\varphi \in L^q(\partial\Omega)$*,* $e^{-tB}\varphi \in D(B)$ *for every* $t > 0$*,* $e^{-\cdot B}\varphi \in W^{1,\infty}([\delta,\infty); L^q(\partial\Omega))$ *for every* $\delta > 0$*,* $e^{-tB}\varphi$ *is right-hand side differentiable in* $L^q(\partial\Omega)$ *at every* $t > 0$*,*

$$\tfrac{d}{dt_+} e^{-t\cdot B}\varphi + Be^{-tB}\varphi = 0 \tag{10}$$

in $L^q(\partial\Omega)$ *for every* $t > 0$ *and there is a* $C > 0$ *such that*

$$\left\|\tfrac{d}{dt_+} e^{-t\cdot B}\varphi\right\|_{L^q(\partial\Omega)} \le C\,\frac{\|\varphi\|_{L^q(\partial\Omega)}}{t} \tag{11}$$

for every $t > 0$*,*

5. *for* $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ *with some* $\varepsilon \in (0,1)$ *if* $p \le d$ *and for* $2 \le q \le \infty$ *if* $p > d$*, there are* $\alpha \in (0,1)$ *and* $c_\alpha > 0$ *such that*

$$\|e^{-t\Lambda_q}\varphi\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left[\left(\frac{\|\varphi\|_{L^q(\partial\Omega)}}{t}\right)^{\frac{1}{p-1}} + \frac{\|\varphi\|^{2/p}_{L^q(\partial\Omega)}}{t^{1/p}} + \|\varphi\|_{L^q(\partial\Omega)}\right] + c_\alpha \tag{12}$$

for every $t > 0$*,* $\varphi \in L^q(\partial\Omega)$*, and in particular,* $e^{-\cdot\,\Lambda_q}\varphi \in C((0,\infty)\times\partial\Omega)$.

Again, the statements of Theorem 1.2 follow from Theorem 5.1 stated and proved in Section 5 as a special case. Under the addition assumption (20) below, we show in Proposition 3.13 that the Dirichlet-to-Neumann operator associated with quasi-linear operators of Leray-Lions type can be realised as subgradient in $L^2(\partial\Omega)$ of a convex, proper, lower semicontinuous and densely defined functional on $L^2(\partial\Omega)$. This result for the Dirichlet-to-Neumann operator associated with the p-Laplace operator has been first established in [20] and revisited in [25]. But to the best of our knowledge, this was not known so far for general Dirichlet-to-Neumann operators associated with second order quasi-linear operators of Leray-Lions type. It was shown in [3] that the entropy solutions operator associated with Λ_1 is m-completely accretive in $L^1(\partial\Omega)$. In this article, we complement this result by establishing in Proposition 3.12 that the part Λ_c of Λ in $C(\partial\Omega)$ is m-completely-accretive in $C(\partial\Omega)$. In addition, claim (4) in Theorem 5.1 (respectively, claim (5) in Theorem 1.2) describes a $L^q - C^{0,\alpha}$-*smoothing effect* of the semigroup $\{e^{-t\,\Lambda_q}\}$ generated by $-\Lambda_q$ on $L^q(\partial\Omega)$. The corresponding estimate (12) is based on the regularity result obtained for the weak solutions of the elliptic equation (2) (see claim (2) of Theorem 1.1) and the positive homogeneity of the operator Λ (cf. condition (31) in Section 2.3).

We conclude this article with the following large time stability result of the semigroup $\{e^{-t\,\overline{\Lambda_q}}\}$ with decay rates and upper estimates of the finite time of extinction.

Theorem 1.3. *Then the following statements hold true:*

1. *(***Stability***) For every $\varphi \in L^q(\partial\Omega)$,*
$$\lim_{t\to+\infty} e^{-tB}\varphi = \overline{\varphi} := \tfrac{1}{\mathcal{H}(\partial\Omega)} \int_{\partial\Omega} \varphi \,\mathrm{d}\mathcal{H} \tag{13}$$
in $L^q(\partial\Omega)$, where $B = \overline{\Lambda_q}$ if $1 \le q < 2$ and $B = \Lambda_q$ if $2 \le q \le \infty$.
2. *(***Stability in $C(\partial\Omega)$***) For $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ for some $\varepsilon \in (0,1)$ if $p \le d$ and $2 \le q \le \infty$ if $p > d$, we have that limit (13) holds in $C(\partial\Omega)$ for all $\varphi \in L^q(\partial\Omega)$.*
3. *(***Decay estimates***) There is a constant $C > 0$ such that for every $\varphi \in L^2(\partial\Omega)$ there is $t_0 \ge 0$ such that*
$$\|e^{-t\Lambda_2}\varphi - \overline{\varphi}\|_{L^2(\partial\Omega)} \le \begin{cases} \|\varphi - \overline{\varphi}\|_{L^2(\partial\Omega)}\, e^{-C\,t} & \text{if } p = 2, \\[2ex] C\, t^{-\frac{1}{p-2}} & \text{if } p > 2, \end{cases} \tag{14}$$
and
$$\|e^{-t\Lambda_2}\varphi - \overline{\varphi}\|_{L^p(\partial\Omega)} \le C\, \|\varphi - \overline{\varphi}\|^{2/p}_{L^2(\partial\Omega)}\, t^{-\frac{1}{p}} \qquad \text{if } 1 < p < 2, \tag{15}$$
for every $t \ge t_0$.
4. *(***Finite time of extinction***) If $(1+\varepsilon) \vee \frac{2d}{d+2} \le p < 2$ for some $\varepsilon > 0$, then for every $\varphi \in L^2(\partial\Omega)$, the function $e^{-\cdot\Lambda_2}\varphi$ extinct in finite time*
$$t_{ext} \le \frac{\|\varphi - \overline{\varphi}\|^{2-p}_{L^2(\partial\Omega)}}{(1-\frac{p}{2})\, C_S^p},$$
where the constant $C_S > 0$ occurs in the Sobolev-type inequality (71). More precisely, we have for every $\varphi \in L^2(\partial\Omega)$ that
$$\|e^{-\cdot\Lambda_2}\varphi - \overline{\varphi}\|_{L^2(\partial\Omega)} \le \Big[(1-\tfrac{p}{2})\, C_S^p\Big]^{\frac{1}{2-p}} \left[\frac{\|\varphi - \overline{\varphi}\|^{2-p}_{L^2(\partial\Omega)}}{(1-\frac{p}{2})\, C_S^p} - t\right]_+^{\frac{1}{2-p}}$$
for every $t \ge 0$.

Again, the statement of Theorem (1.3) follows as a special case from the general Theorem 6.1 stated and proved in Section 6.

In the next section, we fix some notations used throughout this paper and summarise some basic properties about weak solutions of the nonlinear Dirichlet problem (28), state some useful inequalities and briefly recall some basic facts about nonlinear semigroups in Banach spaces.

2. Preliminaries

Throughout this paper, we assume that Ω has a Lipschitz boundary $\partial\Omega$ in the sense of [31, Sect. 1.3] and $1 < p < \infty$. Suppose that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory functions satisfying

$$a(x,\xi)\xi \geq \eta|\xi|^p \tag{16}$$

$$|a(x,\xi)| \leq c|\xi|^{p-1} \tag{17}$$

$$(a(x,\xi_1) - a(x,\xi_1))(\xi_1 - \xi_2) > 0 \tag{18}$$

for a.e. $x \in \Omega$ and all ξ, ξ_1, $\xi_2 \in \mathbb{R}^d$ with $\xi_1 \neq \xi_2$, where c, $\eta > 0$ are constants independent of $x \in \Omega$ and $\xi \in \mathbb{R}^d$, where c and η are positive constance which are independent of $x \in \Omega$ and $\xi \in \mathbb{R}^d$. Under these assumptions on the function a, the second order quasi-linear operator

$$Au := -\operatorname{div}(a(x, \nabla u)) \qquad \text{in } \mathcal{D}'(\Omega) \tag{19}$$

for every $u \in W^{1,p}_{loc}(\Omega)$ belongs to the class of Leray-Lions operators (cf. [27]). In order, to improve the regularity of mild solutions of nonlinear evolution problems, the following additional assumption is very useful: suppose that $\mathcal{A} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory functions satisfying

$$\nabla_\xi \mathcal{A}(x,\xi) = a(x,\xi) \tag{20}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$. By taking $a(x,\xi) = |\xi|^{p-2}\xi$ for $\xi \in \mathbb{R}^d$ and any $x \in \Omega$, one easily sees that the celebrated p-Laplace operator Δ_p belongs to the class of Leray-Lions operators and which satisfies gradient condition (20).

2.1. Frequently used notations and inequalities

For $1 \leq q \leq \infty$, we denote by $L^q(\Omega)$ and $W^{1,q}(\Omega)$ the usual Lebesgue and first Sobolev spaces. We denote by $C^{0,1}(\overline{\Omega})$ the space of all Lipschitz-continuous functions on the closure $\overline{\Omega}$ of Ω. The boundary $\partial\Omega$ of Ω is equipped with the $(d-1)$-dimensional Hausdorff measure $\mathrm{d}\mathcal{H}$ when we work with the Lebesgue space $L^q(\partial\Omega)$. Moreover, $C(\partial\Omega)$ denotes the set of all real-valued continuous functions on $\partial\Omega$.

Since Ω is a Lipschitz domain, the mapping $u \mapsto u_{|\partial\Omega}$ from $C^{0,1}(\overline{\Omega})$ to $C^{0,1}(\partial\Omega)$ has a unique continuous extension mapping

$$\mathcal{T}r : W^{1,p}(\Omega) \to L^{p^*}(\partial\Omega)$$

called *trace operator* with $p^* = \frac{p(d-1)}{d-p}$ if $1 \leq p < d$, $p^* \geq 1$ if $p = d$ and $p^* = \infty$ if $p > d$ (cf. [31, Théorème 4.2, 4.6, and 3.8]). For convenience, we either write $u_{|\partial\Omega}$ or $\mathcal{T}r\, u$ for $u \in W^{1,p}(\Omega)$ even if u does not belong to $C(\overline{\Omega})$ and call $u_{|\partial\Omega}$ and $\mathcal{T}r\, u$ the *trace* of u. The following properties of the trace operator $\mathcal{T}r$ will be used frequently throughout this article:

1. the kernel $\mathit{ker}(\mathcal{T}r) := \left\{ u \in W^{1,p}(\Omega) \,\middle|\, \mathcal{T}r\, u = 0 \right\}$ of $\mathcal{T}r$ coincides with the Sobolev space $W_0^{1,p}(\Omega)$,
2. the range $\mathit{Rg}(\mathcal{T}r) := \left\{ \mathcal{T}r\, u \,\middle|\, u \in W^{1,p}(\Omega) \right\}$ of $\mathcal{T}r$ coincides with the Sobolev-Slobodečki space $W^{1-1/p,p}(\partial\Omega)$ defined as the linear subspace of all $\varphi \in L^p(\partial\Omega)$ with finite semi-norm

$$[f]_p^p := \int_{\partial\Omega}\int_{\partial\Omega} \tfrac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{d-1+p}}\,\mathrm{d}x\mathrm{d}y$$

and equipped with the norm $\|\varphi\|_{W^{1-1/p,p}(\partial\Omega)} := \|\varphi\|_{L^p(\partial\Omega)} + [\varphi]_p$ for every $\varphi \in W^{1-1/p,p}(\partial\Omega)$ (cf. [31, Section 3.8]),
3. the trace operator $\mathcal{T}r$ has a linear bounded right inverse

$$Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega) \tag{21}$$

(cf. [31, Théorème 5.7]).

Here, we denote by $W^{-(1-1/p),p'}(\partial\Omega)$ the dual space of $W^{1-1/p,p}(\partial\Omega)$ and by $\langle\chi,\psi\rangle$ the value of $\chi \in W^{-(1-1/p),p'}(\partial\Omega)$ at $\psi \in W^{1-1/p,p}(\partial\Omega)$.

Another crucial property of a Lipschitz domain Ω is that for $1 \le p < \infty$, the function space $C^\infty(\overline{\Omega})$ lies dense in $W^{1,p}(\Omega)$. We state this standard result explicitly for later use.

Lemma 2.1. *For $1 < p < \infty$ and $1 \le q < \infty$, the space $C^\infty(\overline{\Omega})$ lies dense in $W^{1,p}(\Omega)$ and the set $\left\{ v_{|\partial\Omega} \,\middle|\, v \in C^\infty(\overline{\Omega}) \right\}$ is dense in $W^{1-1/p,p}(\partial\Omega)$ and in $L^q(\partial\Omega)$. In particular, $C^\infty(\overline{\Omega})$ lies dense in*

$$V_{p,q} := \left\{ v \in W^{1,p}(\Omega) \mid v_{|\partial\Omega} \in L^q(\partial\Omega) \right\}$$

equipped with the sum norm.

Proof. First, by the Stone - Weierstraß Theorem, the set $\{v_{|\partial\Omega} \mid v \in C^\infty(\mathbb{R}^d)\}$, which may be identified with a subset of $\{v_{|\partial\Omega} \mid v \in C^\infty(\overline{\Omega})\}$, is dense in $C(\partial\Omega)$. Since the $(d-1)$-dimensional Hausdorff measure is Borel regular (cf. [24, Theorem 1, Sec. 2.1]), the latter set is dense in $L^q(\partial\Omega)$ (cf. [34, Theorem 3.14]). Note that this also means that $W^{1-1/p,p}(\partial\Omega) \cap L^q(\partial\Omega)$ lies dense in $L^q(\partial\Omega)$. Since the set $\{v_{|\overline{\Omega}} \mid v \in C^\infty(\mathbb{R}^d)\}$ is a subset of $C^\infty(\overline{\Omega})$, the last claims follows from [31, Théorème 3.1 in Chapitre 2, §3]. □

Besides Lemma 2.1, we need the following p-variant of Maz'ya's remarkable inequality

$$\|u\|_{L^{pd/(d-1)}(\Omega)} \le C\, \left(\|\nabla u\|_{L^p(\Omega)^d} + \|u_{|\partial\Omega}\|_{L^p(\partial\Omega)}\right) \tag{22}$$

holding for all $u \in W^{1,p}(\Omega)$ provided $1 \le p < \infty$. Here the constant $C > 0$ depends on p, the volume $|\Omega|$, and the isoperimetric constant $C(d)$ (cf. [30, Cor. 3.6.3] and see also [18]). By using inequality (22) and Poincaré's inequality on $W^{1,p}(\Omega)$, on can deduce the following useful inequality

$$\|u\|_{L^p(\Omega)} \le \tilde{C}\Big(\|\nabla u\|_{L^p(\Omega)^d} + \|u_{|\partial\Omega}\|_{L^q(\partial\Omega)}\Big) \tag{23}$$

holding for every $u \in W^{1,p}(\Omega)$ with trace $u_{|\partial\Omega} \in L^q(\partial\Omega)$ for $1 \le q \le \infty$, where $\tilde{C} > 0$ is some constant independent of u. To see this, set $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x$ for given $u \in L^1(\Omega)$. Then Poincaré's inequality on $W^{1,p}(\Omega)$ says that there is a $C_1 > 0$ such that

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C_1\|\nabla u\|_{L^p(\Omega)^d}$$

for every $u \in W^{1,p}(\Omega)$ and so

$$\|u\|_{L^p(\Omega)} \leq C_2\Big(\|\nabla u\|_{L^p(\Omega)^d} + \|u\|_{L^1(\Omega)}\Big)$$

for every $u \in W^{1,p}(\Omega)$. Applying to this Maz'ya's inequality (22) for $p = 1$ and the two inequalities $\|\nabla u\|_{L^1(\Omega)^d} \leq C\,\|\nabla u\|_{L^p(\Omega)^d}$ for $u \in W^{1,p}(\Omega)$ and $\|u\|_{L^1(\partial\Omega)} \leq C\,\|u\|_{L^q(\partial\Omega)}$ for $u \in L^q(\partial\Omega)$ ($1 \leq q \leq \infty$) leads to (23).

2.2. Nonlinear semigroup theory

In this subsection, let $X = L^q(\partial\Omega)$ for $1 \leq q \leq \infty$ or $X = C(\partial\Omega)$. Then we shall usually view operators A on X as relations $A \subseteq X \times X$, but we also use the notation

$$A(u) := \{f \in X \,|\, (u, f) \in A\},$$

which suggests that A is a mapping from X into the power set of X, that is, A is a possibly multivalued operator. Further, we will make use of the notation of the *subgradient*

$$\partial\mathcal{E} := \left\{ (u, f) \in L^2(\partial\Omega) \times L^2(\partial\Omega) \;\middle|\; \begin{array}{c} \mathcal{E}(v) - \mathcal{E}(u) \geq (f, v - u)_{L^2(\partial\Omega)} \\ \text{for all } v \in D(\mathcal{E}) \end{array} \right\}$$

of a given proper, convex functional $\mathcal{E} : L^2(\partial\Omega) \to \mathbb{R} \cup \{+\infty\}$ with effective domain $D(\mathcal{E}) = \{u \in L^2(\partial\Omega) \,|\, \mathcal{E}(u) < +\infty\}$.

In order to keep this article self-contained, we recall some basic definitions and results about nonlinear semigroups in Banach spaces used throughout this paper. An operator A on X is called *accretive* in X if for every (u, v), $(\hat{u}, \hat{v}) \in A$ and every $\lambda \geq 0$ one has

$$\|u - \hat{u}\|_X \leq \|u - \hat{u} + \lambda(v - \hat{v})\|_X$$

and an operator A on X is called *m-accretive in* X if A is accretive in X satisfies the so-called *range condition*

$$Rg(I + \lambda A) = X \qquad \text{for some (or equivalently all) } \lambda > 0. \tag{24}$$

In other words, A is accretive if its *resolvent operator* $J_\lambda := (I+\lambda A)^{-1}$ is a single-valued contraction from the range $Rg(I+\lambda A)$ to the domain $D(A)$ of A for every $\lambda > 0$. Due to the celebrated Crandall-Liggett theorem [16], the condition "A is m-accretive in X" ensures that for all $u_0 \in \overline{D(A)}$, the closure of $D(A)$ in X, the evolution equation

$$\tfrac{du}{dt} + Au \ni 0, \quad u(0) = u_0 \tag{25}$$

is well-posed in the sense of mild solutions. Here, a *mild solution* u of Cauchy problem (25) is a function $u \in C([0, \infty); X)$ with the following property: for every T, $\varepsilon > 0$, for every partition $0 = t_0 < \cdots < t_N = T$ of the interval $[0, T]$ such that $t_i - t_{i-1} < \varepsilon$ for every $i = 1, \ldots, N$, there exists a piecewise constant function $u_{\varepsilon,N} : [0, T] \to X$ given by

$$u_{\varepsilon,N}(t) = u_0\, \mathbb{1}_{\{t_0=0\}}(t) + \sum_{i=1}^{N} u_{\varepsilon,i}\, \mathbb{1}_{(t_{i-1},t_i]}(t)$$

where the values u_i on $(t_{i-1}, t_i]$ solve recursively the finite difference equation

$$u_i + (t_i - t_{i-1})Au_i \ni u_{i-1} \qquad \text{for every } i = 1, \dots, N$$

and

$$\sup_{t\in[0,T]} \|u(t) - u_{\varepsilon,N}(t)\|_X \le \varepsilon.$$

If A is accretive, then every mild solution u of (25) is unique ([8, Theorem 4.1]) and if A, in addition, satisfies the range condition (24), then by the Crandall-Liggett theorem [16], for every element u_0 of $\overline{D(A)}$, there is a unique mild solution u of (25) given by the *exponential formula*

$$u(t) = \lim_{n\to\infty} \left(I + \tfrac{t}{n}A\right)^{-n} u_0 \tag{26}$$

uniformly in t on compact intervals. We call a function u a *strong solution of* (25) *in* X if $u \in W^{1,1}_{loc}((0, +\infty); X) \cap C([0, \infty); X)$, $u(0) = u_0$ and for a.e. $t > 0$ one has $u(t) \in D(A)$ and $-\frac{du}{dt}(t) \in Au(t)$. It is not very difficult to see that if A is an accretive operator in X, then any strong solution of (25) is a mild solution (see [8, p.130]) and hence is unique. For given $u_0 \in \overline{D(A)}$, setting $e^{-tA}u_0 = u(t)$ for every $t \ge 0$, where $t \mapsto u(t)$ denotes the unique mild solution of (25) in X, defines a (non-linear) *strongly continuous semigroup* $\{e^{-tA}\}$ of *contractions* e^{-tA} in $\overline{D(A)}$. More precisely, the family $\{e^{-tA}\}$ of mappings $e^{-tA} : \overline{D(A)} \to \overline{D(A)}$ satisfies

- (*semigroup property*)

$$e^{-(t+s)A} = e^{-tA} \circ e^{-sA} \qquad \text{for every } t,\, s \ge 0,$$

- (*strong continuity*)

$$\lim_{t\to 0+} \|e^{-tA}u - u\|_X = 0 \qquad \text{for every } u \in \overline{D(A)},$$

- (*contraction property*)

$$\|e^{-tA}u - e^{-tA}v\|_X \le \|u - v\|_X \qquad \text{for all } u,\, v \in \overline{D(A)},\, t \ge 0.$$

The family $\{e^{-tA}\}$ is called the *strongly continuous semigroup generated* by $-A$ on $\overline{D(A)}$. On the Banach space X there is a linear ordering "$\le$" defined by $u \le v$ for u, $v \in X$ if and only if $u(x) \le v(x)$ for a.e. (all) $x \in \partial\Omega$. Having this in mind, we call a mapping $S : D(S) \to X$ with domain $D(S) \subseteq X$ a *T-contraction* if

$$\|[Su - Sv]^+\|_X \le \|[u - v]^+\|_X$$

for every u, $v \in D(S)$, where $[u]^+ := \max\{u, 0\}$ and an operator A on X is called *T-accretive* on X if its resolvent operator J_λ is a T-contraction for every $\lambda > 0$. Obviously, every T-contraction S on X is *order preserving*, that is, for every u, $v \in \overline{D(S)}$ with $u \le v$, one has that $Su \le Sv$. Thus by the exponential formula (26) one sees that if A is T-accretive and generates a semigroup $\{e^{-tA}\}$, then each mapping e^{-tA} is *order preserving*. Since we chose either $X = L^q(\partial\Omega)$ for $1 \le q \le \infty$ or $X = C(\partial\Omega)$, we have that T-contractions on X are contractions on X. In general Banach spaces X this result is not true (cf. the appendix of [5]).

Let $\mathcal{J}_0$ denote the set of all convex, lower semicontinuous functions $j : \mathbb{R} \to [0, \infty]$ satisfying $j(0) = 0$ and let $1 \leq q \leq \infty$. We call an operator A in X *completely accretive* if

$$\int_{\partial\Omega} j(u - \hat{u}) \,\mathrm{d}\mu \leq \int_{\partial\Omega} j(u - \hat{u} + \lambda(v - \hat{v})) \,\mathrm{d}\mu \tag{27}$$

for every $j \in \mathcal{J}_0$, (u, v), $(\hat{u}, \hat{v}) \in A$, and $\lambda \geq 0$. Furthermore, we call A *m-completely accretive in X* if A is completely accretive and satisfies the range condition (24). By taking $j(\cdot) = |[\cdot]^+|^q$ for $1 \leq q < \infty$ or $j(\cdot) = [\cdot - k]^+$ for $k \geq 0$ in (27), one easily sees that each completely accretive operator A is T-accretive in $L^q(\partial\Omega)$ for all $1 \leq q \leq \infty$.

Concluding this section, we emphasise that we follow here the convention that constants denoted by C or c_α may vary from line to line.

2.3. The nonlinear Dirichlet problem

In this subsection we review some basic facts about the nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where we assume that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)–(18). We start by recalling the following definition.

Definition 2.2. We call $u \in W^{1,p}_{loc}(\Omega)$ a *weak solution* of equation

$$-\operatorname{div}(a(x, \nabla u)) = 0 \qquad \text{in } \Omega \tag{29}$$

if u satisfies the integral equation

$$\int_\Omega a(x, \nabla u)\nabla v \,\mathrm{d}x = 0 \tag{30}$$

for all $v \in W^{1,p}_0(\Omega)$.

For later use, we need the following compactness result concerning weak solutions of (29), which is an immediate consequence of [11, Theorem 2.1 & Remark 2.1]. We leave the details of the proof of this lemma to the reader as an exercise.

Lemma 2.3. *Suppose that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)–(18). If (u_n) is a bounded sequence in $W^{1,p}(\Omega)$ of weak solution of (29), then there is subsequence (u_{k_n}) of (u_n) and a weak solution $u \in W^{1,p}(\Omega)$ of (29) such that u_{k_n} converges to u weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, ∇u_{k_n} converges to ∇u a.e in Ω and $a(x, \nabla u_{k_n})$ converges to $a(x, \nabla u)$ a.e. in Ω and weakly in $L^{p'}(\Omega)$.*

Now, for any boundary value $\varphi \in W^{1-1/p,p}(\partial\Omega)$, let $\Phi \in W^{1,p}(\Omega)$ be such that $\Phi_{|\partial\Omega} = \varphi$. Then it is well known (cf [29, Exemple 2.3.2]) that under the assumptions (16)-(18), the operator

$$v \mapsto \mathcal{A}v := -\operatorname{div}\left(a(x, \nabla v + \nabla\Phi)\right)$$

satisfies the hypotheses of the Browder-Minty theorem ([29, Théorème 2.1]). Hence equation (29) admits a weak solution $u \in W^{1,p}(\Omega)$ satisfying $u - \Phi \in W^{1,p}_0(\Omega)$. By the strict monotonicity condition (18), the solution u of (29) satisfying $u_{|\partial\Omega} = \varphi$ is unique. Due to the regularity of the solution u, we arrive to the following definition.

Definition 2.4. For given boundary value $\varphi \in W^{1-1/p,p}(\partial\Omega)$, we call a function $u \in W^{1,p}(\Omega)$ a $W^{1,p}$*-solution* of Dirichlet problem (28) on Ω if $u - Z\varphi \in W_0^{1,p}(\Omega)$ and u is a weak solution of equation (29).

In the next lemma, we collect some well-known properties about $W^{1,p}$-solutions of (1), which will be very useful later.

Lemma 2.5. *Suppose that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory functions satisfying (16)-(18). Consider the mapping $P : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ defined by assigning to each $\varphi \in W^{1-1/p,p}(\partial\Omega)$ the unique $W^{1,p}$-solution u of (28) with boundary value φ. Then the following statements are true.*

1. *The mapping P is well-defined and injective.*
2. *The mapping P is continuous.*
3. *Let $\varphi \in W^{1-1/p,p}(\partial\Omega)$. Then for every $\Phi \in W^{1,p}(\Omega)$ with $\Phi_{|\partial\Omega} = \varphi$, there is a unique $u_\Phi \in W_0^{1,p}(\Omega)$ such that $P\varphi = u_\Phi + \Phi$.*
4. *Suppose, in addition, that a satisfies gradient condition (20). Then*

$$\int_\Omega \mathcal{A}(x, \nabla P(\lambda\varphi + \psi))\,\mathrm{d}x \le \int_\Omega \mathcal{A}(x, \lambda\nabla P\varphi + \nabla\Psi)\,\mathrm{d}x$$

   for every φ, $\psi \in W^{1-1/p,p}(\partial\Omega)$ and $\Psi \in W^{1,p}(\Omega)$ such that $\Psi_{|\partial\Omega} = \psi$, and every $\lambda \in \mathbb{R}$.
5. *If the function $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies* homogeneity condition

$$a(x, \lambda\xi) = |\lambda|^{p-2}\lambda\, a(x, \xi) \tag{31}$$

   for every $\lambda \in \mathbb{R}$, $\xi \in \mathbb{R}^d$, and a.e. $x \in \Omega$, then P satisfies

$$P(\lambda\varphi) = \lambda P(\varphi)$$

   for every $\varphi \in W^{1-1/p,p}(\partial\Omega)$ and $\lambda \in \mathbb{R}$.

Proof. Claim (1), (3) and (5) follow from the existence and uniqueness of boundary value problem (28). Also claim (4) is well-known, but for later reference, we outline the details of the proof. Let $\varphi \in W^{1-1/p,p}(\partial\Omega)$ and $\Phi \in W^{1,p}(\Omega)$ be such that $\Phi_{|\partial\Omega} = \varphi$. We set $\mathcal{K}_\Phi := \Phi + W_0^{1,p}(\Omega)$ and consider the functional $\mathcal{F} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_\Omega \mathcal{A}(x, \nabla u)\,\mathrm{d}x \tag{32}$$

for every $u \in W^{1,p}(\Omega)$. The functional $\mathcal{F}$ is continuously differentiable by growth condition (17), the Fréchet-derivate $\mathcal{F}' : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ is given by

$$\langle \mathcal{F}'(u), v\rangle = \int_\Omega a(x, \nabla u)\nabla v\, dx \tag{33}$$

for every u, $v \in W^{1,p}(\Omega)$, and $\mathcal{F}$ is convex by the monotonicity condition (18) (cf. [26, Theorem 6.2.1]). The restriction $\mathcal{F}_{|\mathcal{K}_\Phi}$ of $\mathcal{F}$ on $\mathcal{K}_\Phi$ is coercive by condition (16) and by Poincaré's inequality. Furthermore, the Fréchet-derivative $\mathcal{F}'$ is strictly monotone on the set $\mathcal{K}_\Phi$ by (18). Thus the convex minimisation principle (cf. [39, Theorem 2.E]) implies that there is a unique solution $u \in W^{1,p}(\Omega)$ of minimisation problem:

$$\min\left\{ \int_\Omega \mathcal{A}(x,\nabla u)\,\mathrm{d}x \;\middle|\; u \in W^{1,p}(\Omega) \text{ with } u - \Phi \in W^{1,p}_0(\Omega)\right\}. \tag{34}$$

Note that the uniqueness of the minimiser of (34) is independent of the choice of Φ. Hence we can take $\Phi = Z\varphi$ for every boundary value $\varphi \in W^{1-1/p,p}(\partial\Omega)$, where Z denotes the operator given in (21). Moreover, $u \in W^{1,p}(\Omega)$ is the minimiser of (34) if and only if $u - \Phi \in W^{1,p}_0(\Omega)$ and u is a weak solution of (29) on Ω (cf. [39, Theorem 2.E]). Due to the characterisation of $W^{1,p}$-solutions of Dirichlet problem (28) as the unique minimiser of problem (34), it follows that claim (4) holds.

In particular, claim (2) is well-known, but for the sake of completeness, we give here the proof. Let (φ_n) be a sequence in $W^{1-1/p,p}(\partial\Omega)$ and $\varphi \in W^{1-1/p,p}(\partial\Omega)$ such that φ_n converges to φ in $W^{1-1/p,p}(\partial\Omega)$. Taking $P\varphi_n - Z\varphi_n$ as a test function in (29) and using (16) and (17) together with Hölder's and Young's inequality yield

$$\tfrac{\eta}{2}\int_\Omega |\nabla P\varphi_n|^p\,\mathrm{d}x \le C\int_\Omega |\nabla Z\varphi_n|^p \mathrm{d}x \tag{35}$$

for every n. Since the operator $Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ maps bounded sets into bounded sets, the sequence $(Z\varphi_n)$ is bounded in $W^{1,p}(\Omega)$. Thus, estimate (35) implies that the sequence $(\nabla P\varphi_n)$ is bounded in $L^p(\Omega)^d$. Now, applying Maz'ya's inequality (22) yields the sequence $(P\varphi_n)$ is bounded in $W^{1,p}(\Omega)$. By Lemma 2.3, there is a weak solution $u \in W^{1,p}(\Omega)$ of (29) on Ω and there is a subsequence (φ_{k_n}) of (φ_n) such that $P\varphi_{k_n}$ converges to u weakly in $W^{1,p}(\Omega)$, $P\varphi_{k_n}$ converges to $P\varphi$ in $L^p(\Omega)$, $\nabla P\varphi_{k_n}$ converges to ∇u a.e. on Ω and $a(x,\nabla P\varphi_{k_n})$ converges to $a(x,\nabla P\varphi)$ weakly in $L^{p\prime}(\Omega)^d$. By the compactness of the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ and since φ_{k_n} converges to φ in $L^p(\partial\Omega)$, we can conclude that $u = P\varphi$. Since $P\varphi_{k_n}$ converges to $P\varphi$ in $L^p(\Omega)$, it remains to show that $\nabla P\varphi_{k_n}$ converges to $\nabla P\varphi$ in $L^p(\Omega)^d$. To see this, note that by monotonicity condition (18), for every n,

$$\chi_{k_n}(x) := (a(x,\nabla P\varphi_{k_n}(x)) - a(x,\nabla P\varphi(x)))(\nabla P\varphi_{k_n}(x) - \nabla P\varphi(x)) \tag{36}$$

is non-negative for a.e. $x \in \Omega$. Since $P\varphi_{k_n}$ is a weak solutions of (29),

$$\begin{aligned}\int_\Omega \chi_{k_n}\mathrm{d}x = \int_\Omega a(x,\nabla P\varphi_{k_n})\nabla Z\varphi_{k_n}\mathrm{d}x - \int_\Omega a(x,\nabla P\varphi_{k_n})\nabla P\varphi\mathrm{d}x \\ - \int_\Omega a(x,\nabla P\varphi)(\nabla P\varphi_{k_n} - \nabla P\varphi)\,\mathrm{d}x.\end{aligned}$$

Therefore and by the weak convergence of $a(x,\nabla P\varphi_{k_n})$ to $a(x,\nabla P\varphi)$ in $L^{p\prime}(\Omega)^d$, the strong convergence of $Z\varphi_{k_n}$ to $Z\varphi$ in $W^{1,p}(\Omega)$, and since $P\varphi$ is a weak solutions of (29), it follows that χ_{k_n} converges to 0 in $L^1(\Omega)$. By using the definition of χ_{k_n}, coercivity condition (16), and Hölder's inequality, we see that

$$\begin{aligned}\eta\int_E |\nabla P\varphi_{k_n}|^p\mathrm{d}x &\le \int_E a(x,\nabla P\varphi_{k_n})\nabla P\varphi_{k_n}\mathrm{d}x\\ &= \int_E \chi_{k_n}\mathrm{d}x + \int_E a(x,\nabla P\varphi_{k_n})\nabla P\varphi\mathrm{d}x + \int_E a(x,\nabla P\varphi)(\nabla P\varphi_{k_n} - \nabla P\varphi)\mathrm{d}x\end{aligned}$$

$$\leq \int_E \chi_{k_n} \mathrm{d}x + \|a(\cdot, \nabla P\varphi_{k_n})\|_{L^{p'}(\Omega)} \left(\int_E |\nabla P\varphi|^p \mathrm{d}x\right)^{1/p}$$
$$+ \left(\int_E a(x, \nabla P\varphi)\mathrm{d}x\right)^{1/p'} \|\nabla P\varphi_{k_n} - \nabla P\varphi\|_{L^p(\Omega)}$$

for every measurable subset $E \subseteq \Omega$. Since the measurable set E was arbitrary, and since χ_{k_n} is equi-integrable in $L^1(\Omega)$, $(a(\cdot, \nabla P\varphi_{k_n}))$ is bounded in $L^{p'}(\Omega)$ and $(\nabla P\varphi_{k_n})$ is bounded in $L^p(\Omega)$, our last estimates show that $(|\nabla P\varphi_{k_n}|^p)$ is equi-integrable in $L^1(\Omega)$. Thus and since $\nabla P\varphi_{k_n}$ converges to $\nabla P\varphi$ a.e. on Ω, it follows by Vitali's theorem that $\nabla P\varphi_{k_n}$ converges to $\nabla P\varphi$ in $L^p(\Omega)^d$. Since the same arguments hold, for each subsequence of a convergent sequence (φ_n) in $W^{1-1/p,p}(\partial\Omega)$, we have thereby shown that the operator P is continuous from $W^{1-1/p,p}(\partial\Omega)$ to $W^{1,p}(\Omega)$. □

3. The Dirichlet-to-Neumann operator

This section is devoted to introduce and to bring together some basic properties of the Dirichlet-to-Neumann operator Λ associated with the second order quasi-linear operator A defined in (19). Throughout this section, we assume that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)–(18). This section is subdivided into three subsections.

3.1. General definition and some basic properties

For given boundary value $\varphi \in W^{1-1/p,p}(\partial\Omega)$, let $P\varphi$ be the unique $W^{1,p}$-solution of Dirichlet-problem (28) with respect to boundary value φ as introduced in Lemma 2.5. Furthermore, let A be the second order quasi-linear operator given by (19). Then, under the assumption that $P\varphi$ and $a(\cdot, \nabla P\varphi)$ are smooth enough up to the boundary $\partial\Omega$ and that ν denotes the outward pointing unit normal vector on $\partial\Omega$, then the *co-normal derivative* of $P\varphi$ associated with the operator A on $\partial\Omega$ is *formally* defined by the dot product

$$a(x, \nabla P\varphi) \cdot \nu$$

on $\partial\Omega$. Since the Dirichlet-to-Neumann operator Λ associated with the operator A assigns to each Dirichlet boundary data φ the corresponding co-normal derivative of $P\varphi$, we *formally* set

$$\Lambda\varphi = a(x, \nabla P\varphi) \cdot \nu.$$

Multiplying this equation by some function $\psi \in C^\infty(\overline{\Omega})$ with respect to the inner product on $L^2(\partial\Omega)$ and applying Green's formula yields

$$\int_{\partial\Omega} \Lambda\varphi\, \psi_{|\partial\Omega}\, \mathrm{d}\mathcal{H} = \int_\Omega a(x, \nabla P\varphi)\nabla\psi\, \mathrm{d}x.$$

If, in addition, $\Lambda\varphi \in L^{p'}(\partial\Omega)$, then by an approximation argument, we can conclude that

$$\int_{\partial\Omega} \Lambda\varphi\, \psi\, \mathrm{d}\mathcal{H} = \int_\Omega a(x, \nabla P\varphi)\nabla Z\psi\, \mathrm{d}x$$

for every $\psi \in W^{1-1/p,p}(\partial\Omega)$. Even if φ and ψ merely belong to $W^{1-1/p,p}(\partial\Omega)$, the integral on the right-hand side of this equation exists. Thus, we can use this integral to define the Dirichlet-to-Neumann operator Λ for the more general class of functions $W^{1-1/p,p}(\partial\Omega)$. By linearity of Z and by using Hölder's inequality together with growth condition (17), one easily sees that the functional

$$\psi \mapsto \int_\Omega a(x, \nabla P\varphi)\nabla Z\psi \,\mathrm{d}x \tag{37}$$

belongs to the dual space $W^{-(1-1/p),p'}(\partial\Omega)$. This justifies why our following definition makes sense and is consistent to the case of smooth functions.

Definition 3.1. We call the mapping $\Lambda : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ defined by

$$\langle \Lambda\varphi, \psi\rangle = \int_\Omega a(x, \nabla P\varphi)\nabla Z\psi \,\mathrm{d}x \tag{38}$$

for every φ, $\psi \in W^{1-1/p,p}(\partial\Omega)$ the *Dirichlet-to-Neumann operator* associated with the quasi-linear operator A given by (19).

Remark 3.2. Consider the special case $a(x,\xi) = |\xi|^{p-2}\xi$ for every $x \in \Omega$ and $\xi \in \mathbb{R}^d$. In this case the local operator A defined in (19) reduces to the celebrated p-Laplace operator Δ_p. If the boundary $\partial\Omega$ of Ω is smooth enough (for example, if $\partial\Omega$ is of class $C^{2,\beta}$ with $\beta \in (0,1)$), then one can show by using Hopf's boundary-point Lemma for the p-Laplace operator (see [38, Theorem 5]) that the associated Dirichlet-to-Neumann operator Λ has the character of a nonlocal boundary operator.

The next proposition contains the key properties to establish well-posedness of the elliptic problems associated with the Dirichlet-to-Neumann operator Λ. Some results stated in our proposition are already known but they also complement and improve the known literature (cf. [20, Lema in Section 2] and [3, Lemme 2.1.1]).

Proposition 3.3. *Suppose that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18). Then the operator $\Lambda : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ defined by (38) has the following properties:*

1. *One has*
$$\langle \Lambda\varphi, \psi\rangle = \int_\Omega a(x, \nabla P\varphi)\nabla P\psi \mathrm{d}x \tag{39}$$
for every φ, $\psi \in W^{1-1/p,p}(\partial\Omega)$.
2. *Λ is continuous and monotone, that is, for every*
$$\langle \Lambda\varphi_1 - \Lambda\varphi_2, \varphi_1 - \varphi_2\rangle \geq 0.$$
3. *There are constants $C_1 > 0$ such that*
$$\|\Lambda\varphi\|_{W^{-(1-1/p),p'}(\partial\Omega)} \leq C_1 \|\varphi\|^{p-1}_{W^{1-1/p,p}(\partial\Omega)} \tag{40}$$
for every $\varphi \in W^{1-1/p,p}(\partial\Omega)$.
4. *There is a constant $C_2 > 0$ such that*
$$\langle \Lambda\varphi, \varphi\rangle \geq C_2 \|\varphi\|^{p}_{W^{1-1/p,p}(\partial\Omega)} \tag{41}$$
for every $\varphi \in W^{1-1/p,p}_m(\partial\Omega)$.

Proof. First, we outline that Λ can be written equivalently as in (39). To see this let φ, $\psi \in W^{1-1/p,p}(\partial\Omega)$. By claim (3) of Lemma 2.5, there is a unique $u_{Z\psi} \in W^{1,p}_0(\Omega)$ such that $P\psi = u_{Z\psi} + Z\psi$. Thus

$$\langle \Lambda\varphi, \psi\rangle = \int_\Omega a(x, \nabla P\varphi)\nabla Z\psi \,\mathrm{d}x = \int_\Omega a(x, \nabla P\varphi)\nabla P\psi \,\mathrm{d}x - \int_\Omega a(x, \nabla P\varphi)\nabla u_{Z\psi} \,\mathrm{d}x$$

Since $P\varphi$ is a weak solution of (29), the second integral on the right-hand side equals zero and hence (39) holds. Next, we show that Λ is continuous. Let $(\varphi_n) \subseteq W^{1-1/p,p}(\partial\Omega)$ and $\varphi \in W^{1-1/p,p}(\partial\Omega)$ such that φ_n converges to φ in $W^{1-1/p,p}(\partial\Omega)$. Then by claim (2) of Lemma 2.5, $P\varphi_n$ converges to $P\varphi$ in $W^{1,p}(\Omega)$. Since $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is Carathéodory satisfying growth condition (17), we have

$$\lim_{n\to\infty} a(x, \nabla P\varphi_n) = a(x, \nabla P\varphi) \qquad \text{in } L^{p'}(\Omega)^d. \tag{42}$$

Fix $\psi \in W^{1-1/p,p}(\partial\Omega)$. Then, by Hölder's inequality and since $Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ is linear bounded, we see that

$$|\langle \Lambda\varphi_n - \Lambda\varphi, \psi\rangle| \le C \,\|a(\cdot, \nabla P\varphi_n) - a(\cdot, \nabla P\varphi)\|_{L^{p'}(\Omega)^d} \,\|\psi\|_{W^{1-1/p,p}(\partial\Omega)}$$

for some constant $C > 0$. Thus

$$\|\Lambda\varphi_n - \Lambda\varphi\|_{W^{-(1-1/p),p'}(\partial\Omega)} \le C \,\|a(\cdot, \nabla P\varphi_n) - a(\cdot, \nabla P\varphi)\|_{L^{p'}(\Omega)^d}$$

and so limit (42) implies that $\Lambda\varphi_n$ converges to $\Lambda\varphi$ in $W^{-(1-1/p),p'}(\partial\Omega)$. This proves the continuity of Λ. To see that Λ is monotone, let φ_1, $\varphi_2 \in W^{1-1/p,p}(\partial\Omega)$. Then by (39), we see that

$$\langle \Lambda\varphi_1, \varphi_1 - \varphi_2\rangle = \langle \Lambda\varphi_1, \varphi_1\rangle - \langle \Lambda\varphi_1, \varphi_2\rangle = \int_\Omega a(x, \nabla P\varphi_1)(\nabla P\varphi_1 - \nabla P\varphi_2)\,\mathrm{d}x$$

and similarly,

$$\langle \Lambda\varphi_2, \varphi_1 - \varphi_2\rangle = \langle \Lambda\varphi_2, \varphi_1\rangle - \langle \Lambda\varphi_2, \varphi_2\rangle = \int_\Omega a(x, \nabla P\varphi_2)(\nabla P\varphi_1 - \nabla P\varphi_2)\,\mathrm{d}x.$$

Thus

$$\begin{aligned}\langle \Lambda\varphi_1 - \Lambda\varphi_2, \varphi_1 - \varphi_2\rangle &= \langle \Lambda\varphi_1, \varphi_1 - \varphi_2\rangle - \langle \Lambda\varphi_2, \varphi_1 - \varphi_2\rangle \\ &= \int_\Omega (a(x, \nabla P\varphi_1) - a(x, \nabla P\varphi_2))(\nabla P\varphi_1 - \nabla P\varphi_2)\,\mathrm{d}x,\end{aligned}$$

and hence, monotonicity condition (18) implies that Λ is monotone. Now, let φ, $\psi \in W^{1-1/p,p}(\partial\Omega)$. Then, by growth condition (17), Hölder's inequality, by inequality (35), and by the boundedness of the operator $Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$, we see that

$$|\langle \Lambda\varphi, \psi\rangle| \le \|\nabla P\varphi\|^{p-1}_{L^p(\Omega)^d} \,\|\nabla Z\psi\|_{L^p(\Omega)^d} \le C \,\|\varphi\|^{p-1}_{W^{1-1/p,p}(\partial\Omega)} \,\|\psi\|_{W^{1-1/p,p}(\partial\Omega)},$$

which leads to inequality (40). To see that Λ satisfies (41), let $\varphi \in W^{1-1/p,p}_m(\partial\Omega)$. Then by (39) and coercivity condition (16) yields

$$\langle \Lambda\varphi, \varphi\rangle \ge \eta \,\|\nabla P\varphi\|^p_{L^p(\Omega)^d}. \tag{43}$$

On the other hand, since the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$ is linear bounded and by Maz'ya's inequality (22), we have that

$$\|\varphi\|_{W^{1-1/p,p}(\partial\Omega)} \le C\left(\|\nabla P\varphi\|_{L^p(\Omega)^d} + \|P\varphi\|_{L^p(\Omega)}\right) \le C\left(\|\nabla P\varphi\|_{L^p(\Omega)^d} + \|\varphi\|_{L^p(\partial\Omega)}\right).$$

Applying Poincaré's inequality (44) (stated in Lemma 3.4 below) to the term $\|\varphi\|_{L^p(\partial\Omega)}$ in the latter estimate shows that

$$\|\varphi\|_{W^{1-1/p,p}(\partial\Omega)} \le C\|\nabla P\varphi\|_{L^p(\Omega)^d}.$$

Hence by (43), Λ satisfies inequality (41). This completes the proof of this proposition. $\square$

Lemma 3.4. *For a function $u \in L^1(\partial\Omega)$, we set $\overline{u} = \frac{1}{\mathcal{H}(\partial\Omega)}\int_{\partial\Omega} u\,\mathrm{d}\mathcal{H}$. Then, there is a constant $C > 0$ such that*

$$\int_{\partial\Omega} |u - \overline{u}|^p\,\mathrm{d}\mathcal{H} \le C\int_\Omega |\nabla u|^p\,\mathrm{d}x \tag{44}$$

for every $u \in W^{1,p}(\Omega)$.

The proof of Lemma (3.4) is quiet standard in the literature. Hence we omit it.

Remark 3.5. We underline that the operator Λ is not *strictly monotone* without an additional conditions on $\varphi \in W^{1-1/p,p}(\partial\Omega)$. In other words, for given φ_1, $\varphi_2 \in W^{1-1/p,p}(\partial\Omega)$, the implication

$$\langle \Lambda\varphi_1 - \Lambda\varphi_2, \varphi_1 - \varphi_2\rangle = 0 \quad \Longrightarrow \quad \varphi_1 = \varphi_2$$

does not hold in general as the following counter-example shows. Let $\varphi_1 \equiv c_1$ and $\varphi_2 \equiv c_2$ on $\partial\Omega$ for some c_1, $c_2 \in \mathbb{R}$ such that $c_1 \neq c_2$. Then $P\varphi_i \equiv c_i$ on $\overline{\Omega}$ and hence, $\Lambda\varphi_i \equiv 0$ on $\partial\Omega$ for $i = 1, 2$. Thus,

$$\langle \Lambda\varphi_1 - \Lambda\varphi_2, \varphi_1 - \varphi_2\rangle = 0 \qquad \text{but} \qquad \varphi_1 \neq \varphi_2.$$

An additional condition on $\varphi \in W^{1-1/p,p}(\partial\Omega)$ would be *compatibility condition* (3). As a result, the space $W_m^{1-1/p,p}(\partial\Omega)$ has been introduced by us in Section 1. In fact, Λ restricted on $W_m^{1-1/p,p}(\partial\Omega)$ is strictly monotone. To see this, let φ_1, $\varphi_2 \in W_m^{1-1/p,p}(\partial\Omega)$ such that

$$\langle \Lambda\varphi_1 - \Lambda\varphi_2, \varphi_1 - \varphi_2\rangle = 0.$$

Then by (39) and by the linearity of the operator Z, this equality can be rewritten as

$$\int_\Omega \left(a(x, \nabla P\varphi_1) - a(x, \nabla P\varphi_1)\right)\left(\nabla P\varphi_1 - \nabla P\varphi_2\right)\mathrm{d}x = 0.$$

By the strict monotonicity condition (18), this implies that $\nabla P\varphi_1 = \nabla P\varphi_2$ a.e. on Ω. By [40, Corollary 2.1.9] and since Ω is connected, $P\varphi_1 = P\varphi_2 + c$ on Ω for some $c \in \mathbb{R}$. This means that $P\varphi_{1\,|\partial\Omega} = \varphi_1 = \varphi_2 + c$ on $\partial\Omega$. However, φ_1 and φ_2 satisfy condition (3). This implies that $c = 0$ and thereby we have shown that $\varphi_1 = \varphi_2$ on $\partial\Omega$, proving that Λ is strictly monotone on the space $W_m^{1-1/p,p}(\partial\Omega)$.

With this remark we can conclude the following statement. We leave the easy proof as an exercise for the interested reader.

Corollary 3.6. *Suppose that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18). The restriction of the operator Λ defined by (38) on the space $W_m^{1-1/p,p}(\partial\Omega)$, denoted again by Λ, is a well-defined, continuous, strictly monotone operator from $W_m^{1-1/p,p}(\partial\Omega)$ to the dual space $W_m^{-(1-1/p),p'}(\partial\Omega)$ satisfying the inequalities (40) and (41).*

3.2. The Dirichlet-to-Neumann operator Λ_q on $L^q(\partial\Omega)$ and on $C(\partial\Omega)$

In this subsection, we introduce the Dirichlet-to-Neumann operator Λ_q associated with the differential operator A given by (19) on $L^q(\partial\Omega)$ for $1 \le q \le \infty$ and Λ_c on $C(\partial\Omega)$ under the assumptions that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18).

Under theses assumptions on a, it was first shown in [3] that Λ_1 is completely accretive. In addition, it was proved in [3, Théorème 2.3.1] that the *entropy solutions* operator associated with Λ_1 satisfies the range condition (24) in $X = L^1(\partial\Omega)$. A few year later, it was shown in [1, p.312] that, in fact, the closure $\overline{\Lambda_1}$ (see Definition **??** below) of the Dirichlet-to-Neumann operator Λ_1 in $L^1(\partial\Omega)$ coincides with the corresponding *entropy solution* operator in $L^1(\partial\Omega)$.

For $1 \le q \le \infty$, one has that $\overline{\Lambda_q}$ is contained in $\overline{\Lambda_1}$. Hence, it is not difficult to conclude that $\overline{\Lambda_q}$ is completely accretive for all $1 \le q \le \infty$ and since $(0, 0) \in \Lambda_q$, one has that $\overline{\Lambda_q}$ is m-completely accretive in $L^q(\partial\Omega)$.

However, the result that Λ_c is m-completely accretive in $C(\partial\Omega)$ is not a straight forward consequence of [3, Théorème 2.3.1]. It is one of the main results of this article (see Proposition 3.12).

To keep this article self-contained, we give here the details of the previous mentioned results, but we proceed here in a slightly differently way than in [3] or in [1]. To this end, we begin with the following two definitions.

Definition 3.7. Let $1 \le q \le \infty$ and let Λ be the Dirichlet-to-Neumann operator associated with the quasi-linear operator A defined in (19). Then, the *Dirichlet-to-Neumann operator* Λ_q *on* $L^q(\partial\Omega)$ is defined by the part of Λ in $L^q(\partial\Omega)$, that is,

$$\Lambda_q = \Big\{ (\varphi, \psi) \in L^q(\partial\Omega) \times L^q(\partial\Omega) \,\Big|\, (\varphi, \psi) \in \Lambda \Big\}$$

and the Dirichlet-to-Neumann operator Λ_c on $C(\partial\Omega)$ is defined by the *part of* Λ *in* $C(\partial\Omega)$, that is,

$$\Lambda_c = \Big\{ (\varphi, \psi) \in C(\partial\Omega) \times C(\partial\Omega) \,\Big|\, (\varphi, \psi) \in \Lambda \Big\}.$$

We denote by $D(\Lambda_q)$ (resp., by $D(\Lambda_c)$) the domain of Λ_q (resp., of Λ_c) for every $1 \le q \le \infty$.Further, we define the *closure* $\overline{\Lambda_q}$ *of* Λ_q *in* $L^q(\partial\Omega)$ is by

$$\overline{\Lambda_q} = \left\{ (\varphi, \psi) \in L^q(\partial\Omega) \times L^q(\partial\Omega) \,\middle|\, \begin{array}{c} \text{there exists } \; ((\varphi_n, \psi_n)) \subseteq \Lambda_q \text{ s.t. } \lim\limits_{n\to\infty} \varphi_n = \varphi \\ \& \lim\limits_{n\to\infty} \psi_n = \psi \text{ in } L^q(\partial\Omega) \end{array} \right\}.$$

and we call the operator Λ_q *closed in* $L^q(\partial\Omega)$ if $\Lambda_q = \overline{\Lambda_q}$.

Furthermore, we need the following definition.

Definition 3.8. Let $1 \le q \le \infty$ and ν be the outward pointing unit normal vector on $\partial\Omega$. For a function $u \in W^{1,p}(\Omega)$, we call a function $\psi \in L^q(\partial\Omega)$ the *generalised co-normal derivative of* u *in* $L^q(\partial\Omega)$ if there exists a function $F \in L^1(\Omega)$ satisfying

$$\int_\Omega a(\cdot, \nabla u)\nabla v \,\mathrm{d}x = \int_{\partial\Omega} \psi\, v \,\mathrm{d}\mathcal{H} - \int_\Omega F\, v \,\mathrm{d}x \tag{45}$$

for every $v \in C^\infty(\overline{\Omega})$. By Lemma 2.1, the set $\{v_{|\partial\Omega} \,|\, v \in C^\infty(\overline{\Omega})\}$ lies dense in $L^q(\partial\Omega)$ for $1 \le q < \infty$. Thus the function $\psi \in L^q(\partial\Omega)$ is uniquely determined by equation (45) for the same F, and hence it makes sense to set

$$a(\cdot, \nabla u) \cdot \nu := \psi.$$

If a function $u \in W^{1,p}(\Omega)$ admits a generalised co-normal derivative in $L^q(\partial\Omega)$, then we also write $a(\cdot, \nabla u) \cdot \nu \in L^q(\partial\Omega)$.

Now, we begin investigating the case $q = 2$.

Proposition 3.9. *Suppose $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18) and for $V_0 = W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$ equipped with the sum norm, let $\Lambda_{|V_0}$ be the restriction on V_0 of the Dirichlet-to-Neumann operator Λ defined in (38). Then the following statements hold true.*

1. *The operator Λ_2 is the part of the operator $\Lambda_{|V_0}$ in $L^2(\partial\Omega)$ and the domain $D(\Lambda_2)$ of Λ_2 coincides with the set of all $\varphi \in V_0$ satisfying*
$$a(\cdot, \nabla P\varphi) \cdot \nu \in L^2(\partial\Omega).$$
In particular,
$$\Lambda\varphi = a(\cdot, \nabla P\varphi) \cdot \nu$$
for every $\varphi \in D(\Lambda_2)$ and
$$\int_{\partial\Omega} \Lambda\varphi\, \xi \, \mathrm{d}\mathcal{H} = \langle \Lambda\varphi, \xi \rangle = \int_{\Omega} a(x, \nabla P\varphi)\nabla Z\xi \, \mathrm{d}x \tag{46}$$
for every $\xi \in V_0$.
2. *The operator Λ_2 is a closed operator in $L^2(\partial\Omega)$.*
3. *The operator Λ_2 is m-completely accretive in $L^2(\partial\Omega)$ with dense domain in $L^2(\partial\Omega)$. In particular, for every $\varphi \in V_0$, we have that $J_\lambda\varphi$ converges to φ in $L^2(\partial\Omega)$ as $\lambda \to 0+$, where J_λ denotes the resolvent operator of Λ_2*

To prove this proposition, we need the following lemma.

Lemma 3.10. *Let (φ_n) be a sequence in $L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ and suppose (φ_n) is bounded in $L^2(\partial\Omega)$ such that $(\nabla P\varphi_n)$ is bounded in $L^p(\Omega)^d$. Then there is a $\varphi \in L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ and a subsequence (φ_{k_n}) of (φ_n) such that φ_{k_n} converges weakly to φ in $L^2(\partial\Omega)$, $P\varphi_{k_n}$ converges to $P\varphi$ weakly in $W^{1,p}(\Omega)$. If, in addition, the function $\mathcal{A} : \Omega \times \mathbb{R} \to\ R$ is a Carathéodory function satisfying (20), then, in particular, the subsequence (φ_{k_n}) of (φ_n) satisfies $\mathcal{A}(x, \nabla P\varphi_{k_n})$ converges to $\mathcal{A}(x, \nabla P\varphi)$ a.e. on Ω.*

Proof. To see that the claim of this lemma holds, let (φ_n) be a sequence in $L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ such that (φ_n) is bounded in $L^2(\partial\Omega)$ and $(\nabla P\varphi_n)$ is bounded in $L^p(\Omega)^d$. By inequality (23), we can conclude that $(P\varphi_n)$ is bounded in $W^{1,p}(\Omega)$. Thus by Lemma 2.3 and since $L^2(\partial\Omega)$ is reflexive, there is a subsequence (φ_{k_n}) of (φ_n), a weak solution $u \in W^{1,p}(\Omega)$ of (29) and some $\varphi \in L^2(\partial\Omega)$ such that $P\varphi_{k_n}$ converges to u weakly in $W^{1,p}(\Omega)$, and φ_{k_n} converges weakly to φ in $L^2(\partial\Omega)$. Thus and since the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is compact, it follows that $\varphi \in L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ and $u = P\varphi$. If $\mathcal{A}$ and a are Carathéodory functions satisfying (20), then by growth condition (17) and by Lemma 2.3, we see that $a(x, \nabla P\varphi_{k_n})$ converges to $a(x, \nabla P\varphi)$ a.e. on Ω, weakly in $L^{p'}(\Omega)^d$, and $\mathcal{A}(x, \nabla P\varphi_{k_n})$ converges to $\mathcal{A}(x, \nabla P\varphi)$ a.e. on Ω. □

Now, we can turn to the proof of Proposition 3.9.

Proof of Proposition 3.9. To see that claim (1) holds, we note first that if $V_0 = W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$ is equipped with the sum norm, then the dual space V_0' of is given by the sum space $V_0' = W^{-(1-1/p),p'} + L^2(\partial\Omega)$. By construction, V_0 is continuously embedded into $L^2(\partial\Omega)$ by a bounded linear injection i with a dense image. Hence, the adjoint operator $i' : (L^2(\partial\Omega))' \to V_0'$ is a continuous injection as well. We identify $L^2(\partial\Omega)$ with its dual space $(L^2(\partial\Omega))'$ and so the space V_0 can be considered as a linear subspace of $L^2(\partial\Omega)$ and V_0', where $L^2(\partial\Omega)$ is an intermediate space between V_0 and V_0'. With this preliminaries, we see that the restriction $\Lambda_{|V_0} : V_0 \to V_0'$ of Λ on V_0 remains a continuous and monotone operator and Λ_2 is the part of $\Lambda_{|V_0}$ in $L^2(\partial\Omega)$. Let D be the set of all $\varphi \in V_0$ such that $a(\cdot, \nabla P\varphi)\cdot\nu \in L^2(\partial\Omega)$. Then, one easily sees that the set D is contained in $D(\Lambda_2)$. On the other hand, by definition of Λ_2, for every $\varphi \in D(\Lambda_2)$, one has $\varphi \in V_0$ and there is a $\psi \in L^2(\partial\Omega)$ such that

$$\langle \Lambda\varphi, \xi\rangle = \int_{\partial\Omega} \psi\,\xi \,\mathrm{d}\mathcal{H}$$

for every $\xi \in V_0$. Hence by definition of Λ and by Definition 3.8, we have $\psi = a(x, \nabla P\varphi)\cdot\nu \in L^2(\partial\Omega)$. Therefore the set $D(\Lambda_2)$ is contained in D and, in particular, by definition of Λ, the last equation shows that equation (46) holds. This completes the proof of claim (1) of this proposition.

Next, let φ, $\psi \in L^2(\partial\Omega)$ and (φ_n) and (ψ_n) be two sequences in $L^2(\partial\Omega)$ satisfying

$$(\varphi_n, \psi_n) \in \Lambda_2 \quad \text{for every } n \geq 1, \quad \lim_{n\to\infty} \varphi_n = \varphi \text{ in } L^2(\partial\Omega), \quad \lim_{n\to\infty} \psi_n = \psi \text{ in } L^2(\partial\Omega). \tag{47}$$

Then, by claim (1) of this proposition, each φ_n satisfies

$$\int_\Omega a(x, \nabla P\varphi_n)\nabla Z\xi \,\mathrm{d}x = \int_{\partial\Omega} \psi_n\,\xi\,\mathrm{d}\mathcal{H} \tag{48}$$

for every $\xi \in V_0$. Taking $\xi = \varphi_n$ in this equation and subsequently applying (39), Cauchy-Schwarz's inequality and coercivity condition (16) yield

$$\eta \int_\Omega |\nabla P\varphi_n|^p \,\mathrm{d}x \leq \|\psi_n\|_{L^2(\partial\Omega)}\,\|\varphi_n\|_{L^2(\partial\Omega)}$$

for every n. Therefore and by the assumptions on (φ_n) and (ψ_n), the sequence $(\nabla P\varphi_n)$ is bounded in $L^p(\Omega)^d$. Due to Lemma 3.10, there is a subsequence (φ_{k_n}) of (φ_n) and $\varphi \in W^{1-1/p,p}(\partial\Omega)$ such that $P\varphi_{k_n}$ converges weakly to $P\varphi$ in $W^{1,p}(\Omega)$ and $a(x, \nabla P\varphi_{k_n})$ converges to $a(x, \nabla P\varphi)$ a.e. on Ω and weakly in $L^{p'}(\Omega)^d$. Thus sending $n \to \infty$ equation (48) for general $\xi \in V_0$ and using the limits in (47), shows that $\psi = a(\cdot, \nabla P\varphi) \in L^2(\partial\Omega)$. Hence by claim (1) of this proposition, $(\varphi, \psi) \in \Lambda_2$. This proves that Λ_2 is a closed operator in $L^2(\partial\Omega)$. $\square$

Next, we show that Λ_2 is completely accretive. For this, we make use of the following characterisation (see also [5, Corollary A.43]).

Proposition 3.11 ([9, Proposition 2.2])**.** *Let P_0 denote the set of all functions $T \in C^\infty(\mathbb{R})$ satisfying $0 \leq T' \leq 1$, the support supp(T') of the derivative T' of T is compact, and $x = 0$ is not contained in the support supp(T) of T. An operator A on $L^q(\partial\Omega)$ for $1 \leq q < \infty$ is completely accretive in $L^q(\partial\Omega)$ if and only if*

$$\int_{\partial\Omega} (\psi - \hat{\psi})T(\varphi - \hat{\varphi})\,\mathrm{d}\mathcal{H} \geq 0 \tag{49}$$

for every $T \in P_0$ and every (φ, ψ), $(\hat{\varphi}, \hat{\psi}) \in A$.

Proof of Proposition 3.9 (Continued). Let $T \in P_0$ and $\varphi, \hat{\varphi} \in D(\Lambda_2)$. Since $T : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous such that $T(0) = 0$, $\Phi := T(P\varphi - P\hat{\varphi}) \in W^{1,p}(\Omega)$ and has trace $T(P\varphi - P\hat{\varphi})_{|\partial\Omega} = T(\varphi - \hat{\varphi})$. Therefore, by claim (3) of Lemma 2.5 there is a unique $u_\Phi \in W_0^{1,p}(\Omega)$ satisfying $PT(\varphi - \hat{\varphi}) = u_\Phi + \Phi$. Applying this to formula (46) and using that $P\varphi$ and $P\hat{\varphi}$ are weak solutions of equation (29) yields

$$\begin{aligned}
&\int_{\partial\Omega} (\Lambda_2\varphi - \Lambda_2\hat{\varphi}) T(\varphi_1 - \varphi_2)\, \mathrm{d}\mathcal{H} \\
&\qquad = \int_\Omega (a(x, \nabla P\varphi) - a(x, \nabla P\hat{\varphi})) \nabla PT(\varphi_1 - \varphi_2)\, \mathrm{d}x \\
&\qquad = \int_\Omega (a(x, \nabla P\varphi) - a(x, \nabla P\hat{\varphi})) \nabla T(P\varphi - P\hat{\varphi})\, \mathrm{d}x \\
&\qquad = \int_\Omega (a(x, \nabla P\varphi) - a(x, \nabla P\hat{\varphi}))(\nabla P\varphi - \nabla P\hat{\varphi})\, T'(P\varphi - P\hat{\varphi})\, \mathrm{d}x.
\end{aligned}$$

Since $T'(P\varphi - P\hat{\varphi}) \geq 0$ and by monotonicity condition (18), the integrand in the last integral of this calculation is non-negative a.e. on Ω, proving that the Dirichlet-to-Neumann operator Λ_2 is completely accretive.

Now, consider the operator $\mathcal{N} : V_0 \to V_0'$ defined by

$$\mathcal{N} = \mathrm{I}_{L^2(\partial\Omega)} + \Lambda_{|V_0},$$

where $\Lambda_{|V_0} : V_0 \to V_0'$ is the restriction of Λ on the space V_0. By Proposition 3.3, the operator $\mathcal{N}$ is continuous and monotone. By Hölder's inequality and by inequality (40) in Proposition 3.3, there is a constant $C_1 > 0$ such that

$$\|\mathcal{N}\varphi\|_{V_0'} \leq \|\varphi\|_{L^2(\partial\Omega)} + C_1 \|\varphi\|^{p-1}_{W^{1-1/p,p}(\partial\Omega)}$$

for every $\varphi \in V_0$. Thus the operator $\mathcal{N}$ maps bounded sets of V_0 into bounded sets of V_0'. Moreover, by using the boundedness of the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$, it is not difficult to see that for every $\alpha \in \mathbb{R}$, the set

$$E_\alpha := \left\{ \varphi \in V_0 \,\middle|\, \tfrac{\langle \mathcal{N}\varphi, \varphi\rangle_{V_0', V_0}}{\|\varphi\|_{V_0}} \leq \alpha \right\}$$

is bounded in V_0, which is equivalent to the fact that

$$\lim_{\|\varphi\|_{V_0}} \tfrac{\langle \mathcal{N}\varphi, \varphi\rangle_{V_0', V_0}}{\|\varphi\|_{V_0}} = +\infty.$$

Therefore by [29, Théorème 2.1], the operator $\mathcal{N} : V_0 \to V_0'$ is surjective and since $L^2(\partial\Omega)$ is continuously injected into V_0', we have thereby shown that for every $\psi \in L^2(\partial\Omega)$, there is a $\varphi \in V_0$ such that $\varphi + \Lambda\varphi = \psi$, proving that Λ_2 satisfies the range condition (24) for $X = L^2(\partial\Omega)$.

It remains to show that the domain $D(\Lambda_2)$ lies dense in $L^2(\partial\Omega)$. By Lemma 2.1, V_0 is a dense subspace of $L^2(\partial\Omega)$ and hence we only need to show that for every $\varphi \in V_0$, $\varphi_\lambda := J_\lambda\varphi \in D(\Lambda_2)$ converges to φ in $L^2(\partial\Omega)$ as $\lambda \to 0+$. To see this, take $\varphi \in V_0$, Then for every $\lambda > 0$, φ_λ is the unique weak solution of equation

$$\varphi_\lambda - \varphi + \lambda\, \Lambda_2\varphi_\lambda = 0$$

in $L^2(\partial\Omega)$. Multiplying this equation by $\varphi_\lambda - \varphi$ with respect to the inner product on $L^2(\partial\Omega)$, and then by using (16) and (17) in conjunction with (39), Hölder's and Young's inequality, we see that

$$\|\varphi_\lambda - \varphi\|^2_{L^2(\partial\Omega)} + \eta\,\lambda \|\nabla P\varphi_\lambda\|^p_{L^p(\Omega)^d} \le \tfrac{\lambda\,\eta}{p'}\,\|\nabla P\varphi_\lambda\|^p_{L^p(\Omega)^d} + C\,\lambda\,\|\nabla Z\varphi\|^p_{L^p(\Omega)^d}$$

for some constant $C > 0$ depending on $\eta > 0$ and $c > 0$ from (16) and (17). Rearranging this inequality, yields

$$\|\varphi_\lambda - \varphi\|^2_{L^2(\partial\Omega)} \le C\,\lambda\,\|\nabla Z\varphi\|^p_{L^p(\Omega)}.$$

showing that φ_λ converges to φ in $L^2(\partial\Omega)$ as $\lambda \to 0+$. □

Next, we consider the general case $1 \le q \le \infty$.

Proposition 3.12. *The following statements hold true.*

1. *For $2 \le q \le \infty$, the Dirichlet-to-Neumann operator Λ_q in $L^q(\partial\Omega)$ is closed and m-completely accretive and for $1 \le q < 2$, the closure $\overline{\Lambda_q}$ of the Dirichlet-to-Neumann operator Λ_q in $L^q(\partial\Omega)$ is m-completely accretive.*
2. *The Dirichlet-to-Neumann operator Λ_c in $C(\partial\Omega)$ is closed and m-completely accretive.*
3. *For $1 \le q < \infty$, the domain $D(\Lambda_q)$ lies dense in $L^q(\partial\Omega)$. If Ω has a $C^{1,\beta}$-boundary $\partial\Omega$ for some $\beta \in (0,1)$ then the domain $D(\Lambda_c)$ lies dense in $C(\partial\Omega)$.*

Proof. First, let $2 \le q \le \infty$. First, we show that Λ_q is a closed operator in $L^q(\partial\Omega)$. To this end, note that by the continuous embedding of $L^q(\partial\Omega)$ into $L^2(\partial\Omega)$ and by claim (1) of Proposition 3.9,the operator $\overline{\Lambda_q}$ is contained in $\overline{\Lambda_2} = \Lambda_2$. Thus every $(\varphi,\psi) \in \overline{\Lambda_q}$ satisfies

$$\int_\Omega a(x,\nabla P\varphi)\nabla Z\xi \,\mathrm{d}x = \int_{\partial\Omega} \psi\,\xi_{|\partial\Omega}\,\mathrm{d}\mathcal{H}$$

for every $\xi \in C^\infty(\overline{\Omega})$ and so by Definition 3.8, $\psi = a(\cdot,P\varphi)\cdot\nu \in L^q(\partial\Omega)$, showing that $(\varphi,\psi) \in \Lambda_q$.

By claim (2) of Proposition 3.9, the Dirichlet-to-Neumann operator Λ_2 is completely accretive in $L^2(\partial\Omega)$ and since $\Lambda_q \subseteq \Lambda_2$, respectively, $\Lambda_c \subseteq \Lambda_2$, it is clear that Λ_q and Λ_c are completely accretive in $L^q(\partial\Omega)$ and in $C(\partial\Omega)$, respectively. In the case $1 \le q < 2$, one first shows by the same idea as in the Proposition 3.9 that Λ_q is completely accretive in $L^q(\partial\Omega)$. Then by passing to the limit, one sees that, in particular, the closure $\overline{\Lambda_q}$ of Λ_q is completely accretive in $L^q(\partial\Omega)$.

Next, let $2 < q \le \infty$. Take $\psi \in L^q(\partial\Omega)$. Then by the continuous injection of $L^q(\partial\Omega)$ into $L^2(\partial\Omega)$ and by Proposition 3.13, there is a weak solution $\varphi \in D(\Lambda_2)$ of equation

$$\varphi + \Lambda\varphi = \psi \tag{50}$$

in $L^2(\partial\Omega)$. Since $\Lambda_2 0 = 0$ on $\partial\Omega$ and Λ_2 is L^q-accretive in $L^2(\partial\Omega)$, we have

$$\|\varphi\|_{L^q(\partial\Omega)} \le \|\varphi + \Lambda\varphi - (0 + \Lambda 0)\|_{L^q(\partial\Omega)} = \|\psi\|_{L^q(\partial\Omega)}.$$

Thus, by equation (50) and since $\psi \in L^q(\partial\Omega)$, it follows that $\varphi \in D(\Lambda_q)$ satisfying $\Lambda_q\varphi = \psi - \varphi$, proving that Λ_q satisfies the range condition in $L^q(\Omega)$ for every $2 < q \le \infty$.

Now, take $\psi \in C(\partial\Omega)$. Then, $\psi \in L^\infty(\partial\Omega)$ and since Λ_∞ is m-accretive in $L^\infty(\partial\Omega)$, there is a weak solution $\varphi \in D(\Lambda_\infty)$ of equation (50) in $L^\infty(\partial\Omega)$. Since $\psi - \varphi \in L^q_m(\partial\Omega)$ for any $q \ge 1$, claim (2) of Theorem 1.1 yields $\varphi \in C^{0,\alpha}(\partial\Omega)$. Hence, $\varphi \in D(\Lambda_c)$ with $\Lambda_c\varphi = \psi - \varphi$, proving that Λ_c is m-accretive in $C(\partial\Omega)$.

Now, let $\psi \in L^q(\partial\Omega)$ for $1 \le q < 2$. Then there is a sequence $\psi_n \in L^1(\partial\Omega) \cap L^\infty(\partial\Omega)$ such that ψ_n converges to ψ in $L^q(\partial\Omega)$. Since $L^1(\partial\Omega) \cap L^\infty(\partial\Omega)$ is continuously imbedded into $L^2(\partial\Omega)$, to every ψ_n there is a weak solution $\varphi_n \in D(\Lambda_2)$ of $\varphi_n + \Lambda_2\varphi_n = \psi_n$. Since Λ_2 is L^q-accretive in $L^2(\partial\Omega)$ and $\Lambda_2 0 = 0$, it follows that $\varphi_n \in L^q(\partial\Omega)$. Thus $(\varphi_n, \psi_n - \varphi_n) \in \Lambda_q$ and by the accretivity of Λ_q in $L^q(\partial\Omega)$, we see that

$$\begin{aligned}\|\varphi_n - \varphi_m\|_{L^q(\partial\Omega)} &\le \|(\varphi_n + \Lambda_q\varphi_n) - (\varphi_m + \Lambda_q\varphi_m)\|_{L^q(\partial\Omega)}\\ &\le \|\psi_n - \psi_m\|_{L^q(\partial\Omega)}.\end{aligned}$$

Therefore and since $L^q(\partial\Omega)$ is a Banach space, there is a function $\varphi \in L^q(\partial\Omega)$ such that φ_n converges to φ in $L^q(\partial\Omega)$. This proves that $\varphi \in D(\overline{\Lambda_q})$ satisfying $\varphi + \overline{\Lambda_q}\varphi = \psi$, showing that $\overline{\Lambda_q}$ satisfies the range condition (24) for $X = L^q(\partial\Omega)$.

Next, we show that $D(\Lambda_q)$ lies dense in $L^q(\partial\Omega)$ for $1 \le q < \infty$. Let $\varphi \in L^q(\partial\Omega)$. Then by Lemma 2.1, there is a sequence (φ_n) in $\mathcal{D} := \{v_{|\partial\Omega} \,|v \in C^\infty(\overline{\Omega})\}$ such that φ_n converges to φ in $L^q(\partial\Omega)$. Thus for given $\varepsilon > 0$, there is a $\varphi_{n_0} \in \mathcal{D}$ such that $\|\varphi_{n_0} - \varphi\|_{L^q(\partial\Omega)} \le \varepsilon/3$. For every $\lambda > 0$, let $J_\lambda = (I_{L^2(\partial\Omega)} + \Lambda_2)^{-1}$ be resolvent operator of Λ_2. Let $B = \overline{\Lambda_q}$ if $1 < q < 2$ and $B = \Lambda_q$ if $2 \le q < \infty$. Since B is completely accretive, $B0 = 0$ and since B satisfies the range condition (24) for $X = L^q(\partial\Omega)$, it follows that the resolvent operator J_λ of Λ_2 coincides with the resolvent operator of B on $L^q(\partial\Omega) \cap L^2(\partial\Omega)$, maps $L^q(\partial\Omega) \cap L^2(\partial\Omega)$ into $D(B) \cap D(\Lambda_2)$ and extends to a contractive mapping from $L^q(\partial\Omega)$ to a subset of $L^q(\partial\Omega)$ for every $1 \le q \le \infty$. Thus

$$\begin{aligned}\|J_\lambda\varphi - \varphi\|_{L^q(\partial\Omega)} &\le \|J_\lambda\varphi - J_\lambda\varphi_{n_0}\|_{L^q(\partial\Omega)} + \|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^q(\partial\Omega)} + \|\varphi_{n_0} - \varphi\|_{L^q(\partial\Omega)}\\ &\le 2\,\|\varphi_{n_0} - \varphi\|_{L^q(\partial\Omega)} + \|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^q(\partial\Omega)}\\ &\le 2\,\tfrac{\varepsilon}{3} + \|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^q(\partial\Omega)}.\end{aligned}$$

If $q \ge 2$, then

$$\begin{aligned}\|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^q(\partial\Omega)} &\le \|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^\infty(\partial\Omega)}^{\frac{q-2}{q}}\,\|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^2(\partial\Omega)}\\ &\le (\|\varphi_{n_0}\|_{L^\infty(\partial\Omega)} + \|\varphi_{n_0}\|_{L^\infty(\partial\Omega)})^{\frac{q-2}{q}}\,\|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^2(\partial\Omega)},\end{aligned}$$

where we used that $\varphi_{n_0} \in L^\infty(\partial\Omega)$, J_λ is contractive in $L^\infty(\partial\Omega)$, and the fact that $\Lambda 0 = 0$. If $1 \le q < 2$, then by Hölder's inequality,

$$\|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^q(\partial\Omega)} \le \mathcal{H}(\partial\Omega)^{\frac{1}{q}-\frac{1}{2}}\,\|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^2(\partial\Omega)}$$

Thus, for all $1 \le q < \infty$, there is a constant $C > 0$ such that

$$\|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^q(\partial\Omega)} \le \tfrac{\varepsilon}{2} + C\,\|J_\lambda\varphi_{n_0} - \varphi_{n_0}\|_{L^2(\partial\Omega)}$$

for all $\lambda > 0$. Since for every $1 \le q \le \infty$, the set $\mathcal{D}$ is contained in $L^2(\partial\Omega) \cap L^q(\partial\Omega)$, we have that $J_\lambda\varphi_{n_0} \in D(\Lambda_q)$ for every $\lambda > 0$. Therefore and by claim (3) of Proposition 3.9, the first part of claim (3) of this proposition holds.

Now, suppose that Ω has a $C^{1,\beta}$-boundary $\partial\Omega$ for some $\beta \in (0,1)$. Then by the regularity result [28] concerning the weak solutions of Dirichlet problem (28), the set $\mathcal{D}$ is a subset of the domain $D(\Lambda_c)$. Moreover, by the Stone – Weierstraß Theorem, the set $\mathcal{D}$ lies dense in $C(\partial\Omega)$, proving that $D(\Lambda_c)$ lies dense in $C(\partial\Omega)$.

□

3.3. The Dirichlet-to-Neumann operator realised as a subgradient in $L^2(\partial\Omega)$

Here, we give a sufficient condition ensuring that the Dirichlet-to-Neumann operator Λ_2 can be realised as the subgradient in $L^2(\partial\Omega)$ of a proper, convex, lower semicontinuous functional. The following proposition generalises [20, part (A) of Theorema in Section 2.] and [25, Theorem 2].

Proposition 3.13. *Let $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a Carathéodory function satisfying (16) –(18). In addition, suppose that a satisfies gradient condition (20) for some Carathéodory function $\mathcal{A} : \Omega \times \mathbb{R}^d \to \mathbb{R}$. Then the Dirichlet-to-Neumann operator Λ_2 can be realised as the subgradient $\partial\mathcal{E}$ in $L^2(\partial\Omega)$ of the proper, convex, densely defined and lower semicontinuous functional*

$$\mathcal{E}(\varphi) := \begin{cases} \displaystyle\int_\Omega \mathcal{A}(x, \nabla P\varphi)\,\mathrm{d}x, & \text{if } \varphi \in W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega), \\ +\infty, & \text{if otherwise} \end{cases} \tag{51}$$

for every $\varphi \in L^2(\partial\Omega)$.

The proof of Proposition 3.13 is given by the following two lemmata.

Lemma 3.14. *The functional $\mathcal{E}$ defined by (51) is proper, convex, densely defined, and lower semicontinuous in $L^2(\partial\Omega)$ with domain $D(\mathcal{E}) = W^{1-1/p,p}(\partial\Omega) \cap L^2(\partial\Omega)$.*

Proof. Since $\mathcal{A}$ is a Carathéodory function and by the growth condition (17), the functional $\mathcal{E}$ given by (51) is well-defined and proper. Furthermore by Lemma 2.1, the domain $D(\mathcal{E})$ lies dense in $L^2(\partial\Omega)$. To see that $\mathcal{E}$ is convex, let φ_1, $\varphi_2 \in D(\mathcal{E})$ and $\lambda \in [0,1]$. Then, $\lambda\varphi_1 + (1-\lambda)\varphi_2 \in D(\mathcal{E})$. Thus, by claim (4) of Lemma 2.5 and since $\xi \mapsto \mathcal{A}(x,\xi)$ is convex on $\mathbb{R}^d$ for a.e. $x \in \Omega$,

$$\begin{aligned} \mathcal{E}(\lambda\varphi_1 + (1-\lambda)\varphi_2) &\le \int_\Omega \mathcal{A}(x, \lambda\nabla P\varphi_1 + (1-\lambda)\nabla P(\varphi_2))\,\mathrm{d}x \\ &\le \lambda\,\mathcal{E}(\varphi_1) + (1-\lambda)\,\mathcal{E}(\varphi_2). \end{aligned}$$

It remains to show that $\mathcal{E}$ is lower semicontinuous in $L^2(\partial\Omega)$. For $\alpha \ge 0$, let (φ_n) be a sequence in $D(\mathcal{E})$ and $\varphi \in L^2(\partial\Omega)$ such that $\mathcal{E}(\varphi_n) \le \alpha$ for all n and φ_n converges to φ in $L^2(\partial\Omega)$. Then the sequence (φ_n) satisfies the hypotheses of Lemma 3.10 and hence, $\varphi \in D(\mathcal{E})$ and there is a subsequence (φ_{k_n}) of (φ_n) such that $\mathcal{A}(x, \nabla P\varphi_{k_n})$ converges to $\mathcal{A}(x, \nabla P\varphi)$ a.e on Ω. Thus and since $\mathcal{A}(x, \nabla P\varphi_{k_n})$ are non-negative measurable functions from Ω to $\mathbb{R}$, Fatou's lemma and the assumption $\mathcal{E}(\varphi_{k_n}) \le \alpha$ for all n imply

$$\mathcal{E}(\varphi) \le \liminf_{n\to\infty} \mathcal{E}(\varphi_{k_n}) \le \alpha.$$

As $\alpha \ge 0$ and the convergent sequence $(\varphi_n) \subseteq D(\mathcal{E})$ were arbitrary, we have thereby shown that $\mathcal{E}$ is lower semicontinuous in $L^2(\partial\Omega)$. □

Lemma 3.15. *Let $\mathcal{E}$ be the functional given by (51). Then the subgradient $\partial\mathcal{E}$ in $L^2(\partial\Omega)$ is single-valued and coincides with Λ_2 on $L^2(\partial\Omega)$.*

Proof. First, let $\varphi \in D(\Lambda_2)$. Then by claim (1) of Proposition 3.9, φ belongs to $D(\mathcal{E})$ such that $\Lambda_2\varphi \in L^2(\partial\Omega)$ satisfying (46). Thus by (39), we have that

$$\int_{\partial\Omega} \Lambda_2\varphi\,\xi\,\mathrm{d}\mathcal{H} = \int_\Omega a(x, \nabla P\varphi)\nabla P\xi\,\mathrm{d}x \tag{52}$$

for all $\xi \in L^2(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$. For any $\psi \in D(\mathcal{E})$, $\Phi := P\psi - P\varphi \in W^{1,p}(\Omega)$ and has trace $\Phi_{|\partial\Omega} = \psi - \varphi \in L^2(\partial\Omega)$. By claim (3) of Lemma 2.5, there is a unique $u_\Phi \in W^{1,p}_0(\Omega)$ such that $P(\psi - \varphi) = u_\Phi + (P\psi - P\varphi)$. Hence taking $\xi = \psi - \varphi$ in (52) and using that $P\varphi$ is a weak solution of (29), we see

$$\int_{\partial\Omega} \Lambda_2\varphi\,(\psi - \varphi)\,\mathrm{d}\mathcal{H} = \int_\Omega a(x, \nabla P\varphi)(\nabla P\psi - \nabla P\varphi)\,\mathrm{d}x$$

By the assumptions (18) and (20), $\xi \mapsto \mathcal{A}(x, \xi)$ is convex on $\mathbb{R}^d$ satisfying $\nabla_\xi \mathcal{A}(x, \xi) = a(x, \xi)$ for every $\xi \in \mathbb{R}^d$ and for a.e. $x \in \Omega$. Thus

$$a(x, \xi_0)(\xi - \xi_0) \leq \mathcal{A}(x, \xi) - \mathcal{A}(x, \xi_0)$$

for all ξ, $\xi_0 \in \mathbb{R}^d$ and a.e. $x \in \Omega$. Applying this inequality with $\xi_0 = \nabla P\varphi$ and $\xi = \nabla P\psi$ to the right-hand side of the last equation, we see that

$$\int_{\partial\Omega} \Lambda_2\varphi\,(\psi - \varphi)\,\mathrm{d}\mathcal{H} \leq \mathcal{E}(\psi) - \mathcal{E}(\varphi).$$

As $\psi \in D(\mathcal{E})$ was arbitrary, this shows that $\varphi \in D(\partial\mathcal{E})$ and $\Lambda_2\varphi \in \partial\mathcal{E}(\varphi)$. Now, let $\varphi \in D(\partial\mathcal{E})$ and $\chi \in \partial\mathcal{E}(\varphi)$. By definition of subgradients in $L^2(\partial\Omega)$, $\varphi \in D(\mathcal{E})$ and $\chi \in L^2(\partial\Omega)$ satisfy

$$\int_{\partial\Omega} \chi\,(\psi - \varphi)\,\mathrm{d}\mathcal{H} \leq \int_\Omega \mathcal{A}(x, \nabla P\psi)\,\mathrm{d}x - \int_\Omega \mathcal{A}(x, \nabla P\varphi)\,\mathrm{d}x$$

for every $\psi \in D(\mathcal{E})$. For $\lambda > 0$ and $\zeta \in D(\mathcal{E})$, taking $\psi = \varphi + \lambda\,\zeta$ in the previous inequality and subsequently dividing the resulting inequality by λ yield

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H} \leq \frac{1}{\lambda}\left(\int_\Omega \mathcal{A}(x, \nabla P(\varphi + \lambda\,\zeta))\,\mathrm{d}x - \int_\Omega \mathcal{A}(x, \nabla P\varphi)\,\mathrm{d}x\right).$$

By claim (4) of Lemma 2.5,

$$\int_\Omega \mathcal{A}(x, \nabla P(\varphi + \lambda\,\zeta))\,\mathrm{d}x \leq \int_\Omega \mathcal{A}(x, \nabla P\varphi + \lambda\,P\zeta)\,\mathrm{d}x.$$

Hence

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H} \leq \frac{1}{\lambda}\left(\int_\Omega \mathcal{A}(x, \nabla P\varphi + \lambda\,P\zeta)\,\mathrm{d}x - \int_\Omega \mathcal{A}(x, \nabla P\varphi)\,\mathrm{d}x\right)$$

for all $\lambda > 0$. Recall that the functional $\mathcal{F}$ given by (32) is convex, Gâteaux-differentiable on $W^{1,p}(\Omega)$ and its derivative is given by (33). Thus taking the infimum over all $\lambda > 0$ in the last inequality yields

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H} \leq \int_\Omega a(x, \nabla P\varphi)\nabla P\zeta\,\mathrm{d}x$$

and so by using again (39),

$$\int_{\partial\Omega} \chi\,\zeta\,\mathrm{d}\mathcal{H} \leq \int_\Omega a(x, \nabla P\varphi)\nabla Z\zeta\,\mathrm{d}x.$$

Since $\zeta \in D(\mathcal{E})$ was arbitrary, replacing ζ by $-\zeta$ in the latter inequality shows that $\chi \in L^2(\partial\Omega)$ and $P\varphi \in W^{1,p}(\Omega)$ satisfy equation (45) with $F = 0$. Hence, by claim (1) of Proposition 3.9, $\varphi \in D(\Lambda_2)$ with $\Lambda_2\varphi = \chi$. As $\chi \in \partial\mathcal{E}(\varphi)$ was arbitrary and $\Lambda_2\varphi$ is uniquely determined by (45), we have thereby shown that $\partial\mathcal{E}(\varphi) = \{\Lambda_2\varphi\}$ for every $\varphi \in D(\partial\mathcal{E})$, completing the proof of this lemma. $\square$

4. Elliptic problems associated with Λ

This section is devoted to establish well-posedness of the elliptic problem (2) – (3) for the Dirichlet-to-Neumann operator Λ associated with the second order quasi-linear operator A given by (19) and to prove Hölder-regularity of weak solutions of equation (2). Here, we prove Theorem 1.1 in a more general case (see Theorem 4.2 below).

We begin this section with the following definition.

Definition 4.1. Let $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$. We call a boundary function φ a *weak solution* of the elliptic equation (2) if $\varphi \in W^{1-1/p,p}(\partial\Omega)$ and satisfies

$$\int_\Omega a(x, \nabla P\varphi)\nabla Z\xi \,\mathrm{d}x = \langle \psi, \xi\rangle \tag{53}$$

for all $\xi \in W^{1-1/p,p}(\partial\Omega)$.

Now, we are in a position to formulate our forth main theorem of this article.

Theorem 4.2. *Suppose that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18). Then the following assertions hold.*

1. *For every $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$ there is a unique weak solution $\varphi \in W_m^{1-1/p,p}(\partial\Omega)$ of the elliptic problem (2)-(3). Moreover, the mapping $\psi \mapsto \varphi$ is continuous from $W_m^{-(1-1/p),p'}(\partial\Omega)$ to $W_m^{1-1/p,p}(\partial\Omega)$.*
2. *Let $q = \frac{d-1}{p-1-\varepsilon}$ for some $\varepsilon \in (0,1)$ if $p \le d$ and $q = 1$ if $p > d$. Further, let $\psi \in L^q(\partial\Omega)$ satisfying (3). Then there are $\alpha \in (0,1)$ and $c_\alpha \ge 0$ such that every weak solution $\varphi \in W^{1-1/p,p}(\partial\Omega)$ of equation (2) belongs to $C^{0,\alpha}(\partial\Omega)$ and satisfies inequality (4).*

4.1. Proof of Theorem 4.2

By Corollary 3.6, the operator $\Lambda : W_m^{1-1/p,p}(\partial\Omega) \to W_m^{-(1-1/p),p'}(\partial\Omega)$ is a continuous, strictly monotone, bounded and coercive. The Banach space $W_m^{1-1/p,p}(\partial\Omega)$ is reflexive and separable as a closed subspace of the reflexive and separable Banach space $W^{1-1/p,p}(\partial\Omega)$. Therefore, we can apply [29, Théorème 2.1 & §2.2] and obtain that for every $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$ there is a unique weak solution $\varphi \in W_m^{1-1/p,p}(\partial\Omega)$ of equation (2).

Now, let $(\psi_n) \subseteq W_m^{-(1-1/p),p'}(\partial\Omega)$ and $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$ such that ψ_n converges to ψ in $W_m^{-(1-1/p),p'}(\partial\Omega)$. Then, there are unique weak solutions φ_n and $\varphi \in W_m^{1-1/p,p}(\partial\Omega)$ of equation (2) with right-hand sides ψ_n and ψ, respectively. By coercivity inequality (41) and by Young's inequality, we obtain

$$\tfrac{C_2}{2}\, \|\varphi_n\|^p_{W^{1-1/p,p}(\partial\Omega)} \le C_3\, \|\psi_n\|_{W^{-(1-1/p),p'}(\partial\Omega)}, \tag{54}$$

where $C_2 > 0$ comes from (41) and $C_3 > 0$ is a constant independent of n. Since (ψ_n) is bounded in $W_m^{-(1-1/p),p'}(\partial\Omega)$, inequality (54) implies that (φ_n) is bounded in $W_m^{1-1/p,p}(\partial\Omega)$. Since $W_m^{1-1/p,p}(\partial\Omega)$ is reflexive there is a $\tilde\varphi \in W_m^{1-1/p,p}(\partial\Omega)$ and there is a subsequence (ψ_{k_n}) of (ψ_n) such that for $\varphi_{k_n} = \Lambda^{-1}\psi_{k_n}$, one has

$$\lim_{n\to\infty} \varphi_{k_n} = \tilde\varphi \qquad \text{weakly in } W^{1-1/p,p}(\partial\Omega). \tag{55}$$

Since the operator $Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ is bounded, the sequence $(Z\varphi_{k_n})$ is bounded in $W^{1,p}(\Omega)$ and so inequality (35) implies the boundedness of the sequence $(\nabla P\varphi_{k_n})$ in $L^p(\Omega)^d$. Moreover, (φ_{k_n}) is bounded in $L^p(\partial\Omega)$ and so by Maz'ya's inequality (22), we obtain $(P\varphi_{k_n})$ is bounded in $W^{1,p}(\Omega)$. By Lemma 2.3, there is a weak solution $u \in W^{1,p}(\Omega)$ of (29) on Ω and there is a subsequence of (ψ_{k_n}) denoted again by (ψ_{k_n}) such that for $\varphi_{k_n} = \Lambda^{-1}\psi_{k_n}$, one has

$$\lim_{n\to\infty} P\varphi_{k_n} = u \qquad \text{weakly in } W^{1,p}(\Omega). \tag{56}$$

Since the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is compact, by the convergence of $P\varphi_{k_n|\partial\Omega} = \varphi_{k_n}$ to $u_{|\partial\Omega}$ in $L^p(\partial\Omega)$ and by (55), we can conclude that $u = P\tilde{\varphi}$ and φ_{k_n} converges to $\tilde{\varphi}$ in $L^p(\partial\Omega)$. Thus and since every φ_{k_n} satisfies the compatibility condition (3), we find $\tilde{\varphi} \in W_m^{1-1/p,p}(\partial\Omega)$. It remains to show that $\tilde{\varphi} = \varphi$ and φ_{k_n} converges to φ in $W_m^{1-1/p,p}(\partial\Omega)$ with respect to the norm topology. To see that $\tilde{\varphi} = \varphi$, note by Lemma 2.3, $a(x, \nabla P\varphi_{k_n})$ converges to $a(x, \nabla P\tilde{\varphi})$ weakly in $L^{p'}(\Omega)^d$. Furthermore, every φ_{k_n} is the unique weak solution of (2) with right-hand side ψ_{k_n}. Thus

$$\langle \psi_{k_n}, \xi \rangle = \langle \Lambda\varphi_{k_n}, \xi \rangle = \int_\Omega a(x, \nabla P\varphi_{k_n})\nabla Z\xi \,\mathrm{d}x$$

and

$$\lim_{n\to\infty} \langle \psi_{k_n}, \xi \rangle = \int_\Omega a(x, \nabla P\tilde{\varphi})\nabla Z\xi \,\mathrm{d}x$$

for every $\xi \in W_m^{1-1/p,p}(\partial\Omega)$. On the other hand, $\langle \psi_{k_n}, \xi \rangle$ converges to $\langle \psi, \xi \rangle$ for every $\xi \in W_m^{1-1/p,p}(\partial\Omega)$. Therefore, $\tilde{\varphi}$ is a weak solution of (2) with right-hand side ψ and so by uniqueness, $\tilde{\varphi} = \varphi$. To see that φ_{k_n} converges to φ in $W^{1-1/p,p}(\partial\Omega)$, recall that by Lemma 2.3, $P\varphi_{k_n}$ converges to $P\tilde{\varphi}$ in $L^p(\Omega)$, and $\nabla P\varphi_{k_n}$ converges to $\nabla P\tilde{\varphi}$ a.e. on Ω. Thus it remains to show that $(|\nabla P\varphi_{k_n}|^p)$ is equi-integrable in $L^1(\Omega)$. Following the same idea as given in the proof of Lemma 2.5, it suffices to show that the non-negative function χ_{k_n} defined by (36) converges to 0 in $L^1(\Omega)$. Since φ_{k_n} and φ are the unique weak solutions of (2) with right-hand side ψ_{k_n} and ψ, respectively, we have that

$$\int_\Omega \chi_{k_n}(x), \mathrm{d}x = \langle \psi_{k_n}, \varphi_{k_n} - \varphi \rangle - \langle \psi_{k_n}, \varphi_{k_n} - \varphi \rangle.$$

Therefore and by the convergence of ψ_{k_n} to ψ in $W_m^{-(1-1/p),p'}(\partial\Omega)$ and by the weak limit (55) with $\tilde{\varphi} = \varphi$, we see that χ_{k_n} defined by (36) converges to 0 in $L^1(\Omega)$. Thereby we have shown that for every sequence $(\psi_n) \subseteq W_m^{-(1-1/p),p'}(\partial\Omega)$ converging to some $\psi \in W_m^{-(1-1/p),p'}(\partial\Omega)$, there is a subsequence (ψ_{k_n}) of (ψ_n) such that φ_{k_n} converges to φ in $W_m^{1-1/p,p}(\partial\Omega)$. This proves that the mapping $\psi \mapsto \varphi$ is continuous from $W_m^{-(1-1/p),p'}(\partial\Omega)$ to $W_m^{1-1/p,p}(\partial\Omega)$, completing the proof of claim (1) of Theorem 4.2.

4.2. Preliminaries for the proof of claim (2) of Theorem 4.2

For $1 \le q \le \infty$, let $L_m^q(\partial\Omega)$ denote the set of all $\psi \in L^q(\partial\Omega)$ satisfying (3) and let q' be the Hölder-conjugate of q given by $\frac{1}{q} + \frac{1}{q'} = 1$. The next result seems to be well-known. But for the sake of completeness, we supply the proof here.

Lemma 4.3. *Let $q = \frac{d-1}{p-1}$ if $p \le d$ and $q = 1$ if $p > d$. Then $W_m^{1-1/p,p}(\partial\Omega)$ is continuously embedded into $L_m^{q'}(\partial\Omega)$ by a continuous injection with a dense image. Moreover, $L_m^q(\partial\Omega)$ is continuously embedded into $W_m^{-(1-1/p),p'}(\partial\Omega)$.*

Proof. We prove the claim of this lemma only for $p < d$ since the case $p \geq d$ is similar. Observe that the Lebesgue space $L^{q'}_m(\partial\Omega)$ is a closed linear subspace of $L^{q'}(\partial\Omega)$ and can be identified with the quotient space $L^{q'}(\partial\Omega)/\mathbb{R}$. Here we identify the set of constant functions on $\partial\Omega$ with $\mathbb{R}$. Thus, the dual space $(L^{q'}_m(\partial\Omega))'$ can be identified with $L^q_m(\partial\Omega)$.

Now, we show that $W^{1-1/p,p}_m(\partial\Omega) \hookrightarrow L^{q'}_m(\partial\Omega)$ by a continuous injection for $q = (d-1)/(p-1)$. Note first that $q = (d-1)/(p-1)$ if and only if $q' = (d-1)/(d-p)$. Moreover, the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^{p^*}(\partial\Omega)$ with $p^* = (d-1)p/(d-p)$ (see [31, Théorème 4.2, p. 84]) and its right-inverse $Z : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$ are bounded. Using this and since $\partial\Omega$ has finite Hausdorff measure, we see

$$\|\varphi\|_{L^{q'}(\partial\Omega)} \leq C\|\mathcal{T}r Z\varphi\|_{L^{p^*}(\partial\Omega)} \leq C\|Z\varphi\|_{W^{1,p}(\Omega)} \leq C\|\varphi\|_{W^{1-1/p,p}(\partial\Omega)}$$

for every $\varphi \in W^{1-1/p,p}(\partial\Omega)$. This shows that $W^{1-1/p,p}(\partial\Omega)$ is continuously injected into $L^{q'}(\partial\Omega)$ and hence, in particular, there is a continuous injection $i : W^{1-1/p,p}_m(\partial\Omega) \hookrightarrow L^{q'}_m(\partial\Omega)$. If we can show that $W^{1-1/p,p}_m(\partial\Omega)$ lies dense in $L^{q'}_m(\partial\Omega)$, then the adjoint operator $i' : L^q_m(\partial\Omega) \to W^{-(1-1/p),p'}_m(\partial\Omega)$ is also a continuous injection. For this, we set $e_1 = \mathcal{H}(\partial\Omega)^{-1/2}\mathbb{1}_{\partial\Omega}$. Then $\omega_\varphi := \varphi - (\varphi, e_1)_{L^2(\partial\Omega)}\, e_1 \in W^{1-1/p,p}_m(\partial\Omega)$ for every $\varphi \in W^{1-1/p,p}(\partial\Omega)$. Now, let $g \in L^q_m(\partial\Omega)$ satisfy

$$(g, \varphi)_{L^2(\partial\Omega)} = 0 \qquad \text{for every } \varphi \in W^{1-1/p,p}_m(\partial\Omega). \tag{57}$$

Then, $(g, \omega_\varphi)_{L^2(\partial\Omega)} = 0$ and $(g, e_1)_{L^2(\partial\Omega)} = 0$. Hence,

$$(g, \varphi)_{L^2(\partial\Omega)} = (g, \omega_\varphi)_{L^2(\partial\Omega)} + (\varphi, e_1)_{L^2(\partial\Omega)}\, (g, e_1)_{L^2(\partial\Omega)} = 0$$

for every $\varphi \in W^{1-1/p,p}(\partial\Omega)$. Since the space $W^{1-1/p,p}(\partial\Omega)$ lies dense in $L^{q'}(\partial\Omega)$ (see Lemma 2.1), it follows that $g = 0$. As $g \in L^q_m(\partial\Omega)$ satisfying (57) was arbitrary, we have thereby proved that $W^{1-1/p,p}_m(\partial\Omega)$ lies dense in $L^{q'}_m(\partial\Omega)$, concluding the proof of this Lemma. □

Now, let $q = \frac{d-1}{p-1}$ if $p \leq d$ and $q = 1$ if $p > d$. In order to prove the Hölder regularity of solutions $\varphi \in W^{1-1/p,p}(\partial\Omega)$ of equation (2) with right-hand side $\psi \in L^q_m(\partial\Omega)$, it is crucial to know that $u := P\varphi$ solves the elliptic Neumann boundary value problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = 0 & \text{in } \Omega, \\ \quad a(x, \nabla u) \cdot \nu = \psi & \text{on } \partial\Omega \end{cases} \tag{58}$$

in a weak sense. We recall the definition of a weak solution of problem (58).

Definition 4.4. For $\psi \in L^q_m(\partial\Omega)$, we call a function $u \in W^{1,p}(\Omega)$ a *weak solution* of the elliptic Neumann boundary-value problem (58) if u satisfies

$$\int_\Omega a(x, \nabla u)\nabla v \,\mathrm{d}x = \int_{\partial\Omega} \psi\, v \,\mathrm{d}\mathcal{H} \tag{59}$$

for all $v \in W^{1,p}(\Omega)$.

We have the following characterisation of weak solutions of equation (2).

Lemma 4.5. *For $\psi \in L^q_m(\partial\Omega)$, the function $\varphi \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of equation (2) if and only if $P\varphi$ is a weak solution of (58).*

Proof. Let $\psi \in L^q_m(\partial\Omega)$ arbitrary but fixed and suppose $\varphi \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of equation (2). Then, by Definition 4.1, $P\varphi$ satisfies

$$\int_\Omega a(x, \nabla P\varphi)\nabla Z\xi \,\mathrm{d}x = \int_{\partial\Omega} \psi\,\xi\,\mathrm{d}\mathcal{H} \tag{60}$$

for every $\xi \in W^{1-1/p,p}(\partial\Omega)$. For every $v \in W^{1,p}(\Omega)$, the function $\xi := v_{|\partial\Omega}$ belongs to $W^{1-1/p,p}(\partial\Omega)$ and satisfies $v - Z(v_{|\partial\Omega}) \in W^{1,p}_0(\Omega)$. Thus and since $P\varphi$ is a weak solution of equation (29), equation (60) is equivalent to

$$\int_\Omega a(x, \nabla P\varphi)\nabla v \,\mathrm{d}x = \int_{\partial\Omega} \psi\, v_{|\partial\Omega}\,\mathrm{d}\mathcal{H}$$

for every $v \in W^{1,p}(\Omega)$. Therefore, $\varphi \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of equation (2) if and only if $P\varphi$ satisfies equation (59), meaning $P\varphi$ is a weak solution of Neumann problem (58). □

4.3. Proof of Theorem 4.2 (Continued)

Suppose that $q_\varepsilon := \frac{d-1}{p-1-\varepsilon}$ if $p \le d$ and $q_\varepsilon := 1$ if $p > d$ for $\varepsilon \in [0,1)$. Since $\partial\Omega$ has finite measure, since $q_0 \le q_\varepsilon$, we see by Lemma 4.3 that

$$L^{q_\varepsilon}_m(\partial\Omega) \hookrightarrow L^{q_0}_m(\partial\Omega) \hookrightarrow W^{-(1-1/p),p'}_m(\partial\Omega)$$

respectively by continuous injections. Thus, claim (1) of Theorem 4.2 ensures that for every $\psi \in L^{q_\varepsilon}_m(\partial\Omega)$, there is a unique weak solution $\varphi \in W^{1-1/p,p}_m(\partial\Omega)$ of the elliptic problem (2)-(3).

Now, let $\varepsilon \in (0,1)$ and suppose that $\varphi \in W^{1-1/p,p}(\partial\Omega)$ is a weak solution of (2) for some right-hand side $\psi \in L^{q_\varepsilon}(\partial\Omega)$. By Lemma 4.5, $P\varphi$ is a weak solution of Neumann problem (58). Thus, [32, Theorem 3.7] implies that $P\varphi \in C^{0,\alpha}(\overline{\Omega})$ and satisfies

$$\|P\varphi\|_{C^{0,\alpha}(\overline{\Omega})} \le c_\alpha \left(\|\psi\|^{\frac{1}{p-1}}_{L^{q_\varepsilon}(\partial\Omega)} + \|P\varphi\|_{L^p(\Omega)}\right) + c_\alpha$$

for some $\alpha \in (0,1)$ and $C_\alpha \ge 0$ independent of ψ and φ. Therefore, $\varphi \in C^{0,\alpha}(\partial\Omega)$ and since $\|\varphi\|_{C^{0,\alpha}(\partial\Omega)} \le \|P\varphi\|_{C^{0,\alpha}(\overline{\Omega})}$, the last estimate shows that φ satisfies the desired inequality (4). Thus, claim (2) of Theorem 4.2 holds, completing the proof of this theorem.

5. Parabolic problems associated with $\boldsymbol{\Lambda}$

In this section, we investigate the the well-posedness of initial value problem (5)-(6) for the the Dirichlet-to-Neumann operator Λ associated with the second order quasi-linear operator A defined in (19) under the general assumptions that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)–(18). It is one of the main task of this section to outline the $L^q - C^{0,\alpha}$-regularisation effect of the mild solutions of (5)-(6) depending on the initial value φ_0. Note that Theorem 1.2 follows from the next theorem as a special case.

Theorem 5.1. *Suppose that $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18). Then the following statements hold true:*

1. *The negative Dirichlet-to-Neumann operator $-\Lambda_2$ in $L^2(\partial\Omega)$ generates a strongly continuous order-preserving semigroup $\{e^{-t\,\Lambda_2}\}$ of contractions on $L^2(\partial\Omega)$.*

2. *The semigroup $\{e^{-t\,\Lambda_2}\}$ can be extrapolated to a strongly continuous, order preserving semigroup of contractions on $L^q(\partial\Omega)$ for $1\le q<\infty$, on the closure $\overline{D(\Lambda_\infty)}$ in $L^\infty(\partial\Omega)$ for $q=\infty$, and on the closure $\overline{D(\Lambda_c)}$ in $C(\partial\Omega)$, where $\overline{D(\Lambda_c)}=C(\partial\Omega)$ if Ω has a $C^{1,\beta}$ boundary for some $\beta\in(0,1)$. In particular, the extrapolated semigroup on $L^q(\partial\Omega)$ of $\{e^{-t\,\Lambda_2}\}$ coincides with the semigroup $\{e^{-t\,\overline{\Lambda_q}}\}$ generated by $-\overline{\Lambda_q}$ on $L^q(\partial\Omega)$ if $1\le q<2$ and with the semigroup $\{e^{-t\,\Lambda_q}\}$ generated by $-\Lambda_q$ on $L^q(\partial\Omega)$ (resp., on $\overline{D(\Lambda_\infty)}$) if $2\le q\le\infty$.*
3. *For every $\varphi\in L^q(\partial\Omega)$ $(1\le q\le\infty)$, the mild solution $e^{-tB}\varphi$ satisfies the conservation of mass equality (7), where $B=\overline{\Lambda_q}$ if $1\le q<2$ and $B=\Lambda_q$ if $2\le q\le\infty$.*
4. *For $(2\vee\frac{d-1}{p-1-\varepsilon})\le q<\infty$ with some $\varepsilon\in(0,1)$ if $p\le d$ and for $2\le q<\infty$ if $p>d$, there are $\alpha\in(0,1)$ and $c_\alpha>0$ such that*

$$\|e^{-t\Lambda_q}\varphi\|_{C^{0,\alpha}(\partial\Omega)}\le c_\alpha\left[\|\tfrac{d}{dt}_+e^{-t\Lambda_q}\varphi\|_{L^q(\partial\Omega)}^{\frac{1}{p-1}}+\|P(e^{-t\Lambda_q}\varphi)\|_{L^p(\Omega)}\right]+c_\alpha. \tag{61}$$

for every $t\ge0$ and every $\varphi\in D(\Lambda_q)$. For $\varphi\in D(\Lambda_\infty)$, inequality (61) holds for any $(2\vee\frac{d-1}{p-1-\varepsilon})\le q<\infty$ with some $\varepsilon\in(0,1)$ if $p\le d$ and for any $2\le q<\infty$ if $p>d$.
5. *If a satisfies gradient condition (20) for some Carathéodory function $\mathcal{A}:\Omega\times\mathbb{R}^d\to\mathbb{R}$, then for every $\varphi\in L^q(\partial\Omega)$ $(2\le q<\infty)$ (resp., $\varphi\in\overline{D(\Lambda_c)}$),*
   (a) *the mild solution $t\mapsto e^{-t\Lambda_q}\varphi$ of (5)-(6) in $L^q(\partial\Omega)$ coincides with the strong solution of (5)-(6) in $L^2(\partial\Omega)$ and has regularity*

$$e^{-\cdot\Lambda_q}\varphi\in C((0,\infty);W^{1-1/p,p}(\partial\Omega))\cap W^{1,\infty}([\delta,\infty);L^2(\partial\Omega))\cap C([0,\infty);L^q(\partial\Omega))$$

   for every $\delta>0$ (resp., $e^{-\cdot\Lambda_q}\varphi\in C((0,\infty)\times\partial\Omega)$), $e^{-\cdot\Lambda_q}\varphi$ is right-hand side differentiable in $L^2(\partial\Omega)$ at every $t>0$ and

$$\int_{\partial\Omega}\tfrac{d}{dt}_+e^{-t\Lambda_q}\varphi\,\xi\,\mathrm{d}\mathcal{H}+\int_\Omega a(x,\nabla P(e^{-t\Lambda_q}\varphi))\nabla Z\xi\,\mathrm{d}x=0$$

   for every $\xi\in W^{1-1/p,p}(\partial\Omega)\cap L^2(\partial\Omega)$,
   (b) *the function*

$$t\mapsto\mathcal{E}(e^{-t\Lambda_q}\varphi):=\int_\Omega\mathcal{A}(x,\nabla P(e^{-t\Lambda_q}\varphi))\,\mathrm{d}x \tag{62}$$

   is convex, decreasing, Lipschitz continuous on $[\delta,\infty)$ for every $\delta>0$, and

$$\tfrac{d}{dt}\mathcal{E}(e^{-t\Lambda_q}\varphi)=-\|\tfrac{d}{dt}e^{-t\Lambda_q}\varphi\|^2_{L^2(\partial\Omega)}$$

   for a.e. $t>0$.
6. *Suppose that a satisfies either homogeneity condition (31) if $p\neq2$ or is linear if $p=2$. Then*
   (a) *for $B=\overline{\Lambda_q}$ if $1<q<2$ and $B=\Lambda_q$ if $2\le q<\infty$, we have that for every $\varphi\in L^q(\partial\Omega)$, $e^{-tB}\varphi\in D(B)$ for every $t>0$, $e^{-\cdot B}\varphi\in W^{1,\infty}([\delta,\infty);L^q(\partial\Omega))$ for every $\delta>0$, $e^{-tB}\varphi$ is right-hand side differentiable in $L^q(\partial\Omega)$ at every $t>0$ and satisfies equation (10) in $L^q(\partial\Omega)$ for every $t>0$. In particular, there is a $C>0$ such that $e^{-tB}\varphi$ satisfies inequality (11) for every $t>0$,*
   (b) *for $(2\vee\frac{d-1}{p-1-\varepsilon})\le q\le\infty$ with some $\varepsilon\in(0,1)$ if $p\le d$ and for $2\le q\le\infty$ if $p>d$, there are $\alpha\in(0,1)$ and $c_\alpha>0$ such that inequality (12) holds for every $t>0$, $\varphi\in L^q(\partial\Omega)$, and in particular, $e^{-\cdot\Lambda_q}\varphi\in C((0,\infty)\times\partial\Omega)$.*

5.1. Proof of Theorem 5.1

By Proposition 3.9 and by the Crandall-Liggett theorem [16], it follows that claim (1) of this theorem holds. Furthermore, by Proposition 3.12 and since $\Lambda 0 = 0$, the theory developed in [9] shows that claim (2) holds. To see that claim (3) holds, take $\varphi \in L^q(\partial\Omega)$ if $1 \le q < \infty$ and $\varphi \in \overline{D(\Lambda_c)}$ if $q = \infty$. For $t > 0$ and any integer $N \ge 1$, let $t_i = i\,\frac{t}{N}$ for every $i = 0, \dots, N$. We set $\varphi_0 = \varphi$ and for every $i = 1, \dots, N$, let $\varphi_i \in D(B)$ be the unique solution of equation

$$\varphi_i + \tfrac{t}{N} B\varphi_i = \varphi_{i-1}, \tag{63}$$

where $B = \overline{\Lambda_q}$ if $1 \le q < 2$ and $B = \Lambda_q$ if $2 \le q \le \infty$. By the Crandall-Liggett theorem [16], the step function

$$\Phi_N(t) := \varphi_0\, \mathbb{1}_{\{t_0=0\}}(t) + \sum_{i=1}^{N} \varphi_i\, \mathbb{1}_{(t_{i-1},t_i]}(t)$$

converges to $e^{-tB}\varphi$ in $L^q(\partial\Omega)$ as $N \to \infty$ for every $t > 0$. By (63) and since $Z(\mathbb{1}_{\partial\Omega}) = \mathbb{1}_{\partial\Omega}$,

$$\int_{\partial\Omega} (\varphi_{i-1} - \varphi_i)\, \mathrm{d}\mathcal{H} = 0$$

for all $i = 1, \dots, N$. Therefore sending $N \to \infty$ in equation

$$\int_{\partial\Omega} \Phi_N(t)\, \mathrm{d}\mathcal{H} = \int_{\partial\Omega} \varphi\, \mathrm{d}\mathcal{H}$$

shows that the conservation of mass equality (7) holds. Next, let $\varphi \in D(\Lambda_q)$ for $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q < \infty$ with some $\varepsilon \in (0,1)$ if $p \le d$ and for $2 \le q < \infty$ if $p > d$. Then by the regularity result of mild solution [8, Theorem 4.6], the mild solution $e^{-\cdot\Lambda_q}\varphi$ of (5)-(6) in $L^q(\partial\Omega)$ is a strong solution of (5)-(6) in $L^q(\partial\Omega)$, at every $t \ge 0$ the function $e^{-t\,\Lambda_q}\varphi$ is differentiable form the right, $\frac{d}{dt}_+ e^{-t\Lambda_q}\varphi$ is right continuous, and $\frac{d}{dt}_+ e^{-t\Lambda_q}\varphi + \Lambda_q e^{-t\Lambda_q}\varphi = 0$. In particular, for every $t \ge 0$, $e^{-t\Lambda_q}\varphi$ is a weak solution of the elliptic equation (2) with right-hand side $\psi = -\frac{d}{dt}_+ e^{-t\Lambda_q}\varphi$. Since $\frac{d}{dt}_+ e^{-t\Lambda_q}\varphi_0 \in L^q_m(\partial\Omega)$, we can apply Theorem 4.2, proving that inequality (61) holds. By using this and the continuous embedding of $L^\infty(\partial\Omega)$ in $L^q(\partial\Omega)$, we see that claim (4) holds.

By Proposition 3.13, we can apply the classical theory of non-linear semigroups in Hilbert spaces (more precisely, see [14, Théorème 3.2]) and obtain that the first part of claim (5) holds. It remains to show that for every $\varphi \in L^2(\partial\Omega)$,

$$e^{-\cdot\Lambda_2}\varphi \in C((0,\infty); W^{1-1/p,p}(\partial\Omega)). \tag{64}$$

By the continuity of the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega)$, to see that (64) holds, we only need to show that $P(e^{-\cdot\Lambda_2}\varphi) \in C((0,\infty); W^{1,p}(\Omega))$. To this end, suppose (t_n) is a sequence in $(0,\infty)$ converging to some $t_0 \in (0,\infty)$. Then, $e^{-t_n\Lambda_2}\varphi$ converges to $e^{-t_0\Lambda_2}\varphi$ in $L^2(\partial\Omega)$. Moreover, the function $\mathcal{E}(e^{-\cdot\Lambda_2}\varphi)$ given by (62) is continuous on $(0,\infty)$. Thus by inequality (23), the sequence $(P(e^{-t_n\Lambda_2}\varphi))$ is bounded in $W^{1,p}(\Omega)$. By Lemma 2.3, there is a weak solution $u \in W^{1,p}(\Omega)$ of equation (29) on Ω and there is a subsequence (t_{k_n}) of (t_n) such that $P(e^{-t_{k_n}\Lambda_2}\varphi)$ converges to u weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$, $\nabla P(e^{-t_{k_n}\Lambda_2}\varphi)$ converges to ∇u a.e. on Ω and $a(x, \nabla P(e^{-t_{k_n}\Lambda_2}\varphi))$ converges to $a(x,\nabla u)$ weakly in $L^{p'}(\Omega)^d$ and a.e. on Ω. Thus and since the trace operator $\mathcal{T}r : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is compact, $\mathcal{T}r(P(e^{-t_{k_n}\Lambda_2}\varphi)) = e^{-t_{k_n}\Lambda_2}\varphi$

converges to $\mathcal{T}\!r\, u$ in $L^p(\partial\Omega)$. On the other hand, $e^{-t_{k_n}\Lambda_2}\varphi$ converges to $e^{-t_0\Lambda_2}\varphi$ in $L^2(\partial\Omega)$. Thus $\mathcal{T}\!r\, u = e^{-t_0\Lambda_2}\varphi$ and since $u \in W^{1,p}(\Omega)$ is a weak solution of equation (29) on Ω, we have $u = P(e^{-t_0\Lambda_2}\varphi)$. It remains to show that $\nabla P(e^{-t_{k_n}\Lambda_2}\varphi)$ converges to $\nabla P(e^{-t_0\Lambda_2}\varphi)$ strongly in $L^p(\Omega)^d$. By Lemma 2.3, $\nabla P(e^{-t_{k_n}\Lambda_2}\varphi)$ converges to $\nabla P(e^{-t_0\Lambda_2}\varphi)$ a.e. on Ω. Hence, it remains to show that $(|\nabla P(e^{-t_{k_n}\Lambda_2}\varphi)|^p)$ is equi-integrable in $L^1(\Omega)$. To see this, consider the non-negative function χ_{k_n} defined by (36) with $\varphi_{k_n} := e^{-t_{k_n}\Lambda_2}\varphi$ and $\varphi := e^{-t_0\Lambda_2}\varphi$. Recall that for every $t > 0$,

$$\Lambda_2 e^{-t\Lambda_2}\varphi = -\tfrac{d}{dt_+} e^{-t\Lambda_2}\varphi.$$

Thus and by (39), we see that

$$\begin{aligned}\int_\Omega \chi_{k_n}\,\mathrm{d}x &= (\tfrac{d}{dt_+}e^{-t_0\Lambda_2}\varphi - \tfrac{d}{dt_+}e^{-t_{k_n}\Lambda_2}\varphi, e^{-t_{k_n}\Lambda_2}\varphi - e^{-t_0\Lambda_2}\varphi)_{L^2(\partial\Omega)}\\ &\le \|\tfrac{d}{dt_+}e^{-t_0\Lambda_2}\varphi - \tfrac{d}{dt_+}e^{-t_{k_n}\Lambda_2}\varphi\|_{L^2(\partial\Omega)}\, \|e^{-t_{k_n}\Lambda_2}\varphi - e^{-t_0\Lambda_2}\varphi\|_{L^2(\partial\Omega)}.\end{aligned}$$

Since $\frac{d}{dt_+}e^{-\cdot\Lambda_2}\varphi \in L^\infty([\delta,\infty); L^2(\partial\Omega))$ for every $\delta > 0$ and by the convergence of $e^{-t_{k_n}\Lambda_2}\varphi$ to $e^{-t_0\Lambda_2}\varphi$ in $L^2(\partial\Omega)$, we have that χ_{k_n} converges to 0 in $L^1(\Omega)$. Now, we employ the same arguments as in the proof of Lemma 2.5 to conclude that $(|\nabla P\varphi_{k_n}|^p)$ is equi-integrable in $L^1(\Omega)$. Since the convergent sequence (t_n) in $(0,\infty)$ was arbitrary, we have thereby shown that $P\varphi \in C((0,\infty); W^{1,p}(\Omega))$.

In order to see that statement (a) of claim (6) holds, we first consider the case $p = 2$ and suppose that the Carathéodory function $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is linear in the second variable. Then it is well-known (cf. [6]) that Λ_2 is a positive and self-adjoint linear operator on $L^2(\partial\Omega)$. Moreover, $-\Lambda_2$ generates a strongly continuous semigroup $\{e^{-t\Lambda_2}\}$ of self-adjoint contractions on $L^2(\partial\Omega)$, which is Markovian and has a holomorphic extension on $L^2(\partial\Omega;\mathbb{C})$. Hence, by the classical theory of extrapolation of linear C_0-semigroups (cf. [7, §7.1, §7.2]), the semigroup $\{e^{-t\Lambda_2}\}$ extends to a positive holomorphic C_0-semigroup of bounded linear operators on $L^q(\partial\Omega)$ for $1 < q < \infty$. Thus by claim (2) of this theorem and for $B = \overline{\Lambda_q}$ if $1 < q < 2$ and $B = \Lambda_q$ if $2 \le q < \infty$, the semigroup $\{e^{-tB}\}$ on $L^q(\partial\Omega)$ generated by $-B$ is holomorphic. Hence for every $\varphi \in L^q(\partial\Omega)$, the function $e^{-\cdot B}\varphi$ is a classical solution of (10) in $L^q(\partial\Omega)$ and satisfies inequality (11) (see also [33]).

Now, suppose that $1 < p < \infty$, $p \neq 2$, and the function a satisfies homogeneity condition (31). By Lemma 2.5, the Dirichlet-to-Neumann operator $\Lambda : W^{1-1/p,p}(\partial\Omega) \to W^{-(1-1/p),p'}(\partial\Omega)$ is positive homogeneous of degree $p-1 > 0$, that is,

$$\Lambda(r\varphi) = r^{p-1}\Lambda(\varphi)$$

for all $r \ge 0$ and $\varphi \in W^{1-1/p,p}(\partial\Omega)$. Obviously, this property is also satisfied by Λ_q (respectively, for $\overline{\Lambda_q}$) for every $1 \le q \le \infty$. Since for $1 < q < \infty$, the Lebesgue space $L^q(\partial\Omega)$ is uniformly convex, [10, Theorem 4] (see also [9, Theorem 4.4]) and [8, Theorem 1.16 & Theorem 4.6], imply that statement (a) of claim (6) also holds.

Finally, suppose that $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q < \infty$ with some $\varepsilon \in (0,1)$ if $p \le d$ and for $2 \le q < \infty$ if $p > d$. Then by statement (a) of claim (6), for every $\varphi \in L^q(\partial\Omega)$, the function $e^{-\cdot\Lambda_q}\varphi$ satisfies equation (10) in $L^q(\partial\Omega)$ with $B = \Lambda_q$ for every $t > 0$. Therefore, by claim (4), $e^{-t\Lambda_q}\varphi$ satisfies inequality (61) for every $t > 0$ and some $\alpha \in (0,1)$ and constant $c_\alpha > 0$. Combining inequality (61) with inequality (11) yields

$$\|e^{-t\Lambda_q}\varphi\|_{C^{0,\alpha}(\partial\Omega)} \le c_\alpha \left[\left(\tfrac{\|\varphi\|_{L^q(\partial\Omega)}}{t}\right)^{\frac{1}{p-1}} + \|P(e^{-t\Lambda_q}\varphi)\|_{L^p(\Omega)}\right] + c_\alpha. \tag{65}$$

Since $L^q(\partial\Omega)$ is continuously embedded into $L^2(\partial\Omega)$, $e^{-t\Lambda_q}\varphi = e^{-t\Lambda_2}\varphi$ is the strong solution of (5)-(6) in $L^2(\partial\Omega)$. Thus and by applying inequality (23), Cauchy-Schwarz's inequality and inequality (11) to the term $\|P(e^{-t\Lambda_q}\varphi)\|_{L^p(\Omega)}$ gives

$$\begin{aligned}
\|P(e^{-t\Lambda_q}\varphi)\|_{L^p(\Omega)} &\le C\left((\Lambda_2 e^{-t\Lambda_q}\varphi, e^{-t\Lambda_q}\varphi)^{1/p}_{L^2(\partial\Omega)} + \|e^{-t\Lambda_q}\varphi\|_{L^q(\partial\Omega)}\right)\\
&\le C\left(\|\Lambda_2 e^{-t\Lambda_q}\varphi\|^{1/p}_{L^2(\partial\Omega)}\,\|e^{-t\Lambda_q}\varphi\|^{1/p}_{L^2(\partial\Omega)} + \|e^{-t\Lambda_q}\varphi\|_{L^q(\partial\Omega)}\right)\\
&\le C\left(\frac{\|e^{-t\Lambda_q}\varphi\|^{2/p}_{L^2(\partial\Omega)}}{t^{1/p}} + \|e^{-t\Lambda_q}\varphi\|_{L^q(\partial\Omega)}\right)\\
&\le C\left(\frac{\|\varphi\|^{2/p}_{L^q(\partial\Omega)}}{t^{1/p}} + \|\varphi\|_{L^q(\partial\Omega)}\right)
\end{aligned}$$

where the constant may vary from line to line. Inserting this estimate into inequality (65) shows that inequality (12) holds for every $t > 0$.

By inequality (12), the set $\{e^{-t\Lambda_q}\varphi \,|\, t \ge \delta\}$ is bounded in $C^{0,\alpha}(\partial\Omega)$ for every $\delta > 0$. Thus the theorem of Arzelà-Ascoli implies that $e^{-t\Lambda_q}\varphi \in C((0,\infty)\times\partial\Omega)$. This completes the proof of this theorem.

6. Large time stability of the semigroup and finite time of extinction

In this section, we establish large time stability of the semigroup $\{e^{-t\Lambda_2}\}$ and the phenomenon of finite time of extinction by assuming merely that $a : \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18). Note that the statement of Theorem 1.3 is included in following main result of this section.

Theorem 6.1. *Suppose that $a : \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying (16)-(18). Then the following statements hold true:*

1. *(**Stability**) For every $\varphi \in L^q(\partial\Omega)$, the limit (13) holds in $L^q(\partial\Omega)$, where $B = \overline{\Lambda_q}$ if $1 \le q < 2$ and $B = \Lambda_q$ if $2 \le q < \infty$.*
2. *(**Stability in $C(\partial\Omega)$**) If a satisfies either homogeneity condition (31) if $p \neq 2$ or is linear if $p = 2$, then for $(2 \vee \frac{d-1}{p-1-\varepsilon}) \le q \le \infty$ for some $\varepsilon \in (0,1)$ if $p \le d$ and $2 \le q \le \infty$ if $p > d$, we have that limit (13) holds, in particular, in $C(\partial\Omega)$ for all $\varphi \in L^q(\partial\Omega)$.*
3. *(**Decay estimates**) There is a constant $C > 0$ such that for every $\varphi \in L^2(\partial\Omega)$ there is $t_0 \ge 0$ such that the estimates (14) and (15) hold for every $t \ge t_0$.*
4. *(**Finite time of extinction**) If $(1+\varepsilon) \vee \frac{2d}{d+2} \le p < 2$ for some $\varepsilon > 0$, then for every $\varphi \in L^2(\partial\Omega)$, the function $e^{-\cdot\Lambda_2}\varphi$ extinct in finite time*

$$t_{ext} \le \frac{\|\varphi - \overline{\varphi}\|^{2-p}_{L^2(\partial\Omega)}}{(1-\frac{p}{2})\,\eta\, C_S^p},$$

where the constant $C_S > 0$ occurs in the Sobolev-type inequality (71). More precisely, we have for every $\varphi \in L^2(\partial\Omega)$ that

$$\|e^{-\cdot\Lambda_2}\varphi - \overline{\varphi}\|_{L^2(\partial\Omega)} \le \left[(1-\tfrac{p}{2})\,\eta\, C_S^p\right]^{\frac{1}{2-p}} \left[\frac{\|\varphi-\overline{\varphi}\|^{2-p}_{L^2(\partial\Omega)}}{(1-\frac{p}{2})\,\eta\, C_S^p} - t\right]_+^{\frac{1}{2-p}} \tag{66}$$

for every $t \ge 0$.

Proof of Theorem 6.1. First, we begin to show that the decay estimates (14) and (15) hold. Since $D(\Lambda_2)$ lies dense in $L^2(\partial\Omega)$, a standard density argument shows that it sufficient to show that the decay estimates (14) and (15) hold for any initial values $\varphi \in D(\Lambda_2)$. Note that for every $c \in \mathbb{R}$, one has that $e^{-\cdot\Lambda_2}c = c$ for every $t \geq 1$. Hence the decay estimates (14) and (15) hold for every constant φ. Now, let $\varphi \in D(\Lambda_2)$ be not constant. Then $\varphi \neq \overline{\varphi}$. By claim (1) of Theorem 5.1 and [8, Theorem 4.6], the mild solution $e^{-\cdot\Lambda_2}\varphi$ of (5)-(6) in $L^2(\partial\Omega)$ is a strong solution of (5)-(6) and satisfies equation (10) for $B = \Lambda_2$ in $L^2(\partial\Omega)$ for every $t > 0$. By claim (3) of Theorem 5.1, one has

$$\overline{e^{-t\Lambda_2}\varphi} = \overline{\varphi}$$

for all $t \geq 0$. Thus multiplying equation (10) by $e^{-t\Lambda_2}\varphi - \overline{\varphi}$ with respect to the L^2-inner product and by using coercivity condition (16) yields

$$\tfrac{d}{dt}\|e^{-t\Lambda_2}\varphi - \overline{\varphi}\|^2_{L^2(\partial\Omega)} \leq -\eta\, 2 \int_\Omega |\nabla P(e^{-t\Lambda_2}\varphi)|^p \,\mathrm{d}x \tag{67}$$

and so by Poincaré's inequality (44),

$$\tfrac{d}{dt}\|e^{-t\Lambda_2}\varphi - \overline{\varphi}\|^2_{L^2(\partial\Omega)} \leq -C\, \|e^{-t\Lambda_2}\varphi - \overline{\varphi}\|^p_{L^p(\partial\Omega)} \tag{68}$$

for a.e. $t > 0$ and some constant $C > 0$ containing η and the constant occurring in inequality (44). Now, consider first the case $p \geq 2$. Then by applying the continuous embedding of $L^p(\partial\Omega)$ into $L^2(\partial\Omega)$ to the right-hand side of inequality (68), shows that the non-negative function

$$y(t) := \|e^{-t\Lambda_2}\varphi - \varphi\|^2_{L^2(\partial\Omega)} \tag{69}$$

for every $t \geq 0$ satisfies the differential inequality

$$\tfrac{dy}{dt}(t) \leq -\, C\, y^{p/2}(t) \tag{70}$$

for a.e. $t > 0$, where the constant $C > 0$ may differ from the one given in (68). In addition, since the semigroup $\{e^{-t\Lambda_2}\}$ is contractive on $L^2(\partial\Omega)$ and since $e^{-t\Lambda_2}\overline{\varphi} = \overline{\varphi}$ for all $t \geq 0$, the function y is non-increasing along $[0, \infty)$. In particular, if $y(t_0) = 0$ for some $t_0 \geq 0$, then $y(t) = 0$ for all $t \geq t_0$ and so in this case estimates (14) is true for all $t \geq t_0$. Thus, it remains to consider the case $y(t) > 0$ for all $t \geq 0$. To do so, we divide inequality (70) by $y^{p/2}(t)$ and subsequently integrating over $(0, t)$ for given $t > 0$. Then and since $p \geq 2$, we see that the decay estimates (14) hold for all $t \geq 0$.

Now, consider the case $1 < p < 2$. By the continuous embedding from $L^2(\partial\Omega)$ into $L^p(\partial\Omega)$ and by claim (2) of Theorem 5.1, we have that $e^{-t\Lambda_2}\varphi = e^{-t\overline{\Lambda_p}}\varphi$ for every $\varphi \in D(\Lambda_2)$. Multiplying equation (10) by $e^{-t\Lambda_2}\varphi - \overline{\varphi}$ and then integrating over $(0, t)$ for given $t > 0$ yields

$$\begin{aligned}\int_0^t (\Lambda_2 e^{-s\Lambda_2}\varphi, e^{-s\Lambda_2}\varphi)_{L^2(\partial\Omega)}\,\mathrm{d}s &\leq \tfrac{1}{2}\|e^{-t\Lambda_2}\varphi - \overline{\varphi}\|^2_{L^2(\partial\Omega)} + \int_0^t (\Lambda_2 e^{-s\Lambda_2}\varphi, e^{-s\Lambda_2}\varphi)_{L^2(\partial\Omega)}\,\mathrm{d}s\\ &= \tfrac{1}{2}\|\varphi - \overline{\varphi}\|^2_{L^2(\partial\Omega)}.\end{aligned}$$

Applying to this inequality, coercivity condition (16) and Poincaré's inequality (44), we see that

$$\tfrac{\eta}{C^p}\int_0^t \|e^{-s\overline{\Lambda_p}}\varphi - \overline{\varphi}\|^p_{L^p(\partial\Omega)}\,\mathrm{d}s \leq \eta \int_0^t \|\nabla P(e^{-s\overline{\Lambda_p}}\varphi)\|^p_{L^p(\Omega)^d}\,\mathrm{d}s$$

$$\le \int_0^t (\Lambda_2 e^{-s\overline{\Lambda_p}}\varphi, e^{-s\overline{\Lambda_p}}\varphi)_{L^2(\partial\Omega)}\,\mathrm{d}s$$
$$\le \tfrac{1}{2}\|\varphi-\overline{\varphi}\|^2_{L^2(\partial\Omega)}.$$

Since $\{e^{-t\overline{\Lambda_p}}\}$ is a semigroup of contractions on $L^p(\partial\Omega)$ and since $e^{-t\overline{\Lambda_p}}\overline{\varphi}=\overline{\varphi}$ for all $t\ge 0$, we see that the function $t\mapsto \|e^{-t\overline{\Lambda_p}}\varphi-\overline{\varphi}\|_{L^p(\partial\Omega)}$ is non-increasing along $[0,\infty)$. Applying this to our previous estimate, we obtain

$$t\,\frac{\eta}{C^p}\,\|e^{-t\overline{\Lambda_p}}\varphi-\overline{\varphi}\|^p_{L^p(\partial\Omega)}\le \tfrac{1}{2}\|\varphi-\overline{\varphi}\|^2_{L^2(\partial\Omega)},$$

showing that decay estimate (15) holds, completing the proof of claim (3) of this theorem.

Next, we show that claim (1) of this theorem holds. To see this, we need to treat the cases $p\ge 2$ and $p<2$ separately. Here, we only outline the case $p\ge 2$ since the case $1<p<2$ is shown similarly. Hence, let $p\ge 2$. If $1\le q<2$, then the continuous embedding from $L^2(\partial\Omega)$ into $L^q(\partial\Omega)$ and claim (2) of Theorem 5.1 imply that one has $e^{-t\Lambda_2}\varphi=e^{-t\overline{\Lambda_q}}\varphi$ and

$$\|e^{-t\overline{\Lambda_q}}\varphi-\overline{\varphi}\|_{L^q(\partial\Omega)}\le \mathcal{H}(\partial\Omega)^{\frac{1}{q}-\frac{1}{2}}\,\|e^{-t\Lambda_2}\varphi-\overline{\varphi}\|_{L^2(\partial\Omega)}$$

for every $t\ge 0$ and every $\varphi\in L^2(\partial\Omega)$. Therefore and since by claim (3) of this theorem, limit (13) holds in $L^2(\partial\Omega)$ for $B=\Lambda_2$ for every $\varphi\in L^2(\partial\Omega)$, it follows that limit (13) holds in $L^q(\partial\Omega)$ for $B=\overline{\Lambda_q}$ for every $\varphi\in L^2(\partial\Omega)$. Now, using that $L^2(\partial\Omega)$ lies dense in $L^q(\partial\Omega)$ and the fact that the projection $\varphi\mapsto\overline{\varphi}\,\mathbb{1}_{\partial\Omega}$ is contractive on $L^q(\partial\Omega)$, we obtain that limit (13) holds in $L^q(\partial\Omega)$ for $B=\overline{\Lambda_q}$ for every $\varphi\in L^q(\partial\Omega)$. If $q\ge 2$, then by the continuous injection of $L^\infty(\partial\Omega)$ into $L^2(\partial\Omega)$, by using that $e^{-t\Lambda_2}\varphi=e^{-t\Lambda_q}\varphi$ for every $\varphi\in L^\infty(\partial\Omega)$, Hölder's inequality, and by the L^∞-contractivity of the semigroup $\{e^{-t\Lambda_2}\}$ on $L^2(\Omega)$, we see that

$$\begin{aligned}\|e^{-t\Lambda_q}\varphi-\overline{\varphi}\|_{L^q(\partial\Omega)}&\le \|e^{-t\Lambda_2}\varphi-\overline{\varphi}\|^{\frac{q-2}{q}}_{L^\infty(\partial\Omega)}\,\|e^{-t\Lambda_2}\varphi-\overline{\varphi}\|^{\frac{2}{q}}_{L^2(\partial\Omega)}\\&\le 2^{\frac{q-2}{q}}\,\|\varphi\|^{\frac{q-2}{q}}_{L^\infty(\partial\Omega)}\,\|e^{-t\Lambda_2}\varphi-\overline{\varphi}\|^{\frac{2}{q}}_{L^2(\partial\Omega)}\end{aligned}$$

for every $t\ge 0$ and every $\varphi\in L^\infty(\partial\Omega)$. In the last inequality we used that the map $\varphi\mapsto\overline{\varphi}\mathbb{1}_{\partial\Omega}$ is contractive on $L^\infty(\partial\Omega)$. By this estimate, and since $L^\infty(\partial\Omega)$ lies dense in $L^q(\partial\Omega)$, we can conclude that limit (13) holds in $L^q(\partial\Omega)$ for $B=\Lambda_q$ and $q\ge 2$. This shows that claim (1) of this theorem holds.

In order to show that claim (2) of this theorem holds, note that under the hypotheses of this statement, for every $\varphi\in L^q(\partial\Omega)$, the set $\{e^{-t\Lambda_q}\varphi\; t\ge 1\}$ is bounded in $C^{0,\alpha}(\partial\Omega)$ due to inequality (12). Thus this statements holds by applying the theorem of Arzelà-Ascoli.

To see that claim (4) of this theorem holds, let $(1+\varepsilon)\vee\frac{2d}{d+2}\le p<2$ for some $\varepsilon>0$ and $\varphi\in D(\Lambda_2)$. For such p, the trace operator $\mathcal{T}r: W^{1,p}(\Omega)\to L^2(\partial\Omega)$ is linear bounded. Moreover, the solution operator P introduced in Lemma 2.5 satisfies

$$P(\psi+c\mathbb{1}_{\partial\Omega})=P\psi+c\,\mathbb{1}_{\overline{\Omega}}$$

for every $\psi\in W^{1-1/p,p}(\partial\Omega)$ and every $c\in\mathbb{R}$. Hence, by Maz'ya's inequality (22), we see that

$$\begin{aligned}\|e^{-t\Lambda_q}\varphi-\overline{\varphi}\|_{L^2(\partial\Omega)}&=\|\mathcal{T}r(P(e^{-t\Lambda_q}\varphi-\overline{\varphi}))\|_{L^2(\partial\Omega)}\\&\le C\,\|P(e^{-t\Lambda_q}\varphi-\overline{\varphi})\|_{W^{1,p}(\Omega)}\end{aligned}$$

$$= C\ \big(\|\nabla P(e^{-t\Lambda_q}\varphi)\|_{L^p(\Omega)^d} + \|P(e^{-t\Lambda_q}\varphi) - \overline{\varphi}\,\mathbb{1}_{\overline{\Omega}}\|_{L^p(\Omega)}\big)$$
$$\leq C\ \big(\|\nabla P(e^{-t\Lambda_q}\varphi)\|_{L^p(\Omega)^d} + \|e^{-t\Lambda_q}\varphi - \overline{\varphi}\|_{L^p(\partial\Omega)}\big)$$

for every $t \geq 0$, where the constant $C > 0$ may vary from line to line. By equality (7), we may apply Poincaré's inequality (44) to the term $\|e^{-t\Lambda_q}\varphi - \overline{\varphi}\|_{L^p(\partial\Omega)}$ in the last estimate. Then

$$\|e^{-t\Lambda_q}\varphi - \overline{\varphi}\|_{L^2(\partial\Omega)} \leq C_s\ \|\nabla P(e^{-t\Lambda_q}\varphi)\|_{L^p(\Omega)^d} \tag{71}$$

for every $t \geq 0$ and some constant $C_S > 0$. Now, applying inequality (71) to estimate the right-hand side of inequality (67) yields

$$\tfrac{d}{dt}\|e^{-t\Lambda_2}\varphi - \overline{\varphi}\|^2_{L^2(\partial\Omega)} \leq -\eta\, 2\, C_S^p\ \|e^{-t\Lambda_q}\varphi - \overline{\varphi}\|^p_{L^2(\partial\Omega)}$$

for a.e. $t > 0$, showing that the non-negative and non-increasing function y given by (69) satisfies differential inequality (70) for a.e. $t > 0$. Recall that if $y(t_0) = 0$ for some $t_0 \geq 0$, then $y(t) = 0$ for all $t \geq t_0$. In other words, t_0 describes an extinction time of the function $e^{-\cdot\Lambda_q}\varphi - \overline{\varphi}$. Suppose that for some $t_{ext} > 0$, $y(t) > 0$ for every $t \in [0, t_{ext})$. Then, dividing inequality (70) by $y^{p/2}$, subsequently integrating over $(0, t_{ext})$, and using that $p < 2$ leads to inequality (66) for $\varphi \in D(\Lambda_2)$. By employing a standard density argument, we see that claim (4) of this theorem holds, in particular, for all $\varphi \in L^2(\partial\Omega)$. This completes the proof of Theorem 6.1. □

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7. References

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