Random attractors for the stochastic Navier–Stokes equations on the 2D unit sphere

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Abstract. In this paper we prove the existence of random attractors for the Navier–Stokes equations on 2 dimensional sphere under random forcing irregular in space and time. We also deduce the existence of an invariant measure.

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1. Introduction

Complex three dimensional flows in the atmosphere and oceans are modelled assuming that the Earth's surface is an approximate sphere. Then it is natural to model the global atmospheric circulation on Earth (and large planets) using the Navier-Stokes equations (NSE) on 2-dimensional sphere coupled to classical thermodynamics [33]. This approach is relevant for geophysical flow modeling.

Many authors have studied the deterministic NSEs on the unit sphere. Notably, Il'in and Filatov [31, 29] considered the existence and uniqueness of solutions to these equations and estimated the Hausdorff dimension of their global attractors [30]. Temam and Wang [42] considered the inertial forms of NSEs on sphere while Temam and Ziane [43], see also [4], proved that the NSEs on a 2-dimensional sphere is a limit of NSEs defined on a spherical shell [43]. In other directions, Cao, Rammaha and Titi [15] proved the Gevrey regularity of the solution and found an upper bound on the asymptotic degrees of freedom for the long-time dynamics.

Concerning the numerical simulation of the deterministic NSEs on sphere, Fengler and Freeden [24] obtained some impressive numerical results using the

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spectral method, while the numerical analysis of a pseudo- spectral method for these equations has been carried out in Ganesh, Le Gia and Sloan in [26].

In our earlier paper [10] we analysed the Navier-Stokes equations on the 2-dimensional sphere with Gaussian random forcing. We proved the existence and uniqueness of solutions and continuous dependence on data in various topologies. We also studied qualitative properties of the stochastic NSEs on the unit sphere in the context of random dynamical systems.

Building on those preliminary studies, in the current paper, we prove the existence of random attractors for the stochastic NSEs on the 2-dimensional unit sphere. Let us recall here that, given a probability space, a random attractor is a compact random set, invariant for the associated random dynamical system and attracting every bounded random set in its basis of attraction (see Definition 4.3).

In the area of SPDEs the notions of random and pullback attractors were introduced by Brzeźniak et al. in [7], and by Crauel and Flandoli in [17]. These concepts have been later used to obtain crucial information on the asymptotic behaviour of random (Brzeźniak et al. [7]), stochastic (Arnold [2], Crauel and Flandoli [17], Crauel [18],Flandoli and Schmalfuss [25]) and non-autonomous PDEs (Schmalfuss [37], Kloeden and Schmalfuss [32], Caraballo et al. [14]).

We do not know if our system is dissipative in H^1. Therefore, despite the fact that the embedding $H^1 \hookrightarrow L^2$ is compact, the asymptotic compactness approach seems to be the only method available in the L^2-setting to yield the existence of an attractor, hence of an invariant measure.

The paper is organised as follows. In Section 2, we recall the relevant properties of the deterministic NSEs on the unit sphere, outline key function spaces, and recall the weak formulation of these equations. In Section 3, we define the stochastic NSEs on the sphere and review the key existence and uniqueness results obtained in [10]. In Section 4 we prove the existence of a random attractor of the stochastic NSEs on the 2-d sphere. The paper is concluded with a simple proof of the existence of an invariant measure and some comments on the question of its uniqueness.

In our paper a special attention is given to the noise with low space regularity. While many works on random attractors consider only finite dimensional noise, we follow here the approach from Brzeźniak et al [8] and consider an infinite dimensional driving Wiener process with minimal assumptions on its Cameron-Martin space (known also as the Reproducing Kernel Hilbert Space), see Remark 3.9 and the Introduction to [8] for motivation.

2. The Navier–Stokes equations on a rotating unit sphere

The sphere is a very special example of a compact Riemannian manifold without boundary hence one could recall all the classical tools from differential geometry developed for such manifolds. However we have decided to follow a different path of using the polar coordinates and defining all such objects directly.

2.1. Preliminaries

Let $\mathbb{S}^2$ be a 2-dimensional unit sphere in $\mathbb{R}^3$, i.e. $\mathbb{S}^2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$. An arbitrary point $\mathbf{x}$ on $\mathbb{S}^2$ can be parametrized by the spherical coordinates

$$\mathbf{x} = \widehat{\mathbf{x}}(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi.$$

If $\mathbf{x} = \widehat{\mathbf{x}}(\theta, \phi)$ as above, then the corresponding angles θ and ϕ will be denoted by $\theta(\mathbf{x})$ and $\phi(\mathbf{x})$, or simply by θ and ϕ.

Let $\mathbf{e}_\theta = \mathbf{e}_\theta(\theta, \phi)$ and $\mathbf{e}_\phi = \mathbf{e}_\phi(\theta, \phi)$ be the standard unit tangent vectors to $\mathbb{S}^2$ at point $\widehat{\mathbf{x}}(\theta, \phi) \in \mathbb{S}^2$ in the spherical coordinates, that is

$$\mathbf{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \quad \mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0).$$

Note that

$$\mathbf{e}_\theta = \frac{\partial \widehat{\mathbf{x}}(\theta, \phi)}{\partial \theta}, \quad \mathbf{e}_\phi = \frac{1}{\sin\theta}\frac{\partial \widehat{\mathbf{x}}(\theta, \phi)}{\partial \phi},$$

where the second equality holds whenever $\sin\theta \neq 0$.

The surface gradient for a scalar function f on $\mathbb{S}^2$ is given by

$$\nabla f = \frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi, \quad 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi.$$

For a tangential vector field $\mathbf{u} = (u_\theta, u_\phi)$, i.e. $\mathbf{u} = u_\theta\mathbf{e}_\theta + u_\phi\mathbf{e}_\phi$, we put

$$\operatorname{div}\mathbf{u} = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{\partial}{\partial\phi}u_\phi\right).$$

The tangential velocity field $\mathbf{u}(\mathbf{x}, t) = (u_\theta(\widehat{\mathbf{x}}, t), u_\phi(\widehat{\mathbf{x}}, t))$ of a geophysical fluid flow on a 2-dimensional rotating unit sphere $\mathbb{S}^2$ under the external force $\mathbf{f} = (f_\theta, f_\phi) = f_\theta\mathbf{e}_\theta + f_\phi\mathbf{e}_\phi$ is governed by the Navier-Stokes equations (NSEs), which takes the form [23, 40]

$$\partial_t\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} - \nu\mathbf{L}\mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0. \tag{2.1}$$

Here ν and ρ are two positive constants related to the physical quantities called respectively the viscosity and the density of the fluid. The Coriolis acceleration normal vector field is defined by

$$\boldsymbol{\omega} = 2\Omega\cos\big(\theta(\mathbf{x})\big)\mathbf{x},$$

where $\mathbf{x} = \hat{\mathbf{x}}(\theta(\mathbf{x}), \phi(\mathbf{x}))$ and Ω is a given constant. Note that $\theta(\mathbf{x}) = \cos^{-1}(x_3)$.

In what follows we will identify $\boldsymbol{\omega}$ with the corresponding scalar function ω defined by $\omega(\mathbf{x}) = 2\Omega\cos\big(\theta(\mathbf{x})\big)$.

The operators ∇ and div are the surface gradient and divergence, respectively. The convective acceleration $\nabla_{\mathbf{u}}\mathbf{u}$ is the nonlinear term in the equations. Here, the operator $\mathbf{L}$ is given by [40]

$$\mathbf{L} = \boldsymbol{\Delta} + 2\mathrm{Ric}, \tag{2.2}$$

where $\mathbf{\Delta}$ is the Laplace-de Rham operator, see equality (2.14) below, and Ric denotes the Ricci tensor of the two-dimensional sphere $\mathbb{S}^2$. It is well known that (see e.g. [45, page 75])

$$\mathrm{Ric} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{2.3}$$

We remark that in papers in [15, 29, 31, 42] the authors consider NSEs with $\mathbf{L} = \mathbf{\Delta}$ but the analysis in our paper are still valid in that case.

Let us define the nonlinear term $\mathbf{\nabla}_{\mathbf{u}}\mathbf{u}$. By a proposition in [22] or [21, Definition 3.31], if $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are vector fields defined in some neighbourhood of the surface $\mathbb{S}^2$ and tangent to $\mathbb{S}^2$, i.e. $\tilde{\mathbf{u}}|_{\mathbb{S}^2} = \mathbf{u} : \mathbb{S}^2 \to T\mathbb{S}^2$ and $\tilde{\mathbf{v}}|_{\mathbb{S}^2} = \mathbf{v} : \mathbb{S}^2 \to T\mathbb{S}^2$, then

$$(\mathbf{\nabla}_{\mathbf{v}}\mathbf{u})(\mathbf{x}) = \pi_{\mathbf{x}}\Big(\sum_{i=1}^{3} \tilde{\mathbf{v}}_i(\mathbf{x})\partial_i\tilde{\mathbf{u}}(\mathbf{x})\Big) = \pi_{\mathbf{x}}\big((\tilde{\mathbf{v}}(\mathbf{x})\cdot\tilde{\nabla})\tilde{\mathbf{u}}(\mathbf{x})\big), \quad \mathbf{x} \in \mathbb{S}^2, \tag{2.4}$$

where $\tilde{\nabla}$ is the gradient in $\mathbb{R}^3$ and, for $\mathbf{x} \in \mathbb{S}^2$,

$$\pi_{\mathbf{x}} : \mathbb{R}^3 \ni \mathbf{y} \mapsto \mathbf{y} - (\mathbf{x}\cdot\mathbf{y})\mathbf{x} = -\mathbf{x} \times (\mathbf{x} \times \mathbf{y}) \in T_{\mathbf{x}}\mathbb{S}^2 \tag{2.5}$$

is the orthogonal projection from $\mathbb{R}^3$ onto the tangential space $T_{\mathbf{x}}\mathbb{S}^2$ to $\mathbb{S}^2$ at $\mathbf{x}$.

One should note that the above definition is independent of the choice of extensions $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ of the vector fields $\mathbf{u}$ and $\mathbf{v}$. Indeed, if $\tilde{\tilde{\mathbf{u}}}$ and $\tilde{\tilde{\mathbf{v}}}$ is another pair of such extensions, then since $\tilde{\tilde{\mathbf{u}}} = \tilde{\mathbf{u}}$ on $\mathbb{S}^2$, for every point $\mathbf{x} \in \mathbb{S}^2$, the restrictions to $T_{\mathbf{x}}\mathbb{S}^2$ of the Frechet derivatives at of $\tilde{\tilde{\mathbf{u}}}$ and $\tilde{\mathbf{u}}$ are equal. Since $\tilde{\tilde{\mathbf{u}}}(\mathbf{x}) = \tilde{\mathbf{u}}(\mathbf{x})$ belongs to $T_{\mathbf{x}}\mathbb{S}^2$ and since $(\tilde{\mathbf{v}}(\mathbf{x})\cdot\tilde{\nabla})\tilde{\mathbf{u}}(\mathbf{x}) = [d_{\mathbf{x}}\tilde{\mathbf{u}}](\tilde{\mathbf{v}}(\mathbf{x}))$ (and analogously for the $\tilde{\tilde{}}$ extensions, the claim follows.

Using the well known formula for the vector product in $\mathbb{R}^3$:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}, \ \ \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$$

we easily infer that

$$(\tilde{\mathbf{u}}\cdot\tilde{\nabla})\tilde{\mathbf{u}} = \tilde{\nabla}\frac{|\tilde{\mathbf{u}}|^2}{2} - \tilde{\mathbf{u}} \times (\tilde{\nabla} \times \tilde{\mathbf{u}}).$$

It follows that, using the notation above, that the restrictions to $\mathbb{S}^2$ of $\tilde{\mathbf{u}} \times (\tilde{\nabla} \times \tilde{\mathbf{u}})$ and $\tilde{\tilde{\mathbf{u}}} \times (\tilde{\nabla} \times \tilde{\tilde{\mathbf{u}}})$ coincide.

Lemma 2.1. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are tangential vector fields on* $\mathbb{S}^2$ *and* $\tilde{\mathbf{u}}$ *and* $\tilde{\mathbf{v}}$ *are extensions of those to a neighbourhood of* $\mathbb{S}^2$, *i.e.* $\tilde{\mathbf{u}}|_{\mathbb{S}^2} = \mathbf{u}$ *and* $\tilde{\mathbf{v}}|_{\mathbb{S}^2} = \mathbf{v}$, *then the following identity holds*

$$\pi_{\mathbf{x}}(\tilde{\mathbf{u}} \times \tilde{\mathbf{v}}) = \mathbf{u} \times ((\mathbf{x}\cdot\mathbf{v})\mathbf{x}) + (\mathbf{x}\cdot\mathbf{u})\mathbf{x} \times \mathbf{v} \textit{ for any } \ \mathbf{x} \in \mathbb{S}^2. \tag{2.6}$$

Proof. We can decompose $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ into tangential and normal component as follows

$$\begin{aligned} \tilde{\mathbf{u}} &= \mathbf{u} + \mathbf{u}^{\perp} \quad \text{ with } \mathbf{u} \in T_{\mathbf{x}}\mathbb{S}^2, \ \ \mathbf{u}^{\perp} = (\mathbf{u}\cdot\mathbf{x})\mathbf{x} \\ \tilde{\mathbf{v}} &= \mathbf{v} + \mathbf{v}^{\perp} \quad \text{ with } \mathbf{v} \in T_{\mathbf{x}}\mathbb{S}^2, \ \ \mathbf{v}^{\perp} = (\mathbf{v}\cdot\mathbf{x})\mathbf{x} \end{aligned}$$

Thus,

$$(\tilde{\mathbf{u}} \times \tilde{\mathbf{v}}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{v}^{\perp} + \mathbf{u}^{\perp} \times \mathbf{v}.$$

Since $\mathbf{u} \times \mathbf{v}$ is normal to $T_{\mathbf{x}}\mathbb{S}^2$, $\pi_{\mathbf{x}}(\mathbf{u} \times \mathbf{v}) = 0$. Hence the lemma is proved. $\square$

Using (2.6) for the vector fields $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}} = \tilde{\nabla} \times \tilde{\mathbf{u}}$, we have

$$\pi_{\mathbf{x}}(\tilde{\mathbf{u}} \times (\tilde{\nabla} \times \tilde{\mathbf{u}})) = \mathbf{u} \times ((\mathbf{x} \cdot (\tilde{\nabla} \times \tilde{\mathbf{u}}))\mathbf{x}) + (\tilde{\mathbf{u}} \cdot \mathbf{x})\mathbf{x} \times (\tilde{\nabla} \times \tilde{\mathbf{u}}) \tag{2.7}$$

With a tangential vector field $\mathbf{u}$, since the normal component is zero, (2.7) is reduced to

$$\pi_{\mathbf{x}}(\tilde{\mathbf{u}} \times (\tilde{\nabla} \times \tilde{\mathbf{u}})) = \mathbf{u} \times ((\mathbf{x} \cdot (\tilde{\nabla} \times \tilde{\mathbf{u}}))\mathbf{x}) \tag{2.8}$$

So one can define, for a tangential vector field $\mathbf{u}$

$$\operatorname{curl} \mathbf{u} := \mathbf{x} \cdot (\tilde{\nabla} \times \tilde{\mathbf{u}})_{|\mathbb{S}^2}. \tag{2.9}$$

Given a tangential vector field $\mathbf{v}$, with a slight abuse of notation, we write

$$\mathbf{v} \times \operatorname{curl} \mathbf{u} := \mathbf{v} \times \mathbf{x}(\operatorname{curl} \mathbf{u}).$$

Hence from (2.8), (2.9) we obtain

$$\pi_{\mathbf{x}}[\tilde{\mathbf{u}} \times (\tilde{\nabla} \times \tilde{\mathbf{u}})](\mathbf{x}) = [\mathbf{u}(\mathbf{x}) \times \mathbf{x}]\operatorname{curl} \mathbf{u}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{S}^2,$$

and thus

$$\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{u} = \nabla \frac{|\mathbf{u}|^2}{2} - \mathbf{u} \times \operatorname{curl} \mathbf{u}. \tag{2.10}$$

Let us note that $(\mathbf{x} \cdot (\tilde{\nabla} \times \tilde{\mathbf{u}}))\mathbf{x}$ is just the orthogonal projection of the vector $\tilde{\nabla} \times \tilde{\mathbf{u}}$ onto the normal component of $T_{\mathbf{x}}\mathbb{S}^2$. Furthermore, with slight abuse of notation,

$$\begin{aligned} \mathbf{x} \cdot (\tilde{\nabla} \times \tilde{\mathbf{u}}) &= x_1(\partial_2 u_3 - \partial_3 u_2) + x_2(\partial_3 u_1 - \partial_1 u_3) + x_3(\partial_1 u_2 - \partial_2 u_1) \\ &= \partial_1(x_3 u_2 - x_2 u_3) + \partial_2(x_1 u_3 - x_3 u_1) + \partial_3(x_2 u_1 - x_1 u_2) \\ &= \operatorname{div}(\tilde{\mathbf{u}} \times \mathbf{x}) = -\operatorname{div}(\mathbf{x} \times \tilde{\mathbf{u}}). \end{aligned} \tag{2.11}$$

We have the following well-defined operators [29].

Definition 2.2. *Assume that* $\mathbf{u}$ *is a tangent vector field on* $\mathbb{S}^2$*, and* ψ *a scalar function on* $\mathbb{S}^2$*. We define*

$$[\operatorname{Curl} \psi](\mathbf{x}) = -\mathbf{x} \times \nabla \psi(\mathbf{x}), \quad [\operatorname{curl} \mathbf{u}](\mathbf{x}) = -[\operatorname{div}(\mathbf{x} \times \mathbf{u})](\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2 \tag{2.12}$$

Moreover, given two tangential vector fields $\mathbf{u}$ *and* $\mathbf{v}$*, the tangential vector field*

$$[\mathbf{u}(\mathbf{x}) \times \mathbf{x}]\operatorname{curl} \mathbf{v}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2 \tag{2.13}$$

will be denoted by $\mathbf{u} \times \operatorname{curl} \mathbf{v}$*.*

Often various authors introduce a notion of a normal vector field to $\mathbb{S}^2$ which they then identify with a scalar function on $\mathbb{S}^2$. We have found this to be an unnecessary procedure.

The surface diffusion operator acting on tangential vector fields on $\mathbb{S}^2$ is denoted by $\boldsymbol{\Delta}$ (known as the vector Laplace-Beltrami or Laplace-de Rham operator) and is defined as

$$\boldsymbol{\Delta}\,\mathbf{u} = \nabla \mathrm{div}\,\mathbf{u} - \mathrm{Curl}\,\mathrm{curl}\,\mathbf{u}. \tag{2.14}$$

Using (2.12), one can derive the following relations connecting the above operators:

$$\mathrm{div}\,\mathrm{Curl}\,\psi = 0, \qquad \mathrm{curl}\,\mathrm{Curl}\,\psi = -\Delta\psi, \qquad \boldsymbol{\Delta}\,\mathrm{Curl}\,\psi = \mathrm{Curl}\,\Delta\psi. \tag{2.15}$$

In what follows we denote by dS the Lebesgue integration with respect to the surface measure (or the volume measure when $\mathbb{S}^2$ is seen as a Riemannian manifold). In the spherical coordinates we have, locally, $dS = \sin\theta d\theta d\phi$. For $p \in [1,\infty)$ we will use the notation $L^p = L^p(\mathbb{S}^2)$ for the space $L^p\left(\mathbb{S}^2,\mathbb{R}\right)$ of p-integrable scalar functions on $\mathbb{S}^2$ endowed with the norm

$$\|v\|_{L^p} = \left(\int_{\mathbb{S}^2} |v(\mathbf{x})|^p\, dS(\mathbf{x})\right)^{1/p}.$$

For $p = 2$ the corresponding inner product is denoted by

$$(v_1, v_2) = (v_1, v_2)_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} v_1 v_2\, dS.$$

We will denote by $\mathbb{L}^p = \mathbb{L}^p(\mathbb{S}^2)$ the space $L^p\left(\mathbb{S}^2, T\mathbb{S}^2\right)$ of vector fields $\mathbf{v} : \mathbb{S}^2 \to T\mathbb{S}^2$ endowed with the norm

$$\|\mathbf{v}\|_{L^p} = \left(\int_{\mathbb{S}^2} |\mathbf{v}(\mathbf{x})|^p\, dS(\mathbf{x})\right)^{1/p},$$

where, for $\mathbf{x} \in \mathbb{S}^2$, $|\mathbf{v}(\mathbf{x})|$ stands for the length of $\mathbf{v}(\mathbf{x})$ in the tangent space $T_{\mathbf{x}}\mathbb{S}^2$. For $p = 2$ the corresponding inner product is denoted by

$$(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1, \mathbf{v}_2)_{\mathbb{L}^2} = \int_{\mathbb{S}^2} \mathbf{v}_1 \cdot \mathbf{v}_2(S)\, dS.$$

Throughout the paper, the induced norm on $\mathbb{L}^2(\mathbb{S}^2)$ is denoted by $\|\cdot\|$ and for other inner product spaces, say X with inner product $(\cdot,\ \cdot)_X$, the associated norm is denoted by $\|\cdot\|_X$.

We have the following identities for appropriate scalar and vector fields [29, (2.4)-(2.6)]:

$$(\nabla\psi,\ \mathbf{v}) = -(\psi,\ \mathrm{div}\,\mathbf{v}), \tag{2.16}$$
$$(\mathrm{Curl}\,\psi,\ \mathbf{v}) = (\psi,\ \mathrm{curl}\,\mathbf{v}), \tag{2.17}$$
$$(\mathrm{Curl}\,\mathrm{curl}\,\mathbf{w},\ \mathbf{z}) = (\mathrm{curl}\,\mathbf{w},\ \mathrm{curl}\,\mathbf{z}). \tag{2.18}$$

In (2.17), the $\mathbb{L}^2(\mathbb{S}^2)$ inner product is used on the left hand side and the $L^2(\mathbb{S}^2)$ inner product is used on the right hand side. Using the identity (2.16) the unknown pressure can be eliminated from the first equation in (2.1) through the weak formulation.

2.2. The weak formulation

We now introduce Sobolev spaces $H^s(\mathbb{S}^2)$ and $\mathbb{H}^s(\mathbb{S}^2)$ of scalar functions and vector fields on $\mathbb{S}^2$ respectively.

Let ψ be a scalar function and let $\mathbf{u}$ be a vector field on $\mathbb{S}^2$, respectively. For $s \geq 0$ we define

$$\|\psi\|^2_{H^s(\mathbb{S}^2)} = \|\psi\|^2_{L^2(\mathbb{S}^2)} + \|(-\Delta)^{s/2}\psi\|^2_{L^2(\mathbb{S}^2)}, \tag{2.19}$$

and

$$\|\mathbf{u}\|^2_{\mathbb{H}^s(\mathbb{S}^2)} = \|\mathbf{u}\|^2 + \|(-\boldsymbol{\Delta})^{s/2}\mathbf{u}\|^2, \tag{2.20}$$

where Δ is the Laplace–Betrami and $\boldsymbol{\Delta}$ is the Laplace–de Rham operator on the sphere. In particular, for $s = 1$,

$$\begin{aligned} \|\mathbf{u}\|^2_{\mathbb{H}^1(\mathbb{S}^2)} &= \|\mathbf{u}\|^2 + (\mathbf{u}, -\boldsymbol{\Delta}\,\mathbf{u}) \\ &= \|\mathbf{u}\|^2 + \|\operatorname{div}\mathbf{u}\|^2 + \|\operatorname{Curl}\mathbf{u}\|^2, \end{aligned} \tag{2.21}$$

where we have used formulas (2.14),(2.16)–(2.18).

We note that for $k = 0, 1, 2, \ldots$ and $\theta \in (0, 1)$ the space $H^{k+\theta}(\mathbb{S}^2)$ can be defined as the interpolation space between $H^k(\mathbb{S}^2)$ and $H^{k+1}(\mathbb{S}^2)$. We can apply the same procedure for $\mathbb{H}^{k+\theta}(\mathbb{S}^2)$.

One has the following Poincaré inequality [31, Lemma 2]

$$\lambda_1\|\mathbf{u}\| \leq \|\operatorname{div}\mathbf{u}\| + \|\operatorname{Curl}\mathbf{u}\|, \quad \mathbf{u} \in \mathbb{H}^1(\mathbb{S}^2), \tag{2.22}$$

for some positive constant λ_1.

The space of smooth (C^∞) tangential fields on $\mathbb{S}^2$ can be decomposed into three components, one in the space of all divergence-free fields and the others through the Hodge decomposition theorem [3, Theorem 1.72]:

$$C^\infty(T\mathbb{S}^2) = \mathcal{G} \oplus \mathcal{V} \oplus \mathcal{H}, \tag{2.23}$$

where

$$\mathcal{G} = \{\nabla\psi : \psi \in C^\infty(\mathbb{S}^2)\}, \quad \mathcal{V} = \{\operatorname{Curl}\psi : \psi \in C^\infty(\mathbb{S}^2)\}, \tag{2.24}$$

while $\mathcal{H}$ is the finite-dimensional space of harmonic fields, i.e. $\mathcal{H}$ contains all the vector fields $\mathbf{v}$ so that $\operatorname{Curl}(\mathbf{v}) = \operatorname{div}(\mathbf{v}) = 0$. Since the two dimensional sphere is simply connected, $\mathcal{H} = \{0\}$ [38, page 80]. We introduce the following spaces

$$\begin{aligned} H &= \text{closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\mathbb{S}^2), \\ V &= \text{closure of } \mathcal{V} \text{ in } \mathbb{H}^1(\mathbb{S}^2). \end{aligned}$$

We consider the linear Stokes problem

$$\nu\operatorname{Curl}\operatorname{curl}\mathbf{u} - 2\nu\operatorname{Ric}(\mathbf{u}) + \nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0. \tag{2.25}$$

By taking the inner product of the first equation of (2.25) with $\mathbf{v} \in V$ and then using (2.18), we obtain

$$\nu(\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) - 2\nu(\operatorname{Ric}\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V. \tag{2.26}$$

Next, we define a bilinear form $a : V \times V \to \mathbb{R}$ by

$$a(\mathbf{u}, \mathbf{v}) := (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) - 2\nu(\operatorname{Ric}\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V.$$

In view of (2.21) and (2.3), the bilinear form a satisfies

$$a(\mathbf{u},\mathbf{v}) \le \|\mathbf{u}\|_{\mathbb{H}^1}\|\mathbf{v}\|_{\mathbb{H}^1},$$

and hence it is continuous on V. So by the Riesz Lemma, there exists a unique operator $\mathcal{A}:\ V\to V'$, where V' is the dual of V, such that $a(\mathbf{u},\mathbf{v}) = (\mathcal{A}\mathbf{u},\mathbf{v})$, for $\mathbf{u},\mathbf{v}\in V$. Using the Poincaré inequality (2.22), we also have $a(\mathbf{u},\mathbf{u}) \ge \alpha\|\mathbf{u}\|_V^2$, with $\alpha = \lambda_1 - 2\nu$, which means a is coercive in V whenever $\lambda_1 > 2\nu$. In practice, usually one has $\lambda_1 \gg 2\nu$. Hence by the Lax-Milgram theorem the operator $\mathcal{A}: V\to V'$ is an isomorphism. Furthermore, by using [39, Theorem 2.2.3], we conclude that the operator $\mathcal{A}$ is positive definite, self-adjoint in H and $\mathcal{D}(\mathcal{A}^{1/2}) = V$. Thus, the spectrum all $\mathcal{A}$ consists of an infinite sequence $0<\lambda_1\le\lambda_2\le\ldots\le\lambda_\ell\to\ \infty$, and there exists an orthogonal basis $\{\mathbf{w}_\ell\}_{\ell\ge 1}$ of H consisting of eigenvectors of $\mathcal{A}$.

Hence for each positive integer $\ell = 1,2,\ldots$, the eigenvectors of the operator $\mathcal{A}$ corresponding to the eigenvalue λ_ℓ are given by

$$\mathbf{Z}_{\ell,m}(\theta,\varphi),\quad m=-\ell,\ldots,\ell. \tag{2.27}$$

Since $\{\mathbf{Z}_{\ell,m} : \ell = 1,\ldots; m=-\ell,\ldots,\ell\}$ is an orthonormal basis for H, an arbitrary $\mathbf{v}\in H$ can be written as

$$\mathbf{v} = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\widehat{v}_{\ell,m}\mathbf{Z}_{\ell,m},\qquad \widehat{v}_{\ell,m} = \int_{\mathbb{S}^2}\mathbf{v}\cdot\overline{\mathbf{Z}_{\ell,m}}dS = (\mathbf{v},\ \mathbf{Z}_{\ell,m}). \tag{2.28}$$

Since V is densely and continuously embedded into H and H can be identified with its dual H', we have the following imbeddings:

$$V\subset H\cong H'\subset V'. \tag{2.29}$$

We say that the spaces V, H and V' form a Gelfand triple.

Next we define an operator $\mathbf{A}$ in H as follows:

$$\begin{cases} \mathcal{D}(\mathbf{A}) & := & \{\mathbf{u}\in V : \mathcal{A}\mathbf{u}\in H\}, \\ \mathbf{A}\mathbf{u} & := & \mathcal{A}\mathbf{u}, \quad \mathbf{u}\in\mathcal{D}(\mathbf{A}). \end{cases} \tag{2.30}$$

Let P be the Leray orthogonal projection from $\mathbb{L}^2(\mathbb{S}^2)$ onto H. It can be shown [27] that $\mathcal{D}(\mathbf{A}) = \mathbb{H}^2(\mathbb{S}^2)\cap V$ and $\mathbf{A} = -\mathrm{P}\,(\boldsymbol{\Delta} + 2\mathrm{Ric})$, and $\mathbf{A}^* = \mathbf{A}$. It can also be shown that $V = \mathcal{D}(\mathbf{A}^{1/2})$ and

$$\|\mathbf{u}\|_V^2 \sim (\mathbf{A}\mathbf{u},\mathbf{u}),\quad \mathbf{u}\in\mathcal{D}(\mathbf{A}),$$

where $A\sim B$ indicates that there are two positive constants c_1 and c_2 such that $c_1A\le B\le c_2A$.

Let us now recall how the fractional power of the Stokes operator $\mathbf{A}$ can be defined in our concrete setting. It can be proven that

$$\mathcal{D}(\mathbf{A}^{s/2}) = \left\{\mathbf{v}\in H\ :\mathbf{v} = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\widehat{v}_{\ell,m}\mathbf{Z}_{\ell,m},\quad \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\lambda_\ell^s|\widehat{v}_{\ell,m}|^2<\infty\right\}, \tag{2.31}$$

which is the divergence-free subset of the Sobolev space $\mathbb{H}^s(\mathbb{S}^2)$. For $\mathbf{v} \in \mathcal{D}(\mathbf{A}^{s/2})$, we have

$$\mathbf{A}^{s/2}\mathbf{v} := \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} \lambda_\ell^{s/2}\widehat{v}_{\ell,m}\mathbf{Z}_{\ell,m} \quad \in H. \tag{2.32}$$

The Coriolis operator $\mathbf{C}_1 : \mathbb{L}^2\left(\mathbb{S}^2\right) \to \mathbb{L}^2\left(\mathbb{S}^2\right)$, is defined by the formula

$$(\mathbf{C}_1\mathbf{v})(\mathbf{x}) = (2\Omega\cos\theta(\mathbf{x}))\mathbf{x} \times \mathbf{v}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^2.$$

Clearly, $\mathbf{C}_1$ is linear and bounded in $\mathbb{L}^2\left(\mathbb{S}^2\right)$. In the sequel we will need the operator $\mathbf{C} = \mathrm{P}\,\mathbf{C}_1$ which is well defined and bounded in H. Furthermore, for $\mathbf{u} \in H$

$$(\mathbf{C}\mathbf{u},\mathbf{u}) = (\mathbf{C}_1\mathbf{u},\mathrm{P}\,\mathbf{u}) = \int_{\mathbb{S}^2} 2\Omega\cos\theta(\mathbf{x})\big((\mathbf{x}\times\mathbf{u})\cdot\mathbf{u}(\mathbf{x})\big)\,dS(\mathbf{x}) = 0. \tag{2.33}$$

We consider the trilinear form b on $V \times V \times V$, defined as

$$b(\mathbf{v},\mathbf{w},\mathbf{z}) = (\boldsymbol{\nabla}_{\mathbf{v}}\mathbf{w},\mathbf{z}) = \int_{\mathbb{S}^2} \boldsymbol{\nabla}_{\mathbf{v}}\mathbf{w}\cdot\mathbf{z}\;dS, \qquad \mathbf{v},\mathbf{w},\mathbf{z} \in V. \tag{2.34}$$

Using the following identity

$$2\boldsymbol{\nabla}_{\mathbf{w}}\mathbf{v} = -\operatorname{curl}(\mathbf{w}\times\mathbf{v}) + \nabla(\mathbf{w}\cdot\mathbf{v}) - \mathbf{v}\operatorname{div}\mathbf{w} + \tag{2.35}$$

$$\mathbf{w}\operatorname{div}\mathbf{v} - \mathbf{v}\times\operatorname{curl}\mathbf{w} - \mathbf{w}\times\operatorname{curl}\mathbf{v}. \tag{2.36}$$

and (2.17), for divergence free tangential vector fields $\mathbf{v},\mathbf{w},\mathbf{z}$, the trilinear form can be written as

$$b(\mathbf{v},\mathbf{w},\mathbf{z}) = \frac{1}{2}\int_{\mathbb{S}^2}\left[-\mathbf{v}\times\mathbf{w}\cdot\operatorname{curl}\mathbf{z} + \operatorname{curl}\mathbf{v}\times\mathbf{w}\cdot\mathbf{z} - \mathbf{v}\times\operatorname{curl}\mathbf{w}\cdot\mathbf{z}\right]\,dS. \tag{2.37}$$

Moreover [29, Lemma 2.1]

$$b(\mathbf{v},\mathbf{w},\mathbf{w}) = 0, \qquad b(\mathbf{v},\mathbf{z},\mathbf{w}) = -b(\mathbf{v},\mathbf{w},\mathbf{z}) \qquad \mathbf{v}\in V, \mathbf{w},\mathbf{z} \in \mathbb{H}^1(\mathbb{S}^2). \tag{2.38}$$

We have the following inequality from [31, page 12]

$$\|\mathbf{u}\|_{\mathbb{L}^4(\mathbb{S}^2)} \le C\|\mathbf{u}\|^{1/2}_{\mathbb{L}^2(\mathbb{S}^2)}\|\mathbf{u}\|^{1/2}_V, \quad \mathbf{u} \in \mathbb{H}^1(\mathbb{S}^2). \tag{2.39}$$

Thus, using (2.14), (2.17), (2.30), and (2.37), a *weak solution* of the Navier-Stokes equations (2.1) is a function $\mathbf{u} \in L^2([0,T];V)$ with $\mathbf{u}(0) = \mathbf{u}_0$ that satisfies the weak form of equation (2.1), i.e.

$$(\partial_t\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) + \nu(\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v}) - 2\nu(\operatorname{Ric}\mathbf{u},\mathbf{v}) + (\mathbf{C}\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \qquad \mathbf{v} \in V. \tag{2.40}$$

This weak formulation can be written in operator equation form on V', the dual of V. Let $\mathbf{f} \in L^2([0,T];V')$ and $\mathbf{u}_0 \in H$. We want to find a function $\mathbf{u} \in L^2([0,T];V)$, with $\partial_t\mathbf{u} \in L^2([0,T];V')$ such that

$$\partial_t\mathbf{u} + \nu\mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u},\mathbf{u}) + \mathbf{C}\mathbf{u} = \mathbf{f}, \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{2.41}$$

where the bilinear form $\mathbf{B} : V \times V \to V'$ is defined by

$$(\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w}) = b(\mathbf{u},\mathbf{v},\mathbf{w}) \qquad \mathbf{w} \in V. \tag{2.42}$$

With a slight abuse of notation, we also denote $\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u}, \mathbf{u})$.

The following are some fundamental properties of the trilinear form b; see [24]: There exists a constant $C > 0$ such that

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C \begin{cases} \|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_V^{1/2}\|\mathbf{v}\|_V^{1/2}\|\mathbf{A}\mathbf{v}\|^{1/2}\|\mathbf{w}\|, & \mathbf{u} \in V, \mathbf{v} \in \mathcal{D}(\mathbf{A}), \mathbf{w} \in H, \\ \|\mathbf{u}\|^{1/2}\|\mathbf{A}\mathbf{u}\|^{1/2}\|\mathbf{v}\|_V\|\mathbf{w}\|, & \mathbf{u} \in \mathcal{D}(\mathbf{A}), \mathbf{v} \in V, \mathbf{w} \in H, \\ \|\mathbf{u}\|^{1/2}\|\mathbf{u}\|_V^{1/2}\|\mathbf{v}\|_V\|\mathbf{w}\|^{1/2}\|\mathbf{w}\|_V^{1/2}, & \mathbf{u}, \mathbf{v}, \mathbf{w} \in V. \end{cases} \tag{2.43}$$

We also need the following estimates:

Lemma 2.3. *There exists a positive constant C such that*

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C\|\mathbf{u}\|\|\mathbf{w}\|(\|\operatorname{curl}\mathbf{v}\|_{\mathbb{L}^\infty} + \|\mathbf{v}\|_{\mathbb{L}^\infty}), \quad \mathbf{u} \in H, \mathbf{v} \in V, \mathbf{v} \in H, \tag{2.44}$$

and

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C\|\mathbf{u}\|\|\mathbf{v}\|_V\|\mathbf{w}\|^{1/2}\|\mathbf{A}\mathbf{w}\|^{1/2}, \quad \mathbf{u} \in H, \mathbf{v} \in V, \mathbf{w} \in D(\mathbf{A}). \tag{2.45}$$

and

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C\|\mathbf{u}\|_{\mathbb{L}^4}\|\mathbf{v}\|_V\|\mathbf{w}\|_{\mathbb{L}^4}, \quad \mathbf{v} \in V, \mathbf{u}, \mathbf{w} \in \mathbb{H}^1(\mathbb{S}^2). \tag{2.46}$$

In view of (2.46), b is a bounded trilinear map from $\mathbb{L}^4(\mathbb{S}^2) \times V \times \mathbb{L}^4(\mathbb{S}^2)$ to $\mathbb{R}$. Moreover, we have the following result:

Lemma 2.4. *The trilinear map $b : V \times V \times V \to \mathbb{R}$ has a unique extension to a bounded trilinear map from $\mathbb{L}^4(\mathbb{S}^2) \cap H \times \mathbb{L}^4(\mathbb{S}^2) \times V$ to $\mathbb{R}$.*

It can be seen from (2.46) that b is a bounded trilinear map from $\mathbb{L}^4(\mathbb{S}^2) \times V \times \mathbb{L}^4(\mathbb{S}^2)$ to $\mathbb{R}$. It follows that $\mathbf{B}$ maps $\mathbb{L}^4(\mathbb{S}^2) \cap H$ (and so V) into V' and

$$\|\mathbf{B}(\mathbf{u})\|_{V'} \le C_1\|\mathbf{u}\|^2_{\mathbb{L}^4} \le C_2\|\mathbf{u}\|\|\mathbf{u}\|_V \le C_3\|\mathbf{u}\|^2_V, \quad \mathbf{u} \in V. \tag{2.47}$$

3. The stochastic Navier–Stokes equations on a rotating unit sphere

By adding a white noise term to (2.1) the stochastic NSEs on the sphere is

$$\partial_t\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} - \nu\mathbf{L}\mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} + \nabla p = \mathbf{f} + n(\mathbf{x}, t), \quad \operatorname{div}\mathbf{u} = 0, \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0,$$

where we assume that $\mathbf{u}_0 \in H$, $\mathbf{f} \in V'$ and $n(t, x)$ is a Gaussian random field which is a white noise in time. In the same way as in the deterministic case we apply the operator of projection onto the space of divergence free fields and reformulate the above equation as an Itô type equation

$$d\mathbf{u}(t) + \mathbf{A}\mathbf{u}(t)dt + \mathbf{B}(\mathbf{u}(t), \mathbf{u}(t))dt + \mathbf{C}\mathbf{u} = \mathbf{f}dt + GdW(t), \quad \mathbf{u}(0) = \mathbf{u}_0. \tag{3.1}$$

Here $\mathbf{f}$ is the deterministic forcing term and $\mathbf{u}_0$ is the initial velocity. We assume that W is a cylindrical Wiener process on a certain Hilbert space K defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see [35] and [12]. G is a linear continuous operator from K to H. The space K, which is the RKHS of the

Wiener process, determines the spatial smoothness of the noise term, will satisfy further assumptions to be specified later.

Roughly speaking, a solution to problem (3.1) is a process $\mathbf{u}(t)$, $t \geq 0$, which can be represented in the form

$$\mathbf{u}(t) = \mathbf{v}(t) + \mathbf{z}_\alpha(t), \quad t \geq 0,$$

where $\mathbf{z}_\alpha(t)$, $t \in \mathbb{R}$, is a stationary Ornstein–Uhlenbeck process with drift $-\nu\mathbf{A} - \mathbf{C} - \alpha I$, i.e. a stationary solution of

$$d\mathbf{z}_\alpha + (\nu\mathbf{A} + \mathbf{C} + \alpha)\mathbf{z}_\alpha dt = GdW(t), \quad t \in \mathbb{R}, \tag{3.2}$$

and $\mathbf{v}(t)$, $t \geq 0$, is the solution to the following problem (with $\mathbf{v}_0 = \mathbf{u}_0 - \mathbf{z}_\alpha(0)$):

$$\begin{aligned} \partial_t \mathbf{v} &= -\nu\mathbf{A}\mathbf{v} - \mathbf{B}(\mathbf{v} + \mathbf{z}_\alpha, \mathbf{v} + \mathbf{z}_\alpha) - \mathbf{C}\mathbf{v} + \alpha\mathbf{z}_\alpha + \mathbf{f}, & (3.3)\\ \mathbf{v}(0) &= \mathbf{v}_0. & (3.4) \end{aligned}$$

Definition 3.1. *Suppose that* $\mathbf{z} \in L^4_{\rm loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2))$, $\mathbf{f} \in V'$ *and* $\mathbf{v}_0 \in H$. *A vector field* $\mathbf{v} \in C([0,\infty); H) \cap L^2_{\rm loc}([0,\infty); V') \cap L^4_{\rm loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2))$ *is a solution to problem* (3.3)–(3.4) *if and only if* $\mathbf{v}(0) = \mathbf{v}_0$ *and* (3.3) *holds in the weak sense, i.e. for any* $\phi \in V$,

$$\partial_t(\mathbf{v}, \phi) = -\nu(\mathbf{v}, \mathbf{A}\phi) - b(\mathbf{v} + \mathbf{z}, \mathbf{v} + \mathbf{z}, \phi) - (\mathbf{C}\mathbf{v}, \phi) + (\alpha\mathbf{z} + \mathbf{f}, \phi). \tag{3.5}$$

We remark that for (3.5) to make sense, it is sufficient to assume that $\mathbf{v} \in L^2(0,T;V) \cap L^\infty(0,T;H)$.

We have proved the following major theorems on the existence and uniqueness of the solution of (3.3) – (3.4) in [10].

Theorem 3.2. [10, Theorem 3.1] *Assume that* $\alpha \geq 0$, $\mathbf{z} \in L^4_{\rm loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2)) \cap L^2_{\rm loc}([0,\infty); V')$, $\mathbf{v}_0 \in H$ *and* $\mathbf{f} \in V'$. *Then then there exists a unique solution* $\mathbf{v}$ *of problem* (3.3) – (3.4).

Theorem 3.3. [10, Theorem 3.2] *Assume that* $T > 0$ *is fixed. If* $\mathbf{u}_{0n} \to \mathbf{u}_0$ *in* H,

$$\mathbf{z}_n \to \mathbf{z} \text{ in } L^4([0,T]; \mathbb{L}^4(\mathbb{S}^2)) \cap L^2(0,T;V'), \quad \mathbf{f}_n \to \mathbf{f} \text{ in } L^2(0,T;V').$$

then

$$\mathbf{v}(\cdot, \mathbf{z}_n, \mathbf{f}_n, \mathbf{u}_{0n}) \to \mathbf{v}(\cdot, \mathbf{z}, \mathbf{f}, \mathbf{u}_0) \text{ in } C([0,T]; H) \cap L^2(0,T;V),$$

where $\mathbf{v}(t, \mathbf{z}, \mathbf{f}, \mathbf{u}_0)$ *is the solution of problem* (3.3) – (3.4) *and* $\mathbf{v}(t, \mathbf{z}_n, \mathbf{f}_n, \mathbf{u}_{0n})$ *is the solution of problem* (3.3)–(3.4) *with* $\mathbf{z}, \mathbf{f}, \mathbf{u}_0$ *being replaced by* $\mathbf{z}_n, \mathbf{f}_n, \mathbf{u}_{0n}$. *In particular,* $\mathbf{v}(T, \mathbf{z}_n, \mathbf{u}_{0n}) \to \mathbf{v}(T, \mathbf{z}, \mathbf{u}_0)$ *in* H.
Moreover, then the maps

$$H \ni \mathbf{x} \mapsto \mathbf{v}(\cdot, \mathbf{x}) \in L^2([0,T]; V)$$

is continuous in the weak topologies of H *and* $L^2([0,T]; V)$.

$$H \ni \mathbf{x} \mapsto \mathbf{v}(t, \mathbf{x}) \in H, \quad t \in [0,T]$$

are continuous in the weak topologies of H. *More precisely, if* $\mathbf{x}_n \to \mathbf{x}$ *weakly in* H, *then for any* $\phi \in H$, $(\mathbf{v}(\cdot, \mathbf{x}_n), \phi) \to (\mathbf{v}(\cdot, \mathbf{x}), \phi)$ *uniformly on* $[0,T]$, *as* $n \to \infty$.

3.1. Preliminaries

Let us recall that for a real separable Hilbert space K and a real separable Banach space X, a linear operator $U : K \to X$ is called γ-radonifying iff $\gamma_K \circ U^{-1}$ is σ-additive. Here γ_K is the canonical Gaussian cylindrical measure on K. If a linear map $U : K \to X$ is $\gamma-$radonifying, then $\gamma_K \circ U^{-1}$ has a unique extension to a Borel probability measure denoted by ν_U on X. By $R(K,X)$ we denote the Banach space of γ-radonifying operators from K to X with the norm

$$\|U\|_{R(K,X)} := \left(\int_X |x|_X^2 d\nu_U(x)\right)^{1/2}, \quad U \in R(K,X).$$

From now on we will using freely notation introduced in the former sections. It follows from [13, Theorem 2.3] that for a self adjoint operator $U \geq cI$ in H, where $c > 0$, such that U^{-1} is compact, the operator U^{-s} : $H \to \mathbb{L}^p(\mathbb{S}^2)$ is γ-radonifying iff

$$\int_{\mathbb{S}^2} \left[\sum_\ell \lambda_\ell^{-2s} |\mathbf{e}_\ell(\mathbf{x})|^2\right]^{p/2} dS(\mathbf{x}) < \infty, \tag{3.6}$$

where $\{\mathbf{e}_\ell\}$ is an orthonormal basis of H corresponding to U. This implies the following result.

Lemma 3.4. *Let $\mathbf{\Delta}$ denote the Laplace-de Rham operator on $\mathbb{S}$. Then the operator*

$$(-\mathbf{\Delta})^{-s} : H \to \mathbb{L}^4(\mathbb{S}^2) \quad \textit{is } \gamma- \textit{ radonifying iff } s > 1/2. \tag{3.7}$$

Proof. Let us recall that all the distinct eigenvalues of $-\mathbf{\Delta}$ are given by a sequence $\lambda_\ell = \ell(\ell+1)$, $\ell = 0, 1, \dots$ and the corresponding eigenfunctions are given by the divergence free vector spherical harmonics $\mathbf{Y}_{\ell,m}$ for $|m| \leq \ell$, $\ell \in \mathbb{N}$ [44, page 216]. Let us recall also the addition theorem for vector spherical harmonics [44, formula (81), page 221]

$$\sum_{|m|\leq \ell} |\mathbf{Y}_{\ell,m}(\mathbf{x})|^2 = \frac{2\ell+1}{4\pi} P_\ell(1), \ \mathbf{x} \in \mathbb{S}^2,$$

and the fact that $P_\ell(1) = 1$ with P_ℓ being the Legendre polynomial of degree ℓ. Therefore, (3.6) yields

$$\int_{\mathbb{S}^2} \left[\sum_{\ell=0}^\infty (\ell(\ell+1))^{-2s} \sum_{|m|\leq\ell} |\mathbf{Y}_{\ell,m}(\mathbf{x})|^2\right]^{4/2} dS(\mathbf{x}) \tag{3.8}$$

$$= \int_{\mathbb{S}^2} \left[\sum_{\ell=0}^\infty (\ell(\ell+1))^{-2s} \frac{2\ell+1}{4\pi} P_\ell(1)\right]^2 dS(\mathbf{x}) < \infty$$

if and only if $s > \frac{1}{2}$ and the lemma follows. □

Let
$$X = \mathbb{L}^4(\mathbb{S}^2) \cap H$$
denote the Banach space endowed with the norm
$$\|x\|_X = \|x\|_H + \|x\|_{\mathbb{L}^4(\mathbb{S}^2)}.$$

Remark 3.5. It follows from Lemma 3.4 that the operator
$$\mathbf{A}^{-s} : H \to \mathbb{L}^4(\mathbb{S}^2) \cap H \quad \text{is } \gamma-\text{radonifying if } s > 1/2. \tag{3.9}$$

Let us recall, that X is an M-type 2 Banach space, see [6] for details. The Stokes operator $-\mathbf{A}$ restricted to X is an infinitesimal generator of an analytic semigroup. We will consider an operator in X defined by the formula
$$\hat{\mathbf{A}} = \nu\mathbf{A} + \mathbf{C}, \quad \mathrm{dom}(\hat{\mathbf{A}}) = \mathrm{dom}(\mathbf{A}),$$
where $\nu > 0$, and $\mathbf{C}$ is the Coriolis operator. For the reader's convenience we recall a result presented in [10].

Proposition 3.6. [10, Proposition 5.2] *The operator $\hat{\mathbf{A}}$ with the domain dom$(\hat{\mathbf{A}})$ = dom$(\mathbf{A})$ generates an analytic C_0-semigroup $\left(e^{-t\hat{\mathbf{A}}}\right)$ in X. Moreover, there exist constants $\mu > 0$, such that for any $\delta \geq 0$ there exists $M_\delta \geq 1$ such that*
$$\|\hat{\mathbf{A}}^\delta e^{-t\hat{\mathbf{A}}}\|_{\mathcal{L}(X,X)} \leq M_\delta t^{-\delta} e^{-\mu t} \quad t > 0.$$

The next lemma is a special case of Lemma 5.1 in [10]

Lemma 3.7. *Suppose, there exists a separable Hilbert space $K \subset X$ such that $\hat{\mathbf{A}}^{-\delta} X \subset K$ and the operator $\hat{\mathbf{A}}^{-\delta} : K \to X$ is γ-radonifying for some $\delta > 0$. Then*
$$\int_0^\infty \|e^{-t\hat{\mathbf{A}}}\|_{R(K,X)} dt < \infty.$$

Let E denote the completion of X with respect to the image norm $\|\mathbf{v}\|_E = \|\mathbf{A}^{-\delta}\mathbf{v}\|_X$, $\mathbf{v} \in X$. For $\xi \in (0, 1/2)$ we set
$$C^\xi_{1/2}(\mathbb{R}, E) := \{\omega \in C(\mathbb{R}, E) : \omega(0) = 0, \sup_{t,s\in\mathbb{R}} \frac{|\omega(t) - \omega(s)|_E}{|t-s|^\xi (1 + |t| + |s|)^{1/2}} < \infty\}.$$

The space $C^\xi_{1/2}(\mathbb{R}, E)$ equipped with the the norm
$$\|\omega\|_{C^\xi_{1/2}(\mathbb{R},E)} = \sup_{t\neq s\in\mathbb{R}} \frac{|\omega(t) - \omega(s)|_E}{|t-s|^\xi (1 + |t| + |s|)^{1/2}}$$
is a nonseparable Banach space. However, the closure of $\{\omega \in C_0^\infty(\mathbb{R}) : \omega(0) = 0\}$ in $C^\xi_{1/2}(\mathbb{R}, E)$, denoted by $\Omega(\xi, E)$, is a separable Banach space.

Let us denote by $C_{1/2}(\mathbb{R}, X)$ the space of all continuous functions $\omega : \mathbb{R} \to X$ such that
$$\|\omega\|_{C_{1/2}(\mathbb{R},E)} = \sup_{t\in\mathbb{R}} \frac{|\omega(t)|_E}{1 + |t|^{1/2}} < \infty.$$
The space $C_{1/2}(\mathbb{R}, E)$ endowed with the norm $\|\cdot\|_{C_{1/2}(\mathbb{R},E)}$ is a nonseparable Banach space.

We denote by $\mathcal{F}$ the Borel σ-algebra on $\Omega(\xi, E)$. One can show [5] that for $\xi \in (0, 1/2)$, there exists a Borel probability measure $\mathbb{P}$ on $\Omega(\xi, E)$ such that the canonical process w_t, $t \in \mathbb{R}$, defined by

$$w_t(\omega) := \omega(t), \quad \omega \in \Omega(\xi, E), \tag{3.10}$$

is a two-sided Wiener process. The Cameron-Martin (or Reproducing Kernel Hilbert space) of the Gaussian measure $\mathcal{L}(w_1)$ on E is equal to K. For $t \in \mathbb{R}$, let $\mathcal{F}_t := \sigma\{w_s : s \leq t\}$. Since for each $t \in \mathbb{R}$ the map $z \circ i_t : E^* \to L^2(\Omega(\xi, E), \mathcal{F}_t, \mathbb{P})$, where $i_t : \Omega(\xi, E) \ni \gamma \mapsto \gamma(t) \in E$, satisfies $\mathbb{E}|z \circ i_t|^2 = t|z|_K^2$, there exists a unique extension of $z \circ i_t$ to a bounded linear map $W_t : K \to L^2(\Omega(\xi, E), \mathcal{F}_t, \mathbb{P})$. Moreover, the family $(W_t)_{t\in\mathbb{R}}$ is an H-cylindrical Wiener process on a filtered probability space $(\Omega(\xi, E), \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \big(\mathcal{F}_t)_{t\in\mathbb{R}}$ in the sense of e.g. [12].

3.2. Ornstein-Uhlenbeck process

The following is our standing assumption.

Assumption 3.8. *Suppose $K \subset H \cap \mathbb{L}^4(\mathbb{S}^2)$ is a Hilbert space such that*

$$\mathbf{A}^{-\delta} : K \to H \cap \mathbb{L}^4(\mathbb{S}^2) \text{ is } \gamma\text{-radonifying} \tag{3.11}$$

for some $\delta \in (0, \frac{1}{2})$,

Remark 3.9. It follows from Remark 3.5 that if $K = D\big(\mathbf{A}^s\big)$ with $s > 0$, then Assumption 3.8 is satisfied. See also Remark 6.1 in [11].

On the space $\Omega(\xi, E)$ we consider a flow $\vartheta = (\vartheta_t)_{t\in\mathbb{R}}$ defined by

$$\vartheta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega(\xi, E), \;\; t \in \mathbb{R}.$$

For $\xi \in (\delta, 1/2)$ and $\tilde{\omega} \in C^{\xi}_{1/2}(\mathbb{R}, X)$ we define

$$\hat{z}(t) = \hat{z}(\hat{\mathbf{A}}; \tilde{\omega})(t) = \int_{-\infty}^{t} \hat{\mathbf{A}}^{1+\delta} e^{-(t-r)\hat{\mathbf{A}}}(\tilde{\omega}(t) - \tilde{\omega}(r))dr, \quad t \in \mathbb{R}. \tag{3.12}$$

By Proposition 3.6, for each $\delta > 0$ there exists $C = C(\delta) > 0$ such that

$$\|\hat{\mathbf{A}}^{\delta} e^{-t\hat{\mathbf{A}}}\|_{\mathcal{L}(X,X)} \leq Ct^{-\delta}e^{-\mu t}, \quad t \geq 0. \tag{3.13}$$

This was an assumption in [11, Proposition 6.2]. Rewriting that proposition in a slightly more general form we have

Proposition 3.10. *For any $\alpha \geq 0$, the operator $-(\hat{\mathbf{A}} + \alpha I)$ is a generator of an analytic semigroup $\{e^{-t(\hat{\mathbf{A}}+\alpha I)}\}_{t\geq 0}$ in X such that*

$$\|\hat{\mathbf{A}}^{\delta} e^{-t(\hat{\mathbf{A}}+\alpha I)}\|_{\mathcal{L}(X,X)} \leq Ct^{-\delta}e^{-(\mu+\alpha)t}, \quad t \geq 0.$$

If $t \in \mathbb{R}$, then $\hat{z}(t)$ defined in (3.12) is a well-defined element of X and for each $t \in \mathbb{R}$ the mapping $\tilde{\omega} \mapsto \hat{z}(t)$ is continuous from $C^{\xi}_{1/2}(\mathbb{R}, X)$ to X. Moreover, the map $\hat{z} : C^{\xi}_{1/2}(\mathbb{R}, X) \to C_{1/2}(\mathbb{R}, X)$ is well defined, linear and bounded. In particular, there exists a constant $C < \infty$ such that for any $\tilde{\omega} \in C^{\xi}_{1/2}(\mathbb{R}, X)$

$$|\hat{z}(\tilde{\omega})(t)| \leq C(1 + |t|^{1/2})\|\tilde{\omega}\|_{C^{1/2}(\mathbb{R},X)}. \tag{3.14}$$

The following results for the operator $\hat{\mathbf{A}}$ follow from Corollary 6.4, Theorem 6.6 and Corollary 6.8 in from [11], respectively.

Corollary 3.11. *For all $-\infty < a < b < \infty$ and $t \in \mathbb{R}$, for $\tilde{\omega} \in C^{\xi}_{1/2}(\mathbb{R}, X)$ the map*

$$\tilde{\omega} \mapsto (\hat{z}(\tilde{\omega})(t), \hat{z}(\tilde{\omega})) \in X \times L^4(a, b; X)$$

is continuous. Moreover, the above result is valid with the space $C^{\xi}_{1/2}(\mathbb{R}, X)$ being replaced by $\Omega(\xi, X)$.

Theorem 3.12. *For any $\omega \in C^{\xi}_{1/2}(\mathbb{R}, X)$,*

$$\hat{z}(\vartheta_s \omega(t)) = \hat{z}(\omega)(t + s), \quad t, s \in \mathbb{R}.$$

In particular, for any $\omega \in \Omega$ and all $t, s \in \mathbb{R}$, $\hat{z}(\vartheta_s \omega)(0) = \hat{z}(\omega)(s)$.

For $\xi \in C_{1/2}(\mathbb{R}, X)$ we put

$$(\tau_s \zeta) = \zeta(t + s), \quad t, s \in \mathbb{R}.$$

Thus, τ_s is a linear a bounded map from $C_{1/2}(\mathbb{R}, X)$ into itself. Moreover, the family $(\tau_s)_{s \in \mathbb{R}}$ is a C_0 group on $C_{1/2}(\mathbb{R}, X)$.

Using this notation Theorem 3.12 can be rewritten in the following way.

Corollary 3.13. *For $s \in \mathbb{R}$, $\tau_s \circ \hat{z} = \hat{z} \circ \vartheta_s$, i.e.*

$$\tau_s(\hat{z}(\omega)) = \hat{z}(\vartheta_s(\omega)), \quad \omega \in C^{\xi}_{1/2}(\mathbb{R}, X).$$

We define

$$\mathbf{z}_\alpha(\omega) := \hat{z}(\hat{\mathbf{A}} + \alpha I; (\hat{\mathbf{A}} + \alpha I)^{-\delta} \omega) \in C_{1/2}(\mathbb{R}, X),$$

i.e. for any $t \geq 0$,

$$\begin{aligned} \mathbf{z}_\alpha(\omega)(t) \quad := \quad & \int_{-\infty}^{t} (\hat{\mathbf{A}} + \alpha I)^{1+\delta} e^{-(t-r)(\hat{\mathbf{A}}+\alpha I)} \\ & [(\hat{\mathbf{A}} + \alpha I)^{-\delta} \omega(t) - (\hat{\mathbf{A}} + \alpha I)^{-\delta} \omega(r)] dr \end{aligned} \tag{3.15}$$

For $\omega \in C_0^\infty(\mathbb{R})$ with $\omega(0) = 0$, by the fundamental theorem of calculus, we obtain

$$\begin{aligned} \frac{d\mathbf{z}_\alpha(t)}{dt} \quad = \quad & -(\hat{\mathbf{A}} + \alpha I) \int_{-\infty}^{t} (\hat{\mathbf{A}} + \alpha I)^{1+\delta} e^{-(t-r)(\hat{\mathbf{A}}+\alpha I)} \\ & [(\hat{\mathbf{A}} + \alpha I)^{-\delta} \omega(t) - (\hat{\mathbf{A}} + \alpha I)^{-\delta} \omega(r)] dr + \dot{\omega}(t), \end{aligned}$$

where $\dot{\omega}(t) = d\omega(t)/dt$. Hence $\mathbf{z}_\alpha(t)$ is the solution of the following equation

$$\frac{d\mathbf{z}_\alpha(t)}{dt} + (\hat{\mathbf{A}} + \alpha I)\mathbf{z}_\alpha = \dot{\omega}(t), \quad t \in \mathbb{R}. \tag{3.16}$$

It follows from Theorem 3.12 that

$$\mathbf{z}_\alpha(\vartheta_s \omega)(t) = \mathbf{z}_\alpha(\omega)(t + s), \quad \omega \in C^{\xi}_{1/2}(\mathbb{R}, X),\ t, s \in \mathbb{R}. \tag{3.17}$$

Similar to our definition (3.10) of the Wiener process $w(t)$, $t \in \mathbb{R}$, we can view the formula (3.15) as a definition of a process $\mathbf{z}_\alpha(t)$, $t \in \mathbb{R}$, on the probability space $(\Omega(\xi, E), \mathcal{F}, \mathbb{P})$. Equation (3.16) suggests that this process is an Ornstein-Uhlenbeck process.

Proposition 3.14. *The process* $\mathbf{z}_\alpha(t)$, $t \in \mathbb{R}$, *is a stationary Ornstein-Uhlenbeck process. It is the solution of the equation*

$$d\mathbf{z}_\alpha(t) + (\hat{\mathbf{A}} + \alpha I)\mathbf{z}_\alpha dt = dw(t), \quad t \in \mathbb{R},$$

i.e. for all $t \in \mathbb{R}$, *a.s.*

$$\mathbf{z}_\alpha(t) = \int_{-\infty}^{t} e^{-(t-s)(\hat{\mathbf{A}}+\alpha I)} dw(s), \tag{3.18}$$

where the integral is the Itô integral on the M*-type 2 Banach space* X *in the sense of* [6].

In particular, for some constant C *depending on* X,

$$\mathbb{E}\|\mathbf{z}_\alpha(t)\|_X^2 \le C \int_0^\infty e^{-2\alpha s} \|e^{-s\hat{\mathbf{A}}}\|^2_{R(K,X)} ds.$$

Moreover, $\mathbb{E}\|\mathbf{z}_\alpha(t)\|_X^2$ *tends to* 0 *as* $\alpha \to \infty$.

Proof. Stationarity of the process $\mathbf{z}_\alpha$ follows from equation (3.17). The equality (3.18) follows by finite-dimensional approximation.

From [6] we have

$$\begin{aligned} \mathbb{E}\|\mathbf{z}_\alpha(t)\|_X^2 &= \mathbb{E}\left\| \int_{-\infty}^{t} e^{-(\hat{\mathbf{A}}+\alpha I)(t-s)} dw(s) \right\|_X^2 \\ &\le C \int_{-\infty}^{t} \|e^{-(\hat{\mathbf{A}}+\alpha I)(t-s)}\|^2_{R(K,X)} ds \qquad (3.19) \\ &\le C \int_0^\infty e^{-2\alpha s} \|e^{-s\hat{\mathbf{A}}}\|^2_{R(K,X)} ds. \qquad (3.20) \end{aligned}$$

Using [10, Lemma 5.1] with $\hat{\mathbf{A}} = -\mathbf{\Delta}$, $V = -2\nu \mathrm{Ric} + \mathbf{C}$ and observation (3.7), we conclude that

$$\int_0^\infty \|e^{-s\hat{\mathbf{A}}}\|^2_{R(K,X)} ds < \infty. \tag{3.21}$$

Hence, we conclude that the last integral (3.20) is finite. Finally, the last statement follows from (3.20) by applying the Lebesgue Dominated Convergence Theorem. □

By Proposition 3.14, $\mathbf{z}_\alpha(t)$, $t \in \mathbb{R}$, is a stationary and ergodic X-valued process, hence by the Strong Law for Large Numbers (see Da Prato and Zabczyk [36] for a similar argument),

$$\lim_{t\to\infty} \frac{1}{t} \int_{-t}^{0} \|\mathbf{z}_\alpha(s)\|_X^2 ds = \mathbb{E}\|\mathbf{z}_\alpha(0)\|_X^2, \quad \mathbb{P}\text{-a.s. on } C^\xi_{1/2}(\mathbb{R}, X). \tag{3.22}$$

It also follows from Proposition 3.14 that we can find α_0 such that for all $\alpha \ge \alpha_0$,

$$\mathbb{E}\|\mathbf{z}_\alpha(0)\|_X^2 < \frac{\nu^2 \lambda_1}{6C^2}, \tag{3.23}$$

where λ_1 is the constant appearing in the Poincaré inequality (2.22) and $C > 0$ is a certain universal constant.

Denote by $\Omega_\alpha(\xi, E)$ the set of those $\omega \in \Omega(\xi, E)$ for which the equality (3.22) holds true. It follows from Corollary 3.13 that this set is invariant with respect to the flow ϑ, i.e. for all $\alpha \geq 0$ and all $t \in \mathbb{R}$, $\vartheta_t(\Omega_\alpha(\xi, E)) \subset \Omega_\alpha(\xi, E)$. Therefore, the same is true for a set

$$\hat{\Omega}(\xi, E) = \bigcap_{n=0}^{\infty} \Omega_n(\xi, E).$$

It follows that as a model for a metric dynamical system we can take either the quadruple $(\Omega(\xi, E), \mathcal{F}, \mathbb{P}, \vartheta)$ or the quadruple $(\hat{\Omega}(\xi, E), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta})$, where $\hat{\mathcal{F}}$,$\hat{\mathbb{P}}$, and $\hat{\vartheta}$ are respectively the natural restrictions of $\mathcal{F}, \mathbb{P}$ and ϑ to $\hat{\Omega}(\xi, E)$.

Proposition 3.15. *The quadruple* $(\hat{\Omega}(\xi, E), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta})$ *is a metric DS. For each* $\omega \in \hat{\Omega}(\xi, E)$ *the limit in* (3.22) *exists.*

4. Attractors for random dynamical systems generated by the stochastic NSEs on the sphere

4.1. Preliminaries

A measurable dynamical system (DS) is a triple

$$\mathfrak{T} = (\Omega, \mathcal{F}, \vartheta),$$

where $(\Omega, \mathcal{F})$ is a measurable space and $\vartheta : \mathbb{R} \times \Omega \ni (t, \omega) \mapsto \vartheta_t \omega \in \Omega$ is a measurable map such that for all $t, s \in \mathbb{R}$, $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$. A metric DS is a quadruple

$$\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(\Omega, \mathcal{F}, \vartheta)$ is a measurable DS such that for each $t \in \mathbb{R}$, $\vartheta_t : \Omega \to \Omega$ preserves $\mathbb{P}$.

Suppose also that (X, d) is a Polish space (i.e. complete separable metric space) and $\mathcal{B}$ is its Borel $\sigma-$field. Let $\mathbb{R}^+ = [0, \infty)$.

Definition 4.1. *Given a metric DS* $\mathfrak{T}$ *and a Polish space* X*, a map* $\varphi : \mathbb{R}^+ \times \Omega \times X(t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ *is called a measurable random dynamical system (RDS) (on* X *over* $\mathfrak{T}$*) iff*

(i) φ *is* $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B})$*-measurable.*

(ii) $\varphi(t + s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega)$ *for all* $s, t \in \mathbb{R}^+$ *and* $\varphi(0, \omega) = id$*, for all* $\omega \in \Omega$*. (Cocycle property)*

An RDS φ is said to be continuous or differentiable iff for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$, $\varphi(t, \cdot, \omega) : X \to X$ is continuous or differentiable, respectively. Similarly, an RDS φ is said to be time continuous iff for all $\omega \in \Omega$ and for all $x \in X$, $\varphi(\cdot, x, \omega) : \mathbb{R}^+ \to X$ is continuous.

For two nonempty sets $A, B \subset X$, we put

$$d(A, B) = \sup_{x \in A} d(x, B) \quad \text{and} \quad \rho(A, B) = \max\{d(A, B), d(B, A)\}.$$

In fact, ρ restricted to the family $\mathfrak{CB}$ of all nonempty closed subsets on X is a metric, and it is called the Hausdorff metric. From now on, let $\mathcal{X}$ be the

σ-field on $\mathfrak{CB}$ generated by open sets with respect to the Hausdorff metric ρ; see [16].

A set-valued map $C : \Omega \to \mathfrak{CB}$ is said to be measurable iff C is $(\mathcal{F}, \mathcal{X})$-measurable. Such a map is often called a *closed random set*.

For a given closed random set B, the Ω-*limit set* of B is defined to be the set

$$\Omega(B,\omega) = \Omega_B(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t,\vartheta_{-t}\omega)B(\vartheta_{-t}\omega)}. \tag{4.1}$$

Definition 4.2. *A closed random set $K(\omega)$ is said to (a) attract, (b) absorb, (c) ρ-attract another closed random set $B(\omega)$ iff for all $\omega \in \Omega$, respectively,*

(a) $\lim_{t\to\infty} d(\varphi(t,\vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)) = 0$;

(b) *there exists a time $t_B(\omega)$ such that*

$$\varphi(t,\vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subset K(\omega) \text{ for all } t \geq t_B(\omega).$$

(c)

$$\lim_{t\to\infty} \rho(\varphi(t,\vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)) = 0.$$

We denote by $\mathcal{F}^u$ the $\sigma-$algebra of universally measurable sets associated to the measurable space $(\Omega, \mathcal{F})$. As far as we are aware, the following definition appeared for the first time as Definition 3.4 in the fundamental work by Fladoli and Schmalfuss [25], see also [8].

Definition 4.3. *A random set $A : \Omega \to \mathfrak{CB}(X)$ is a* **random $\mathfrak{D}$-attractor** *iff*

(i) *A is a compact random set,*

(ii) *A is φ-invariant, i.e., $\mathbb{P}$-a.s.*

$$\varphi(t,\omega)A(\omega) = A(\vartheta_t\omega)$$

(iii) *A is $\mathfrak{D}$-attracting, in the sense that, for all $D \in \mathfrak{D}$ it holds*

$$\lim_{t\to\infty} d(\varphi(t,\vartheta_{-t}\omega)D(\vartheta_{-t}\omega), A(\omega)) = 0.$$

Definition 4.4. *We say that an RDS ϑ-cocycle φ defined on a separable Banach space X is $\mathfrak{D}$-asymptotically compact iff for each $D \in \mathfrak{D}$, for every $\omega \in \Omega$, for any positive sequence (t_n) such that $t_n \to \infty$ and for any sequence $\{x_n\}$ such that*

$$x_n \in D(\vartheta_{-t_n}\omega), \qquad \text{for all } n \in \mathbb{N},$$

the set $\{\varphi(t_n, \vartheta_{-t_n}\omega)x_n : n \in \mathbb{N}\}$ is relatively compact in X.

Now we need to state a result on the existence of a random $\mathfrak{D}$-attractor, see Theorem 2.8 in [8] and references therein.

Theorem 4.5. *Assume that $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric DS,* X *is a Polish space, $\mathfrak{D}$ is a nonempty class of closed and bounded random sets on X and φ is a continuous, $\mathfrak{D}$-asymptotically compact RDS on* X *(over $\mathfrak{T}$). Assume that there exists a $\mathfrak{D}$-absorbing closed and bounded random set B on X, i.e. for any given $D \in \mathfrak{D}$ there exists $t(D) \geq 0$ such that $\varphi(t,\vartheta_t\omega)D(\vartheta_{-t}\omega) \subset B(\omega)$ for all $t \geq t(D)$. Then, there exists $\mathfrak{D}$-attractor A given by*

$$A(\omega) = \Omega_B(\omega), \quad \omega \in \Omega, \tag{4.2}$$

with

$$\Omega_B(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t,\vartheta_{-t}\omega, B(\vartheta_{-t}\omega))}, \quad \omega\in\Omega.$$

which is $\mathcal{F}^u$-measurable[1].

Remark 4.6. If $\mathfrak{D}$ contains every bounded and closed nonempty deterministic subsets of X, then as a consequence of this theorem, of Theorem 2.1 in [20], and of Corollary 5.8 in [19] we obtain that the random attractor A is given by

$$A(\omega) = \overline{\bigcup_{C\subset X} \Omega_C(\omega)} \quad \mathbb{P}-\text{ a.s }, \tag{4.3}$$

where the union in (4.3) is made for all bounded and closed nonempty deterministic subsets C of X.

4.2. Random dynamical systems generated by the NSEs

We fix $\delta < 1/2$ and $\xi \in (\delta, 1/2)$ and put $\Omega = \Omega(\xi, E)$. Then we define a map $\varphi = \varphi_\alpha : \mathbb{R}_+ \times \Omega \times H \to H$ by

$$\varphi : \mathbb{R}_+ \times \Omega \times H \ni (t,\omega,\mathbf{x}) \mapsto \mathbf{v}(t,\mathbf{z}(\omega),\mathbf{x}-\mathbf{z}(\omega)(0)) + \mathbf{z}(\omega)(t) \in H, \tag{4.4}$$

where $\mathbf{v}(t,\omega,\mathbf{v}_0) = \mathbf{z}_\alpha(t,\omega,\mathbf{v}_0)$ is the solution to problem (3.3-3.4). Because $\mathbf{z}(\omega) \in C_{1/2}(\mathbb{R}, X)$, $\mathbf{z}(\omega)(0)$ is a well-defined element of H and hence φ is well defined. Furthermore, we have the main result of this section.

Theorem 4.7. *The couple (φ, ϑ) is a random dynamical system.*

Proof. All properties with the exception of the cocycle one of a random dynamical system follow from Theorem 3.3. Hence we only need to show that for any $\mathbf{x} \in H$,

$$\varphi(t+s,\omega)\mathbf{x} = \varphi(t,\vartheta_s\omega)\varphi(s,\omega)\mathbf{x}, \quad t,s\in\mathbb{R}^+.$$

The proof can be completed by applying similar techniques to those used in [11, Theorem 6.15] in the case of the stochastic Navier-Stokes equations in a 2-dimensional unbounded domain. □

Suppose that Assumption 3.8 is satisfied. If $u_s \in H$, $s\in\mathbb{R}$, $f\in V'$ and W_t, $t\in\mathbb{R}$ is a two-sided Wiener process introduced after (3.10) such that the Cameron-Martin (or Reproducing Kernel Hilbert) space of the Gaussian measure $\mathcal{L}(w_1)$ is equal to K. A process $\mathbf{u}(t)$, $t\geq 0$, with trajectories in $C([s,\infty);H) \cap L^2_{\text{loc}}([s,\infty);V) \cap L^2_{\text{loc}}([s,\infty);\mathbb{L}^4(\mathbb{S}^2))$ is a solution to problem (3.1) iff $\mathbf{u}(s) = \mathbf{u}_s$ and for any $\phi\in V$, $t > s$,

$$\begin{aligned}(\mathbf{u}(t),\phi) = (\mathbf{u}(s),\phi) - \nu\int_s^t (\mathbf{A}\mathbf{u}(r),\phi)dr - \int_s^t b(\mathbf{u}(r),\mathbf{u}(r),\phi)dr \\ - \int_s^t (\mathbf{C}\mathbf{u}(r),\phi)dr + \int_s^t (\mathbf{f},\phi)dr + \int_s^t \langle\phi, dW_r\rangle.\end{aligned} \tag{4.5}$$

[1] By $\mathcal{F}^u$ we understand the σ-algebra of universally measurable sets associated to the measurable space $(\Omega,\mathcal{F})$, see the monograph [18] by Crauel.

Proposition 4.8. *In the framework as above, suppose that* $\mathbf{u}(t) = \mathbf{z}_\alpha(t) + \mathbf{v}_\alpha(t)$, $t \geq s$, *where* $\mathbf{v}_\alpha$ *is the unique solution to problem* (3.3)–(3.4) *with initial data* $\mathbf{u}_0 - \mathbf{z}_\alpha(s)$ *at time* s*. If the process* $\mathbf{u}(t)$, $t \geq s$, *has trajectories in* $C([s,\infty);H) \cap L^2_{\rm loc}([s,\infty);V) \cap L^2_{\rm loc}([s,\infty);\mathbb{L}^4(\mathbb{S}^2))$, *then it is a solution to problem* (3.1)*. Vice-versa, if a process* $\mathbf{u}(t)$, $t \geq s$, *with trajectories in* $C([s,\infty);H) \cap L^2_{\rm loc}([s,\infty);V) \cap L^2_{\rm loc}([s,\infty);\mathbb{L}^4(\mathbb{S}^2))$ *is a solution to problem* (3.1)*, then for any* $\alpha \geq 0$*, a process* $\mathbf{v}_\alpha(t)$, $t \geq s$, *defined by* $\mathbf{z}_\alpha(t) = \mathbf{u}(t) - \mathbf{v}_\alpha(t)$, $t \geq s$, *is a solution to* (3.3) *on* $[s,\infty)$.

Our previous results yield the existence and the uniqueness of solutions to problem (3.1) as well as its continuous dependence on the data (in particular on the initial value $\mathbf{u}_0$ and the force $\mathbf{f}$). Moreover, if we define, for $\mathbf{x} \in H$, $\omega \in \Omega$, and $t \geq s$,

$$\mathbf{u}(t,s;\omega,\mathbf{u}_0) := \varphi(t-s;\vartheta_s\omega)\mathbf{u}_0 = \mathbf{v}(t,s;\omega,\mathbf{u}_0 - \mathbf{z}(s)) + \mathbf{z}(t), \tag{4.6}$$

then for each $s \in \mathbb{R}$ and each $\mathbf{u}_0 \in H$, the process $\mathbf{u}(t)$, $t \geq s$, is a solution to problem (3.1).

Before presenting the main results of this section, let us recall the weak continuity of the RDS generated by stochastic NSEs on the sphere as stated in the last part of Theorem 3.3.

We have the Poincaré inequalities

$$\begin{aligned} \|\mathbf{u}\|_V^2 &\geq \lambda_1 \|\mathbf{u}\|^2, \quad \text{for all } \mathbf{u} \in V, \\ \|\mathbf{A}\mathbf{u}\|^2 &\geq \lambda_1 \|\mathbf{u}\|^2, \quad \text{for all } \mathbf{u} \in \mathcal{D}(A) \cap V. \end{aligned} \tag{4.7}$$

For any $\mathbf{u}, \mathbf{v} \in V$, we define a new scalar product $[\cdot,\cdot] : V \times V \to \mathbb{R}$ by the formula $[\mathbf{u},\mathbf{v}] = \nu(\mathbf{u},\mathbf{v})_V - \nu\frac{\lambda_1}{2}(\mathbf{u},\mathbf{v})$. Clearly, $[\cdot,\cdot]$ is bilinear and symmetric. From (2.22), we can prove that $[\cdot,\cdot]$ define an inner product in V with the norm $[\cdot] = [\cdot,\cdot]^{1/2}$, which is equivalent to the norm $\|\cdot\|_V$.

Lemma 4.9. *Suppose that* $\mathbf{v}$ *is a solution to problem* (3.3) *on the time interval* $[a,\infty)$ *with* $\mathbf{z} \in L^4_{\rm loc}(\mathbb{R}^+,\mathbb{L}^4(\mathbb{S}^2)) \cap L^2_{\rm loc}(\mathbb{R}^+,V')$ *and* $\alpha \geq 0$*. Denote* $\mathbf{g}(t) = \alpha\mathbf{z}(t) - \mathbf{B}(\mathbf{z}(t),\mathbf{z}(t))$, $t \in [a,\infty)$*. Then, for any* $t \geq \tau \geq a$,

$$\begin{aligned} \|\mathbf{v}(t)\|^2 &\leq \|\mathbf{v}(\tau)\|^2 e^{-\nu\lambda_1(t-\tau) + \frac{3C^2}{\nu}\int_\tau^t \|\mathbf{z}(s)\|^2_{\mathbb{L}^4})ds} \\ &\quad + \frac{3}{\nu}\int_\tau^t (\|\mathbf{g}(s)\|^2_{V'} + \|\mathbf{f}\|^2) e^{-\nu\lambda_1(t-\tau) + \frac{3C^2}{\nu}\int_s^t \|\mathbf{z}(\xi)\|^2_{\mathbb{L}^4})d\xi} ds \end{aligned} \tag{4.8}$$

$$\begin{aligned} \|\mathbf{v}(t)\|^2 &= \|\mathbf{v}(\tau)\|^2 e^{-\nu\lambda_1(t-\tau)} \\ &\quad + 2\int_\tau^t e^{-\nu\lambda_1(t-s)} (b(\mathbf{v}(s),\mathbf{z}(s),\mathbf{v}(t)) + \langle \mathbf{g}(s),\mathbf{v}(s)\rangle + \langle \mathbf{f},\mathbf{v}(s)\rangle - [\mathbf{v}(s)]^2) ds \end{aligned} \tag{4.9}$$

Proof. By [41, Lemma III.1.2], we have $\frac{1}{2}\partial_t\|\mathbf{v}(t)\| = (\mathbf{v}(t),\mathbf{v}(t))$. Hence

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 &= \nu(\mathbf{A}\mathbf{v},\mathbf{v}) - (\mathbf{C}\mathbf{v},\mathbf{v}) - (\mathbf{B}(\mathbf{v},\mathbf{v}),\mathbf{v}) - (B(\mathbf{z},\mathbf{v}),\mathbf{v}) \\ &\qquad - (B(\mathbf{v},\mathbf{z}),\mathbf{v}) + \langle \mathbf{g},\mathbf{v}\rangle + \langle \mathbf{f},\mathbf{v}\rangle \\ &= \nu\|\mathbf{v}\|_V^2 - b(\mathbf{v},\mathbf{z},\mathbf{v}) + \langle \mathbf{g},\mathbf{v}\rangle + \langle \mathbf{f},\mathbf{v}\rangle. \end{aligned} \tag{4.10}$$

From (2.46) and invoking the Young inequality, we have

$$\begin{aligned}|b(\mathbf{v},\mathbf{z},\mathbf{v})| &\le C\|\mathbf{v}\|_{\mathbb{L}^4}\|\mathbf{v}\|_V\|\mathbf{z}\|_{\mathbb{L}^4}\\ &\le \frac{\nu}{6}\|\mathbf{v}\|_V^2 + \frac{3C^2}{2\nu}\|\mathbf{v}\|^2\|\mathbf{z}\|_{\mathbb{L}^4}^2,\end{aligned}$$

and

$$\begin{aligned}|\langle \mathbf{g},\mathbf{v}\rangle + \langle \mathbf{f},\mathbf{v}\rangle| &\le \|\mathbf{g}\|_{V'}\|\mathbf{v}\|_V + \|\mathbf{f}\|_{V'}\|\mathbf{v}\|_V\\ &\le \frac{\nu}{3}\|\mathbf{v}\|^2 + \frac{3}{2\nu}\|g\|_{V'}^2 + \frac{3}{2\nu}\|\mathbf{f}\|_{V'}^2.\end{aligned}$$

Hence from (4.10) and (4.7), we get

$$\begin{aligned}\frac{d}{dt}\|\mathbf{v}(t)\|^2 &\le -\nu\|\mathbf{v}(t)\|^2 + \frac{3C^2}{\nu}\|\mathbf{z}(t)\|_{\mathbb{L}^4}^2\|\mathbf{v}(t)\|^2 + \frac{3}{\nu}\|\mathbf{g}(t)\|_{V'}^2 + \frac{3}{\nu}\|\mathbf{f}\|_{V'}^2\\ &\le \left(-\nu\lambda_1 + \frac{3C^2}{\nu}\|\mathbf{z}(t)\|_{\mathbb{L}^4}^2\right)\|\mathbf{v}(t)\|^2 + \frac{3}{\nu}\|\mathbf{g}(t)\|_{V'}^2 + \frac{3}{\nu}\|\mathbf{f}\|_{V'}^2.\end{aligned}$$

Next, using the Gronwall Lemma, we arrive at (4.8).

By adding and subtracting $\nu\frac{\lambda_1}{2}\|\mathbf{v}(t)\|^2$ from (4.10) we find that

$$\frac{d}{dt}\|\mathbf{v}(t)\|^2 + \nu\lambda_1\|\mathbf{v}(t)\|^2 + 2[\mathbf{v}(t)]^2 \tag{4.11}$$

$$= 2b(\mathbf{v}(t),\mathbf{z}(t),\mathbf{v}(t)) + 2\langle \mathbf{g}(t),\mathbf{v}(t)\rangle + 2\langle \mathbf{f}(t),\mathbf{v}(t)\rangle. \tag{4.12}$$

Hence (4.9) follows by the variation of constants formula. □

Lemma 4.10. *Under the above assumptions, for each $\omega \in \Omega(\xi, E)$,*

$$\lim_{t\to-\infty}\|\mathbf{z}(\omega)(t)\|^2 e^{\nu\lambda_1 t + \int_t^0 \frac{3C^2}{\nu}\|\mathbf{z}(\omega)(s)\|_{\mathbb{L}^4}^2 ds} = 0.$$

Lemma 4.11. *Under the above assumptions, for each $\omega \in \Omega(\xi, E)$,*

$$\int_{-\infty}^0 [1 + \|\mathbf{z}(\omega)(t)\|_{\mathbb{L}^4}^2 + \|\mathbf{z}(\omega)(t)\|_{\mathbb{L}^4}^4] e^{\nu\lambda_1 t + \int_t^0 \frac{3C^2}{\nu}\|\mathbf{z}(\omega)(s)\|_{\mathbb{L}^4}^2 ds} < \infty.$$

Definition 4.12. *A function $r : \Omega \to (0,\infty)$ belongs to the class $\mathfrak{R}$ if and only if*

$$\limsup_{t\to\infty} r(\vartheta_{-t}\omega)^2 e^{-\nu\lambda_1 t + \int_t^0 \frac{3C^2}{\nu}\|\mathbf{z}(\omega)(s)\|_{\mathbb{L}^4}^2 ds} = 0,$$

where $C > 0$ is the constant appearing in (3.23).

We denote by $\mathfrak{DR}$ the class of all closed and bounded random sets D on H such that the function $\omega \mapsto r(D(\omega)) := \sup\{\|\mathbf{x}\|_H : \mathbf{x} \in D(\omega)\}$ belongs to the class $\mathfrak{R}$.

Proposition 4.13. *Define functions $r_i : \Omega \to (0,\infty)$, $i = 1,2,3,4,5$ by the following formulae, for $\omega \in \Omega$,*

$$\begin{aligned}
r_1^2(\omega) &:= \|\mathbf{z}(\omega)(0)\|_H^2, \\
r_2^2(\omega) &:= \sup_{s\le 0} \|\mathbf{z}(\omega)(s)\|_H^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr} \\
r_3^2(\omega) &:= \int_{-\infty}^0 \|\mathbf{z}(\omega)(s)\|_H^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}\,ds \\
r_4^2(\omega) &:= \int_{-\infty}^0 \|\mathbf{z}(\omega)(s)\|_{\mathbb{L}^4}^4 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}\,ds \\
r_5^2(\omega) &:= \int_{-\infty}^0 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}\,ds.
\end{aligned}$$

Then all these functions belong to the class $\mathfrak{R}$.

Proof. Since by Theorem 3.12, $\mathbf{z}(\vartheta_{-t}\omega)(s) = \mathbf{z}(\omega)(s-t)$, we have

$$\begin{aligned}
r_2^2(\vartheta_{-t}\omega) &= \sup_{s\le 0} \|\mathbf{z}(\vartheta_{-t}\omega)(s)\|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\vartheta_{-t}\omega)(r)\|_{\mathbb{L}^4}^2\,dr} \\
&= \sup_{s\le 0} \|\mathbf{z}(\omega)(s-t)\|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\omega)(r-t)\|_{\mathbb{L}^4}^2\,dr} \\
&= \sup_{s\le 0} \|\mathbf{z}(\omega)(s-t)\|^2 e^{\nu\lambda_1 (s-t) + \frac{3C^2}{\nu}\int_{s-t}^{-t} \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr} e^{\nu\lambda_1 t} \\
&= \sup_{\sigma\le -t} \|\mathbf{z}(\omega)(\sigma)\|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{-t} \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr} e^{\nu\lambda_1 t}
\end{aligned}$$

Hence, multiplying the above by $e^{-\nu\lambda_1 t} e^{\frac{3C^2}{\nu}\int_{-t}^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}$ we get

$$r_2^2(\vartheta_{-t}\omega) e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr} \le \sup_{\sigma\le -t} \|\mathbf{z}(\omega)(\sigma)\|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}.$$

This, together with Lemma 4.10 concludes the proof in the case of function r_2. In the case of r_1, we have

$$r_1^2(\vartheta_{-t}\omega) e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr} = \|\mathbf{z}(\omega)(-t)\|^2 e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}.$$

Thus, by Lemma 4.10 we infer that r_1 also belongs to the class $\mathfrak{R}$. The argument in the case of function r_3 is similar but for the sake of the completeness we include it here. From the first part of the proof we infer that

$$r_3^2(\vartheta_{-t}\omega) e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr} \le \int_{-\infty}^{-t} \|\mathbf{z}(\omega)(\sigma)\|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}\,d\sigma.$$

Since by Lemma 4.11 $\int_{-\infty}^0 \|\mathbf{z}(\omega)(\sigma)\|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}\,d\sigma$ is finite, by the Lebesgue Monotone Convergence Theorem we conclude that

$$\int_{-\infty}^{-t} \|\mathbf{z}(\omega)(\sigma)\|^2 e^{\nu\lambda_1 \sigma + \frac{3C^2}{\nu}\int_{\sigma}^{0} \|\mathbf{z}(\omega)(r)\|_{\mathbb{L}^4}^2\,dr}\,d\sigma \to 0 \text{ as } t\to\infty.$$

The proof in the other cases is analogous. $\square$

We have the following trivial results.

Proposition 4.14. *The class $\mathfrak{R}$ is closed with respect to sum, multiplication by a constant and if $r \in \mathfrak{R}$, $0 \le \bar{r} \le r$, then $\bar{r} \in \mathfrak{R}$. The class $\mathfrak{R}$ is closed with respect to sum, multiplication by a constant and if $r \in \mathfrak{R}$, $0 \le \bar{r} \le r$, then $\bar{r} \in \mathfrak{R}$.*

Now we are ready to state and prove the main result of this paper. A result of similar type for the Navier–Stokes equations on some 2-dimensional unbounded domain has been discussed in [8].

Theorem 4.15. *Consider the metric DS $\mathfrak{T} = \left(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\right)$ from Proposition* 3.15*, and the RDS φ on H over $\mathfrak{T}$ generated by the stochastic Navier-Stokes equations on the* 2*-dimensional unit sphere with additive noise* (3.1) *satisfying Assumption* 3.8*. Then the following properties hold.*

(i) *there exists a $\mathfrak{DR}$-absorbing set $B \in \mathfrak{DR}$;*
(ii) *the RDS φ is $\mathfrak{DR}$-asymptotically compact;*
(iii) *the family A of sets defined by $A(\omega) = \Omega_B(\omega)$ for all $\omega \in \Omega$, is the minimal $\mathfrak{DR}$-attractor for φ, is $\hat{\mathcal{F}}$-measurable, and*

$$A(\omega) = \overline{\bigcup_{C \subset H} \Omega_C(\omega)} \quad \hat{\mathbb{P}} - a.s., \tag{4.13}$$

where the union in (4.13) *is made for all bounded and closed nonempty deterministic subsets C of H.*

Proof. In view of Theorem 4.5 and Remark 4.6, it is enough to show (i) and (ii). The proof of (ii) will be done in the next proposition.
Proof of (i)

With a fixed $\omega \in \Omega$, let $D(\omega)$ be a random set from the class $\mathfrak{DR}$ with radius $r_D(\omega)$, i.e. $r_D(\omega) := \sup\{|\mathbf{x}|_H : x \in D(\omega)\}$.

For given $s \le 0$ and $\mathbf{x} \in H$, let $\mathbf{v}$ be the solution of (3.3) on time interval $[s, \infty)$ with the initial condition $\mathbf{v}(s) = \mathbf{x} - \mathbf{z}(s)$. By applying (4.8) with $t = 0, \tau = s \le 0$, we get

$$\begin{aligned} \|\mathbf{v}(0)\|^2 \quad &\le \quad 2\|\mathbf{x}\|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 \|\mathbf{z}(r)\|_{\mathbb{L}^4}^2 \, dr} + 2\|\mathbf{z}(s)\|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 \|\mathbf{z}(r)\|_{\mathbb{L}^4}^2 \, dr} \\ &+ \quad \frac{3}{\nu} \int_s^0 \{\|\mathbf{g}(t)\|_{\mathrm{V}'}^2 + \|\mathbf{f}\|_{\mathrm{V}'}^2\} e^{\nu\lambda_1 t + \frac{3C^2}{\nu} \int_t^0 \|\mathbf{z}(r)\|_{\mathbb{L}^4}^2 \, dr} \, dt. \end{aligned} \tag{4.14}$$

Set, for $\omega \in \Omega$,

$$\begin{aligned} r_{11}(\omega)^2 = 2 + \sup_{s \le 0} \Big\{ & 2\|\mathbf{z}(s)\|^2 e^{\nu\lambda_1 s + \frac{3C^2}{\nu} \int_s^0 \|\mathbf{z}(r)\|_{\mathbb{L}^4}^2 \, dr} \\ + \frac{3}{\nu} \int_s^0 & \{\|\mathbf{g}(t)\|_{\mathrm{V}'}^2 + \|\mathbf{f}\|_{\mathrm{V}'}^2\} e^{\nu\lambda_1 t + \frac{3C^2}{\nu} \int_t^0 \|\mathbf{z}(r)\|_{\mathbb{L}^4}^2 \, dr} \, dt \Big\}, \end{aligned} \tag{4.15}$$

and

$$r_{12}(\omega) = \|\mathbf{z}(0)(\omega)\|_H. \tag{4.16}$$

Using Lemma 4.11 and Proposition 4.13 we conclude that both r_{11} and r_{12} belong to $\mathfrak{R}$ and that $r_{13} := r_{11} + r_{12}$ belongs to $\mathfrak{R}$ as well. Therefore, the random set B defined by $B(\omega) := \{\mathbf{u} \in H : \|\mathbf{u}\| \le r_{13}(\omega)\}$ belongs to the family $\mathfrak{DR}$.

Now we will show that B absorbs D. Let $\omega \in \Omega$ be fixed. Since $r_D \in \mathfrak{R}$ there exists $t_D(\omega) \ge 0$, such that

$$r_D(\vartheta_{-t}\omega)^2 e^{-\nu\lambda_1 t + \frac{3C^2}{\nu}\int_{-t}^0 \|\mathbf{z}(\omega)(s)\|_{\mathbb{L}^4}^2\,ds} \le 1, \text{ for } t \ge t_D(\omega).$$

Thus, if $\mathbf{x} \in D(\vartheta_{-t}\omega)$ and $s \ge t_D(\omega)$, then by (4.14), $\|\mathbf{v}(0,s;\omega,\mathbf{x}-\mathbf{z}(s))\| \le r_{11}(\omega)$. Thus we infer that

$$\|\mathbf{u}(0,s;\omega,\mathbf{x})\| \le \|\mathbf{v}(0,s;\omega,\mathbf{x}-\mathbf{z}(s))\| + \|\mathbf{z}(0)(\omega)\| \le r_{13}(\omega).$$

In other words, $\mathbf{u}(0,s;\omega,\mathbf{x}) \in B(\omega)$, for all $s \ge t_D(\omega)$. This proves that B absorbs D. □

Proposition 4.16. *Assume that for each random set D belonging to $\mathfrak{DR}$, there exists a random set B belonging to $\mathfrak{DR}$ such that B absorbs D. Then the RDS φ is $\mathfrak{DR}$-asymptotically compact.*

Let us recall that the RDS φ is independent of the auxiliary parameter $\alpha \in \mathbb{N}$. For reasons that will become clear later, we choose α such that $\mathbb{E}\|\mathbf{z}_\alpha(0)\|_{\mathbb{L}^4}^2 \le \frac{\nu^2\lambda_1}{6C^2}$, where $\mathbf{z}_\alpha(t)$, $t \in \mathbb{R}$, is the Ornstein-Uhlenbeck process from subsection 3.2, $C > 0$ is a certain universal constant, λ_1 is the constant from (4.7) and $\nu > 0$ is the viscosity. It follows from Proposition 3.14 that such a choice is possible.

The proof of the proposition is adapted from [8], in which a RDS generated by NSEs on some 2-dimensional unbounded domain was considered. The proposition generalises the asymptotically compactness of the RDS in [11, Proposition 8.1] to the $\mathfrak{DR}$- asymptotically compactness of the RDS.

Proof. Suppose that B is a closed random set from the class $\mathfrak{DR}$ and $K \in \mathfrak{DR}$ is a close random set which absorbs B. We fix $\omega \in \Omega$. Let us take an increasing sequence of positive numbers $(t_n)_{n=1}^\infty$ such that $t_n \to \infty$ and an H-valued sequence $(\mathbf{x}_n)_n$ such that $\mathbf{x}_n \in B(\vartheta_{-t_n}\omega)$, for all $n \in \mathbb{N}$.

Step I. Reduction. Since $K(\omega)$ absorbs B, for $n \in \mathbb{N}$ sufficiently large, $\varphi(t_n, \vartheta_{-t_n}\omega)B \subset K(\omega)$. Since $K(\omega)$ is closed and bounded, and hence weakly compact, without loss of generality we may assume that $\varphi(t_n, \vartheta_{-t_n}\omega)B \subset K(\omega)$ for all $n \in \mathbb{N}$ and, for some $\mathbf{y}_0 \in K(\omega)$,

$$\varphi(t_n, \vartheta_{-t_n}\omega)\mathbf{x}_n \to \mathbf{y}_0 \quad \text{weakly in } H. \tag{4.17}$$

Since $\mathbf{z}(0) \in H$, we also have

$$\varphi(t_n, \vartheta_{-t_n}\omega)\mathbf{x}_n - \mathbf{z}(0) \to \mathbf{y}_0 - \mathbf{z}(0) \quad \text{weakly in } H.$$

In particular,

$$\|\mathbf{y}_0 - \mathbf{z}(0)\| \le \liminf_{n\to\infty} \|\varphi(t_n, \vartheta_{-t_n}\omega)\mathbf{x}_n - \mathbf{z}(0)\|. \tag{4.18}$$

We claim that it is enough to prove that for some subsequence $\{n'\} \subset \mathbb{N}$

$$\|\mathbf{y}_0 - \mathbf{z}(0)\| \ge \limsup_{n'\to\infty} \|\varphi(t_{n'}, \vartheta_{-t_{n'}}\omega)\mathbf{x}_{n'} - \mathbf{z}(0)\|. \tag{4.19}$$

Indeed, since H is a Hilbert space, (4.18) in conjunction with (4.19) imply that

$$\varphi(t_n, \vartheta_{-t_n}\omega)\mathbf{x}_n - \mathbf{z}(0) \to \mathbf{y}_0 - \mathbf{z}(0) \quad \text{strongly in } H$$

which implies that

$$\varphi(t_n, \vartheta_{-t_n}\omega)\mathbf{x}_n \to \mathbf{y}_0 \quad \text{strongly in } H.$$

Therefore, in order to show that $\{\varphi(t_n, \vartheta_{-t_n}\omega)\mathbf{x}_n\}_n$ is relatively compact in H we need to prove that (4.19) holds true.

Step II. Construction of a negative trajectory, i.e. a sequence $(\mathbf{y}_n)_{n=-\infty}^0$ such that $\mathbf{y}_n \in K(\vartheta_n\omega)$, $n \in \mathbb{Z}^-$, and $\mathbf{y}_k = \varphi(k-n, \vartheta_n\omega)\mathbf{y}_n$, $n < k \leq 0$.

Since $K(\vartheta_{-1}\omega)$ absorbs B, there exists a constant $N_1(\omega) \in \mathbb{N}$, such that

$$\{\varphi(-1+t_n, \vartheta_{1-t_n}\vartheta_{-1}\omega)\mathbf{x}_n : n \geq N_1(\omega)\} \subset K(\vartheta_{-1}\omega).$$

Hence we can find a subsequence $\{n'\} \subset \mathbb{N}$ and $\mathbf{y}_{-1} \in K(\vartheta_{-1}\omega)$ such that

$$\varphi(-1+t_{n'}, \vartheta_{-t_{n'}}\omega)\mathbf{x}_{n'} \to \mathbf{y}_{-1} \text{ weakly in } H. \tag{4.20}$$

We observe that the cocycle property, with $t = 1$, $s = t_{n'} - 1$, and ω being replaced by $\vartheta_{-t_{n'}}\omega$, reads as follows:

$$\varphi(t_{n'}, \vartheta_{-t_{n'}}\omega) = \varphi(1, \vartheta_{-1}\omega)\varphi(-1+t_{n'}, \vartheta_{t_{n'}}\omega).$$

Hence, by the last part of Theorem 3.3, from (4.17) and (4.20) we infer that $\varphi(1, \vartheta_{-1}\omega)\mathbf{y}_{-1} = \mathbf{y}_0$. By induction, for each $k = 1, 2, \ldots$, we can construct a subsequence $\{n^{(k)}\} \subset \{n^{(k-1)}\}$ and $\mathbf{y}_{-k} \in K(\vartheta_{-k}\omega)$, such that $\varphi(1, \vartheta_{-k}\omega)\mathbf{y}_{-k} = \mathbf{y}_{-k+1}$ and

$$\varphi(-k+t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}} \to \mathbf{y}_{-k} \text{ weakly in } H, \text{ as } n^{(k)} \to \infty. \tag{4.21}$$

As above, the cocycle property with $t = k$, $s = t_{n^{(k)}}$ and ω being replaced by $\vartheta_{-t_{n^{(k)}}}\omega$ yields

$$\varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega) = \varphi(k, \vartheta_{-k}\omega)\varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega), \quad k \in \mathbb{N}. \tag{4.22}$$

Hence, from (4.21) and by applying the last part of Theorem 3.3, we get

$$\begin{aligned}
\mathbf{y}_0 &= \mathrm{w} - \lim_{n^{(k)} \to \infty} \varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}} \\
&= \mathrm{w} - \lim_{n^{(k)} \to \infty} \varphi(k, \vartheta_{-k}\omega)\varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}} \\
&= \varphi(k, \vartheta_{-k}\omega)(\mathrm{w} - \lim_{n^{(k)} \to \infty} \varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}) \\
&= \varphi(k, \vartheta_{-k}\omega)\mathbf{y}_{-k},
\end{aligned} \tag{4.23}$$

where w-lim denotes the limit in the weak topology on H. The same proof yields a more general property:

$$\varphi(j, \vartheta_{-k}\omega)\mathbf{y}_{-k} = \mathbf{y}_{-k+j} \text{ if } 0 \leq j \leq k.$$

Before continuing with the proof, let us point out that (4.23) means precisely that $\mathbf{y}_0 = \mathbf{u}(0, -k; \omega, \mathbf{y}_{-k})$, where $\mathbf{u}$ is defined in (4.6).

Step III. Proof of (4.19). From now on, unless explicitly stated, we fix $k\in\mathbb{N}$, and we will consider problem (3.1) on the time interval $[-k,0]$. From (4.6) and (4.22), with $t=0$ and $s=-k$, we have

$$\begin{aligned} &\|\varphi(t_{n^{(k)}},\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(0)\|^2\\ &\quad=\|\varphi(k,\vartheta_{-k}\omega)\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(0)\|^2\\ &\quad=\|\mathbf{v}(0,-k;\omega,\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(-k))\|^2. \end{aligned} \tag{4.24}$$

Let $\mathbf{v}$ be the solution to (3.3) on $[-k,\infty)$ with $\mathbf{z}=\mathbf{z}_\alpha(\cdot,\omega)$ and the initial condition at time $-k$: $\mathbf{v}(-k)=\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(-k)$. In other words,

$$\mathbf{v}(s)=\mathbf{v}\big(s,-k;\omega,\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(-k)\big),\quad s\geq -k.$$

From (4.24) and (4.9) with $t=0$ and $\tau=-k$ we infer that

$$\begin{aligned} &\|\varphi(t_{n^{(k)}},\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(0)\|^2=e^{-\nu\lambda_1 k}\|\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(-k)\|^2\\ &+2\int_{-k}^0 e^{\nu\lambda_1 s}(b(\mathbf{v}(s),\mathbf{z}(s),\mathbf{v}(s))+\langle \mathbf{g}(s),\mathbf{v}(s)\rangle+\langle \mathbf{f},\mathbf{v}(s)\rangle-[\mathbf{v}(s)]^2)ds. \end{aligned} \tag{4.25}$$

It is enough to find a nonnegative function $h\in L^1(-\infty,0)$ such that

$$\limsup_{n^{(k)}\to\infty}\|\varphi(t_{n^{(k)}},\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(0)\|^2\leq\int_{-\infty}^{-k}h(s)ds+\|\mathbf{y}_0-\mathbf{z}(0)\|^2. \tag{4.26}$$

For, if we define the diagonal process $(m_j)_{j=1}^\infty$ by $m_j=j^{(j)}$, $j\in\mathbb{N}$, then for each $k\in\mathbb{N}$, the sequence $(m_j)_{j=k}^\infty$ is a subsequence of the sequence $(n^{(k)})$ and hence by (4.26), $\limsup_j\|\varphi(t_{m_j},\vartheta_{-t_{m_j}}\omega)\mathbf{x}_{m_j}-\mathbf{z}(0)\|^2\leq\int_{-\infty}^{-k}h(s)ds+\|\mathbf{y}_0-\mathbf{z}(0)\|^2$. Taking the $k\to\infty$ limit in the last inequality we infer that

$$\limsup_j\|\varphi(t_{m_j},\vartheta_{-t_{m_j}}\omega)\mathbf{x}_{m_j}-\mathbf{z}(0)\|^2\leq\|\mathbf{y}_0-\mathbf{z}(0)\|^2,$$

which proves claim (4.19).

Step IV. Proof of (4.26). We begin with estimating the first term on the RHS of (4.25). If $-t_{n^{(k)}}<-k$, then by (4.6) and (4.8) we infer that

$$\begin{aligned} &\|\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}}-\mathbf{z}(-k)\|^2\\ &\quad=\|\mathbf{v}(-k,-t_{n^{(k)}};\vartheta_{-k}\omega,\mathbf{x}_{n^{(k)}}-\mathbf{z}(-t_{n^{(k)}}))\|^2e^{-\nu\lambda_1 k}\\ &\leq e^{-\nu\lambda_1 k}\big\{\|\mathbf{x}_{n^{(k)}}-\mathbf{z}(-t_{n^{(k)}})\|^2e^{-\nu\lambda_1(t_{n^{(k)}}-k)+\frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k}\|\mathbf{z}(s)\|^2_{\mathbb{L}^4}ds}\\ &\quad+\frac{3}{\nu}\int_{-t_{n^{(k)}}}^{-k}[\|\mathbf{g}(s)\|^2_{V'}+\|\mathbf{f}\|^2_{V'}]e^{-\nu\lambda_1(-k-s)+\frac{3C^2}{\nu}\int_s^{-k}\|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4}d\zeta}\big\}\\ &\leq 2I_{n^{(k)}}+2II_{n^{(k)}}+\frac{3}{\nu}III_{n^{(k)}}+\frac{3}{\nu}IV_{n^{(k)}}, \end{aligned} \tag{4.27}$$

where

$$
\begin{aligned}
I_{n^{(k)}} &= \|\mathbf{x}_{n^{(k)}}\|^2 e^{-\nu\lambda_1 t_{n^{(k)}} + \frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k} \|\mathbf{z}(s)\|^2_{\mathbb{L}^4} ds} \\
II_{n^{(k)}} &= \|\mathbf{z}(t_{n^{(k)}})\|^2 e^{-\nu\lambda_1 t_{n^{(k)}} + \frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k} \|\mathbf{z}(s)\|^2_{\mathbb{L}^4} ds} \\
III_{n^{(k)}} &= \int_{-t_{n^{(k)}}}^{-k} \|\mathbf{g}(s)\|^2_{V'} e^{-\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^{-k} \|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4} d\zeta} \\
IV_{n^{(k)}} &= \int_{-t_{n^{(k)}}}^{-k} \|\mathbf{f}(s)\|^2_{V'} e^{-\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^{-k} \|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4} d\zeta}
\end{aligned}
$$

First we will find a nonnegative function $h \in L^1(-\infty, 0)$ such that

$$
\limsup_{n^{(k)}\to\infty} \|\varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}} - \mathbf{z}(-k)\|^2 e^{-\nu\lambda_1 k} \le \int_{-\infty}^{-k} h(s)ds, \quad k \in \mathbb{N}. \tag{4.28}
$$

This will be accomplished as soon as we prove the following four lemmas.

Lemma 4.17. $\limsup_{n^{(k)}\to\infty} I_{n^{(k)}} = 0.$

Lemma 4.18. $\limsup_{n^{(k)}\to\infty} II_{n^{(k)}} = 0.$

Lemma 4.19. $\int_{-\infty}^0 \|\mathbf{g}(s)\|^2_{V'} e^{-\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4} d\zeta} < \infty.$

Lemma 4.20. $\int_{-\infty}^0 e^{-\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4} d\zeta} < \infty.$

Proof of Lemma 4.17. We recall that for $\alpha \in \mathbb{N}$, $\mathbf{z}(t) = \mathbf{z}_\alpha(t)$, $t \in \mathbb{R}$, being the Ornstein-Uhlenbeck process from subsection 3.2, one has

$$
\mathbb{E}\|\mathbf{z}(0)\|^2_X = \mathbb{E}\|\mathbf{z}_\alpha(0)\|^2_X < \frac{\nu^2\lambda_1}{6C^2}.
$$

Let us recall that the space $\hat{\Omega}(\xi, E)$ was constructed in such a way that

$$
\lim_{n^{(k)}\to\infty} \frac{1}{-k - (-t_{n^{(k)}})} \int_{t_{n^{(k)}}}^{-k} \|\mathbf{z}_\alpha(s)\|^2_X ds = \mathbb{E}\|\mathbf{z}(0)\|^2_X < \infty.
$$

Therefore, since the embedding $X \hookrightarrow \mathbb{L}^4(\mathbb{S}^2)$ is a contraction, we have for $n^{(k)}$ sufficiently large,

$$
\frac{3C^2}{\nu}\int_{t_{n^{(k)}}}^{-k} \|\mathbf{z}_\alpha(s)\|^2_{\mathbb{L}^4} ds < \frac{\nu\lambda_1}{2}(t_{n^{(k)}} - k). \tag{4.29}
$$

Since the set B is bounded in H, there exists $\rho_1 > 0$ such that for all $n^{(k)}$, $\|\mathbf{x}_{n^{(k)}}\| \le \rho_1$. Hence

$$
\limsup_{n^{(k)}\to\infty} \|\mathbf{x}_{n^{(k)}}\|^2 e^{-\nu\lambda_1 t_{n^{(k)}} + \frac{3C^2}{\nu}\int_{-t_{n^{(k)}}}^{-k} \|\mathbf{z}(s)\|^2_{\mathbb{L}^4} ds} \le \limsup_{n^{(k)}\to\infty} \rho_1^2 e^{-\frac{\nu\lambda_1}{2}(t_{n^{(k)}} - k)} = 0. \tag{4.30}
$$

$\square$

Proof of Lemma 4.20. We denote by $p(s) = \nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(s)\|^2_{\mathbb{L}^4}$. As in the proof of Lemma 4.17 we have, for $s \le s_0$, $p(s) < \frac{\nu\lambda_1}{2}s$. Hence $\int_{-\infty}^0 e^{p(s)}ds < \infty$, as required. □

Proof of Lemma 4.18. Because of (3.14), we can find $\rho_2 \ge 0$ and $s_0 < 0$, such that,

$$\max\left(\frac{\|\mathbf{z}(s)\|}{|s|}, \frac{\|\mathbf{z}(s)\|_{V'}}{|s|}, \frac{\|\mathbf{z}(s)\|_{\mathbb{L}^4}}{|s|}\right) \le \rho_2, \qquad \text{for } s \le s_0. \tag{4.31}$$

Hence by (4.29) we infer that

$$\begin{aligned}
&\limsup_{n^{(k)}\to\infty} \|\mathbf{z}(-t_{n^{(k)}})\|^2 e^{\int_{-t_{n^{(k)}}}^{-k}(-\nu\lambda_1 + \frac{3C^2}{\nu}\|\mathbf{z}(s)\|^2)ds} \\
&\quad \le \limsup_{n^{(k)}\to\infty} \frac{\|\mathbf{z}(-t_{n^{(k)}})\|^2}{|t_{n^{(k)}}|^2} \limsup_{n^{(k)}\to\infty} |t_{n^{(k)}}|^2 e^{-\frac{\nu\lambda_1}{2}(t_{n^{(k)}}-k)} \le 0.
\end{aligned} \tag{4.32}$$

This concludes the proof of Lemma 4.18. □

Proof of Lemma 4.19. Since $\|\mathbf{g}(s)\|^2_{V'} = \|\alpha\mathbf{z}(s) + 2\mathbf{B}(\mathbf{z}(s))\|^2_{V'} \le 2\alpha^2\|\mathbf{z}(s)\|^2_{V'} + 2C\|\mathbf{z}(s)\|^4_{\mathbb{L}^4}$, we only need to show that the integrals

$$\int_{-\infty}^0 \|\mathbf{z}(s)\|^4_{\mathbb{L}^4} e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4}d\zeta}ds \text{ and } \int_{-\infty}^0 \|\mathbf{z}(s)\|^2_{V'} e^{\nu\lambda_1 s + \frac{3C^2}{\nu}\int_s^0 \|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4}d\zeta}ds$$

are finite.

It is enough to consider the case of $\|\mathbf{z}(s)\|^4_{\mathbb{L}^4}$ since the proof will be similar for the remaining case. Reasoning as in (4.29), we can find $t_0 \ge 0$ such that for $t \ge t_0$,

$$\int_{-t}^{-t_0}\left(-\nu\lambda_1 + \frac{3C^2}{\nu}\|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4}\right)d\zeta \le -\frac{\nu\lambda_1}{2}(t - t_0).$$

Taking into account the inequality (4.31), we have $\|\mathbf{z}(t)\| \le \rho_2(1 + |t|)$, $t \in \mathbb{R}$. Therefore, with $\rho_3 := \exp(\int_{-t_0}^0(-\nu\lambda_1 + \frac{3C^2}{\nu}\|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4})d\zeta$, we have

$$\begin{aligned}
&\int_{-\infty}^{-t_0} \|\mathbf{z}(s)\|^4_{\mathbb{L}^4} e^{\int_s^0(\nu\lambda_1 + \frac{3C^2}{\nu}\|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4})d\zeta}ds \\
&\qquad = \rho_3 \int_{-\infty}^{-t_0} \|\mathbf{z}(s)\|^4_{\mathbb{L}^4} e^{\int_s^{-t_0}(\nu\lambda_1 + \frac{3C^2}{\nu}\|\mathbf{z}(\zeta)\|^2_{\mathbb{L}^4})d\zeta}ds \\
&\qquad \le \rho_2^4\rho_3 e^{\nu\lambda_1 t_0/2}\int_{-\infty}^{t_0} |s|^4 e^{\nu\lambda_1 s/2}ds < \infty.
\end{aligned}$$

By the continuity of all relevant functions, we can let $t_0 \to 0$ to get the result. □

Therefore, the proof of (4.28) is concluded, and it only remains to finish the proof of (4.26). Let us denote by

$$\begin{aligned}
\mathbf{v}_{n^{(k)}}(s) &= \mathbf{v}(s, -k; \omega, \varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}} - \mathbf{z}(-k)), \quad s \in (-k, 0), \\
\mathbf{v}_k(s) &= \mathbf{v}(s, -k; \omega, \mathbf{y}_{-k} - \mathbf{z}(-k)), \quad s \in (-k, 0).
\end{aligned}$$

From (4.21) and the last part of Theorem 3.3 we infer that

$$\mathbf{v}_{n^{(k)}} \to \mathbf{v}_k \text{ weakly in } L^2(-k,0;V). \tag{4.33}$$

Since $e^{\nu\lambda_1 \cdot}\mathbf{g},\ e^{\nu\lambda_1 \cdot}\mathbf{f} \in L^2(-k,0;V')$, we get

$$\lim_{n^{(k)}\to\infty} \int_{-k}^{0} e^{\nu\lambda_1 s}\langle \mathbf{g}(s), \mathbf{v}_{n^{(k)}}(s)\rangle ds = \int_{-k}^{0} e^{\nu\lambda_1 s}\langle \mathbf{g}(s), \mathbf{v}_k(s)\rangle ds \tag{4.34}$$

and

$$\lim_{n^{(k)}\to\infty} \int_{-k}^{0} e^{\nu\lambda_1 s}\langle \mathbf{f}, \mathbf{v}_{n^{(k)}}(s)\rangle ds = \int_{-k}^{0} e^{\nu\lambda_1 s}\langle \mathbf{f}, \mathbf{v}_k(s)\rangle ds. \tag{4.35}$$

On the other hand, using the same methods as those in the proof of Theorem 3.2, there exists a subsequence of $\{\mathbf{v}_{n^{(k)}}\}$, which, for the sake of simplicity of notation, is denoted as the old one which satisfies

$$\mathbf{v}_{n^{(k)}} \to \mathbf{v}_k \text{ strongly in } L^2(-k,0;\mathbb{L}^2_{\mathrm{loc}}(\mathbb{S}^2)). \tag{4.36}$$

Next, since $\mathbf{z}(t)$ is an $\mathbb{L}^4$-valued process, so is $e^{\nu\lambda_1 t}\mathbf{z}(t)$. Thus by [10, Corollary 4.1], (4.33) and (4.36), we infer that

$$\begin{aligned}\lim_{n^{(k)}\to\infty} &\int_{-k}^{0} e^{\nu\lambda_1 s} b(\mathbf{v}_{n^{(k)}}(s), \mathbf{z}(s), \mathbf{v}_{n^{(k)}}(s)) ds \\ &= \int_{-k}^{0} e^{\nu\lambda_1 s} b(\mathbf{v}_k(s), \mathbf{z}(s), \mathbf{v}_k(s)) ds.\end{aligned} \tag{4.37}$$

Moreover, since the norms $[\cdot]$ and $\|\cdot\|_V$ are equivalent on V, and since for any $s \in (-k,0]$, $e^{-\nu k\lambda_1} \leq e^{\nu\lambda_1 s} \leq 1$, $(\int_{-k}^{0} e^{\nu\lambda_1 s}[\cdot]^2 ds)^{1/2}$ is a norm in $L^2(-k,0;V)$ equivalent to the standard one. Hence, from (4.33) we obtain,

$$\int_{k}^{0} e^{\nu\lambda_1 s}[\mathbf{v}_k(s)]^2 ds \leq \liminf_{n^{(k)}\to\infty} \int_{-k}^{0} e^{\nu\lambda_1 s}[\mathbf{v}_{n^{(k)}}(s)]^2 ds.$$

In other words,

$$\limsup_{n^{(k)}\to\infty} \left(-\int_{-k}^{0} e^{\nu\lambda_1 s}[\mathbf{v}_{n^{(k)}}(s)]^2 ds \right) \leq -\int_{-k}^{0} e^{\nu\lambda_1 s}[\mathbf{v}_k(s)]^2 ds. \tag{4.38}$$

From (4.25), (4.28), (4.37) and (4.38) we infer that

$$\begin{aligned}&\limsup_{n^{(k)}\to\infty} \|\varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}} - \mathbf{z}(0)\|^2 \\ &\quad\leq \int_{-\infty}^{-k} h(s) ds + 2\int_{-k}^{0} e^{\nu\lambda_1 s}\big\{ b(\mathbf{v}_k(s), \mathbf{z}(s), \mathbf{v}_k(s)) \\ &\qquad + \langle \mathbf{g}(s), \mathbf{v}_k(s)\rangle + \langle \mathbf{f}, \mathbf{v}_k(s)\rangle - [\mathbf{v}_k(s)]^2 \big\} ds\end{aligned} \tag{4.39}$$

On the other hand, from (4.23) and (4.9), we have

$$
\begin{aligned}
\|\mathbf{y}_0 - \mathbf{z}(0)\|^2 &= \|\varphi(k, \vartheta_{-k}\omega)\mathbf{y}_k - \mathbf{z}(0)\|^2 = \|\mathbf{v}(0, -k; \omega, \mathbf{y}_k - \mathbf{z}(-k))\|^2 \\
&= \|\mathbf{y}_k - \mathbf{z}(-k)\|^2 e^{-\nu\lambda_1 k} + 2\int_{-k}^{0} e^{\nu\lambda_1 s}\big\{\langle \mathbf{g}(s), \mathbf{v}_k(s)\rangle \\
&\quad + b(\mathbf{v}_k(s), \mathbf{z}(s), \mathbf{v}_k(s)) + \langle \mathbf{f}, \mathbf{v}_k(s)\rangle - [\mathbf{v}_k(s)]^2\big\} ds.
\end{aligned} \tag{4.40}
$$

Hence, by combining (4.39) with (4.40), we get

$$
\begin{aligned}
&\limsup_{n^{(k)}\to\infty} \|\varphi(t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega)\mathbf{x}_{n^{(k)}} - \mathbf{z}(0)\|^2 \\
&\quad \le \int_{-\infty}^{-k} h(s)ds + \|\mathbf{y}_0 - \mathbf{z}(0)\|^2 - \|\mathbf{y}_k - \mathbf{z}(-k)\|^2 e^{-\nu\lambda_1 k} \\
&\quad \le \int_{-\infty}^{-k} h(s)ds + \|\mathbf{y}_0 - \mathbf{z}(0)\|^2,
\end{aligned}
$$

which proves (4.26), and hence the proof of Proposition 4.16 is finished. □

5. Invariant measure

In this section we consider the existence of an invariant measure. The main result in this section, i.e. Theorem 5.2 is a direct consequence of Corollary 4.4 [17] and our Theorem 4.15about the existence of an attractor for the RDS generated by the stochastic Navier-Stokes equations (3.1).

Let φ be the RDS corresponding to the SNSEs (3.1) and defined in (4.4). We define the transition operator P_t by a standard formula. For $g \in \mathcal{B}_b(\mathrm{H})$, we put

$$
P_t g(\mathbf{x}) \quad = \quad \int_\Omega [g(\varphi(t, \omega, \mathbf{x}))]\, d\mathbb{P}(\omega), \quad \mathbf{x} \in H. \tag{5.1}
$$

As in [11, Proposition 3.8] we have the following result whose proof is simply a repetition of the proof from [11]

Proposition 5.1. *The family $(P_t)_{t\ge 0}$ is Feller, i.e. $P_t g \in C_b(H)$ if $g \in C_b(H)$. Moreover, for any $g \in C_b(\mathrm{X})$, $P_t g(\mathbf{x}) \to g(\mathbf{x})$ as $t \searrow 0$.*

Following [17] one can prove that φ is a Markov RDS, i.e. $P_{t+s} = P_t P_s$ for all $t, s \ge 0$. Hence by [11, Corollary 3.10] which says that a time-continuous and continuous asymptotically compact, Markov RDS φ admits a Feller invariant probability measure μ, i.e. a Borel probability measure μ

$$
P_t^* \mu = \mu, \; t \ge 0, \tag{5.2}
$$

where

$$
P_t^* \mu(\Gamma) = \int_{\mathrm{H}} P_t(x, \Gamma)\, \mu(dx), \quad \Gamma \in \mathcal{B}(\mathrm{H}),
$$

and $P_t(x, \cdot)$ is the transition probability, $P_t(x, \Gamma) = P_t 1_\Gamma(x)$, $x \in H$.
A Feller invariant probability measure for a Markov RDS φ on H is, by

definition, an invariant probability measure for the semigroup $(P_t)_{t\geq 0}$ defined by (5.1). Therefore, we obtain the following result.

Theorem 5.2. *There exists an invariant measure for the stochastic NSE* (3.1).

Remark 5.3. We believe that the uniqueness of an invariant measure for non-degenerate noise will follow from the classical procedure and as in [9]. If the noise is degenerate and spatially smooth, it seems that the results from a recent paper by Hairer and Mattingly [28] should be applicable in our setting. In particular, [28, Theorem 8.4], which gives a sufficient conditions for uniqueness in terms of controllability, should be applicable. Details will be published elsewhere. One should point out that these authors use the "vorticity" formulation and their initial data belongs to the L^2 space. This corresponds to our approach with the initial data belonging to the finite enstrophy space H^1. However, we work in the space of finite energy, which seems to be physically more natural. On the other hand, verifying the sufficient conditions could be more challenging. For the NSE without the Coriolis force this problems has been investigated in [1]. Corresponding NSE with the Coriolis force study is postponed till the next publication.

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