A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials

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Abstract

In this paper, we consider semilinear elliptic equations of the form

$$-\Delta u - \frac{\lambda}{|x|^2} u + b(x)\, h(u) = 0 \quad \text{in } \Omega \setminus \{0\}, \tag{0.1}$$

where λ is a parameter with $-\infty < \lambda \leq (N-2)^2/4$ and Ω is an open subset in $\mathbb{R}^N$ with $N \geq 3$ such that $0 \in \Omega$. Here, $b(x)$ is a positive continuous function on $\overline{\Omega} \setminus \{0\}$ which behaves near the origin as a regularly varying function at zero with index θ greater than -2. The nonlinearity h is assumed continuous on $\mathbb{R}$ and positive on $(0, \infty)$ with $h(0) = 0$ such that $h(t)/t$ is bounded for small $t > 0$. We completely classify the behaviour near zero of all positive solutions of (0.1) when h is regularly varying at ∞ with index q greater than 1 (that is, $\lim_{t\to\infty} h(\xi t)/h(t) = \xi^q$ for every $\xi > 0$). In particular, our results apply to (0.1) with $h(t) = t^q (\log t)^{\alpha_1}$ as $t \to \infty$ and $b(x) = |x|^\theta (-\log |x|)^{\alpha_2}$ as $|x| \to 0$, where α_1 and α_2 are any real numbers.

We reveal that the solutions of (0.1) generate a very complicated dynamics near the origin, depending on the interplay between q, N, θ and λ, on the one hand, and the position of λ with respect to 0 and $(N-2)^2/4$, on the other hand. Our main results for $\lambda = (N-2)^2/4$ appear here for the first time, as well as for the case $\lambda < 0$. We establish a trichotomy of positive solutions of (0.1) under optimal conditions, hence generalizing and improving through a different approach a previous result with Chaudhuri on (0.1) with $0 < \lambda < (N-2)^2/4$ and $b = 1$. Moreover, recent results of the author with Du on (0.1) with $\lambda = 0$ are here sharpened and extended to any $-\infty < \lambda < (N-2)^2/4$. In addition, we unveil a new single-type behaviour of the positive solutions of (0.1) specific to $0 < \lambda < (N-2)^2/4$. We also provide necessary and sufficient conditions for the existence of positive solutions of (0.1) that are comparable with the fundamental solutions of

$$-\Delta u - \frac{\lambda}{|x|^2} u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

In particular, for $b = 1$ and $\lambda = 0$, we find a sharp condition on h such that the origin is a removable singularity for all non-negative solutions of (0.1), thus addressing an open question of Vázquez and Véron.

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CHAPTER 1

Introduction

1.1. Background and aims

The local behaviour of solutions (sub-super-solutions) of second-order, quasilinear elliptic, divergence structure, partial differential equations has been the main theme of investigation by many authors. A question of basic importance in the study of partial differential equations is to understand the behaviour of all possible solutions near an isolated singularity. As recalled in [**10**], Bôcher's Theorem in harmonic analysis states that a positive harmonic function u in the punctured unit ball $B_1(0) \setminus \{0\}$ in $\mathbb{R}^N$ with $N \geq 2$ must be of the form

$$\begin{cases} a \log(1/|x|) + g(x) & \text{if } N = 2, \\ a|x|^{2-N} + g(x) & \text{if } N \geq 3, \end{cases}$$

where a is a non-negative constant and g is a harmonic function in the ball $B_1(0)$. However, it is difficult in general to give as complete a description of the behaviour of solutions near an isolated singularity for nonlinear partial differential equations. Serrin [**34, 35**] obtained the earliest general results on the isolated singularities of solutions for quasilinear elliptic equations in divergence form

$$\operatorname{div} \mathbf{A}(x, u, \nabla u) = B(x, u, \nabla u) \tag{1.1}$$

where $\mathbf{A}$ (respectively, B) is a given vector (respectively, scalar) function of the variables $x, u, \nabla u$ such that the growth of B is dominated by that of $\mathbf{A}$. Continuing Serrin's work [**34**], Kichenassamy and Véron [**25**] studied the isolated singularities for the m-Laplace equation

$$\Delta_m u = \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right) = 0 \quad \text{with } m > 1.$$

We refer to Véron [**45**] for the development of the singularity theory for solutions to nonlinear second-order differential equations of elliptic (and parabolic) type up to 1996. The topic of isolated singularities continues to attract a lot of attention. New results on universal estimates of spatial singularities for quasilinear elliptic equations of the form $-\Delta_m u = f(u)$ and also for semilinear systems of Lane–Emden type are obtained in [**29**] based on Liouville type theorems. Other recent progress includes the classification of singularities for non-negative viscosity solutions for the infinite Laplace equation

$$\Delta_\infty u =: \sum_{i,j=1}^{N} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

(see [**32**]) and, more generally, for the Aronsson equation (see [**23**]).

The most intricate situations in the study of the singularity problem for quasilinear elliptic equations such as (1.1) arise when the growth of B is bigger than that of **A** (cf., Véron [**45**]). Understanding the possible behaviour near the singularity of all solutions to such problems is mainly limited to particular classes of nonlinear models. Much research originated in attempts to generalize the well-known classification results on $\Delta u = |u|^{q-1}u$ in $B_1(0) \setminus \{0\}$ due to Véron [**43, 44**] for $1 < q < N/(N-2)$ (any $q > 1$ if $N = 2$) and Brezis–Véron [**9**] for $q \geq N/(N-2)$. We refer to Friedman–Véron [**19**] and Vázquez–Véron [**40**] for the classification of the isolated singularities of solutions of $\Delta_m u = |u|^{q-1}u$ with $q > 1$ and $1 < m \leq N$. These results were extended in the recent paper [**15**] to weighted quasilinear elliptic equations by developing new techniques relying on the regular variation theory.

After the groundbreaking paper [**9**], much research was devoted to the removability of singularities of solutions to elliptic partial differential equations. Labutin [**26**] obtained a removability result for fully nonlinear uniformly elliptic equations of the form $F(D^2u) + f(u) = 0$, where f satisfies certain sharp conditions depending on F. Recently, Felmer and Quaas [**18**] extended the results of Brezis–Véron [**9**] and Labutin [**26**] to a large class of nonlinear second order elliptic differential operators for which a fundamental solution can be constructed. However, up to this point, it is still open the following question of Vázquez–Véron [**42**] on the removability of singularities for the equation

$$-\Delta u + h(u) = 0 \quad \text{in } \Omega^* := \Omega \setminus \{0\}, \tag{1.2}$$

where h is a continuous non-decreasing real function. From now on, Ω denotes an open subset of $\mathbb{R}^N$ with $N \geq 3$ such that $0 \in \Omega$.

Question (Vázquez–Véron, [42]): *What is the weakest assumption on h such that any isolated singularity of a non-negative solution of* (1.2) *is removable?*

In this paper we are motivated by [**22**], [**15**] and [**11**] to give a complete classification of the singular solutions for a broader class of nonlinear elliptic equations than (1.2). As a byproduct, we resolve Vázquez–Véron's question when h is regularly varying at ∞ of index greater than 1 (see §1.3). An important feature of our study lies in the incorporation of inverse square potentials and weighted nonlinearities, thereby embracing classes of time-independent nonlinear Schrödinger equations. We consider semilinear elliptic equations of the form

$$-\Delta u - \frac{\lambda}{|x|^2} u + b(x)\, h(u) = 0 \quad \text{in } \Omega^* := \Omega \setminus \{0\}, \tag{1.3}$$

where λ is a real parameter such that $-\infty < \lambda \leq (N-2)^2/4$. Throughout this paper, all solutions are understood in the sense of distributions (see Definition 1.1 in §1.2). The precise assumptions on b and h are given in the next section.

Our Theorems 2.1, 2.2 and 2.4 on (1.3) with $-\infty < \lambda < (N-2)^2/4$ will refine and generalize the main results in [**15**] on $\Delta u = b(x)h(u)$ in Ω^* with $N \geq 3$, as well as Theorem 1.1 in [**11**] on (1.3) with $b = 1$ and $0 < \lambda < (N-2)^2/4$. The method of proof outlined in [**11**] relies essentially on the fact that every positive solution of (1.3) blows-up at zero for $0 < \lambda < (N-2)^2/4$. Since this is not the case when $\lambda \leq 0$, the approach in [**11**] is not applicable to our problem. In this paper, we treat through a different approach the more general setting of (1.3) for the whole range $-\infty < \lambda \leq (N-2)^2/4$ and completely describe the asymptotic behaviour near zero for all positive solutions of (1.3). Specifically, our Theorem 2.4 improves and generalizes the main result in [**11**] and [**14**] by establishing a trichotomy of positive solutions for (1.3) under optimal conditions. Our structural assumptions on b and h will rely on regular variation theory as in [**15**].

For the first time appears here Theorem 2.3 specific to $0 < \lambda < (N-2)^2/4$, as well as Theorem 2.6 and Theorem 2.7, which fully classify the isolated singularities of (1.3) for $\lambda = (N-2)^2/4$. Moreover, we establish necessary and sufficient conditions for the existence of positive solutions of (1.3) that are comparable with the fundamental solutions of (2.2) (see Theorem 2.1 when $\lambda < (N-2)^2/4$ and Theorem 2.5 for $\lambda = (N-2)^2/4$). These results with respect to the dominant fundamental solution of (2.2) generalize theorems of Guerch and Véron [**22**] for positive solutions of (1.3) with $b = 1$ and h a continuous non-decreasing real function. We use $\Phi_\lambda^\pm$ (respectively $\Psi^\pm$) to denote the fundamental solutions of (2.2) when $\lambda < (N-2)^2/4$ (respectively, $\lambda = (N-2)^2/4$). Their definition is given by (2.3) and (2.24), respectively. As a novelty, Theorem 2.1 and Theorem 2.5 give sharp conditions for (1.3) to admit positive solutions comparable with the other fundamental solution of (2.2) (i.e., Φ_λ^- and Ψ^-, respectively).

Our main results are stated in Chapter 2 and illustrated in Chapter 7. We mention here (and further explain in Chapter 2) that we differentiate our results at two levels. First, we need to reason differently according to whether $\lambda < (N-2)^2/4$ (see Theorems 2.1–2.4) or $\lambda = (N-2)^2/4$ (see Theorems 2.5–2.7). Second, it is vital to distinguish the case $\lambda \leq 0$ from $0 < \lambda < (N-2)^2/4$. This distinction arises from the viewpoint of critical exponents vis-à-vis the index q of regular variation for h in (1.5): we have only one such exponent q^* if $\lambda \leq 0$ versus two critical exponents q^* and q^{**} if $0 < \lambda < (N-2)^2/4$. We define q^* and q^{**} in (1.11) as we need the precise assumptions on b and h in §1.2. It turns out that $q^* = q^{**}$ if $\lambda = (N-2)^2/4$. Another feature of this paper is to reveal a new asymptotic behaviour of the solutions of (1.3) in regard to the critical exponents in Theorems 2.3 and 2.4 for $\lambda < (N-2)^2/4$, respectively Theorems 2.6 and 2.7 for $\lambda = (N-2)^2/4$.

1.2. Our framework

1.2.1. The concept of solution of (1.3). Unless otherwise stated, we always impose the following.

ASSUMPTION A. *The function h is continuous on $\mathbb{R}$ and positive on $(0,\infty)$ with $h(0) = 0$ such that $h(t)/t$ is bounded for small $t > 0$, while $b(x)$ is a positive continuous function on $\overline{\Omega} \setminus \{0\}$.*

By a solution of (1.3) we mean a $C^1(\Omega^*)$-solution of (1.3) in the sense of distributions in Ω^* (in $\mathcal{D}'(\Omega^*)$). More precisely, we give the following.

DEFINITION 1.1. A function u is said to be a *solution* (*sub-solution, super-solution*) of (1.3) if $u(x) \in C^1(\Omega^*)$ and for all functions (non-negative functions) $\phi(x)$ in the space $C_c^1(\Omega^*)$, we have

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx - \int_\Omega \frac{\lambda}{|x|^2} u \, \phi \, dx + \int_\Omega b(x) h(u) \, \phi \, dx = 0 \qquad (\leq 0, \ \geq 0). \tag{1.4}$$

By $C_c^1(\Omega^*)$, we denote the space of $C^1(\Omega^*)$–functions with compact support in Ω^*. We say that a solution of (1.3) can be extended to a solution of (1.3) in all Ω if (1.4) holds for every $\phi \in C_c^1(\Omega)$.

1.2.2. On regular variation theory. The structure conditions on the lower order term in (1.3) involve the regular variation theory, a key ingredient in our approach similar to earlier work without potential in [**14, 15**]. Next, we recall the definition of a slowly varying function.

DEFINITION 1.2. A positive measurable function L defined on an interval (A,∞) with $A>0$ is called *slowly varying at* ∞ if

$$\lim_{t\to\infty}\frac{L(\xi t)}{L(t)}=1 \quad \text{for every } \xi>0.$$

A function ϕ is called *regularly varying* at ∞ with real index m, or $\phi\in RV_m(\infty)$ in short, if $\phi(t)=t^m L(t)$ for some function L that is slowly varying at ∞. Hence, a slowly varying function at ∞ is a regularly varying function at ∞ with index 0.

EXAMPLE 1.3. Any positive constant function is trivially a slowly varying function. Other non-trivial examples of slowly varying functions at ∞ include:

(a) The logarithm $\log t$, its m-iterates $\log_m t$ (defined as $\log\log_{m-1} t$) and powers of $\log_m t$ for any integer $m\geq 1$.
(b) $\exp\left(\frac{\log t}{\log\log t}\right)$.
(c) $\exp((\log t)^\nu)$ with $\nu\in(0,1)$.
(d) $\exp\{(\log t)^{1/3}\cos((\log t)^{1/3})\}$.

These examples show that the limit at ∞ of a slowly varying function (at ∞), say L, cannot be determined in general, and it may not even exist. Indeed, in Example 1.3(d) above, we have

$$\liminf_{t\to\infty} L(t)=0 \quad\text{and}\quad \limsup_{t\to\infty} L(t)=+\infty.$$

In contrast, if $\phi\in RV_m(\infty)$ with $m>0$ (respectively, $m<0$), then $\lim_{t\to\infty}\phi(t)=\infty$ (respectively, 0). The concept of regular variation can be given at zero.

DEFINITION 1.4 (see [**33**]). We say that $r\longmapsto L(r)$ is *slowly varying at (the right of) zero* if the function $t\longmapsto L(1/t)$ is slowly varying at ∞

For properties of regularly varying functions used in this paper, see Appendix A.

1.2.3. Notation and main assumption. By $f_1(t)\sim f_2(t)$ as $t\to t_0$ for $t_0\in\mathbb{R}\cup\{\pm\infty\}$, we mean that $\lim_{t\to t_0} f_1(t)/f_2(t)=1$. We shall assume we are given two functions L_h and L_b which are *slowly varying*[1] at ∞ and 0, respectively. Their subscripts indicate the functions they are associate with, namely:

$$\begin{cases} h(t)\sim\tilde{h}(t):=t^q L_h(t) & \text{as } t\to\infty \quad \text{for some } q>1,\\ b(x)\sim |x|^\theta L_b(|x|) & \text{as } |x|\to 0 \quad \text{for some } \theta>-2. \end{cases} \tag{1.5}$$

This means that h is regularly varying at ∞ with index q, while b behaves near the origin as a regularly varying function at 0 with index θ. The above asymptotic information is crucial in obtaining a complete classification of the behaviour near the origin for all positive solutions of (1.3). This classification is intimately connected with the inequalities $q>1$ and $\theta>-2$ that appear in (1.5). In particular, our

[1]We shall follow the custom of denoting a slowly varying function by L, possibly with subscripts, the letter L being a reminder of the French word *lentement* since one of the first works on Karamata's theory was published in French [**24**].

results will be very different from those pertaining to nonlinearities h with sublinear growth treated elsewhere (see, for example, Bidaut-Véron and Grillot [**3**], where $h(t) = t^q$ with $0 < q < 1$ and $b(x) = |x|^\theta$ with $\theta > -2$).

1.2.4. Smoothness properties. By (1.5) and Proposition A.2 in the Appendix A, we can assume, without loss of generality, that $L_h \in C^2[t_0, \infty)$ and $L_b \in C^2(0, r_0]$ for some positive constants t_0 and r_0, such that:

$$\lim_{t\to\infty} \frac{tL_h'(t)}{L_h(t)} = \lim_{t\to\infty} \frac{t^2L_h''(t)}{L_h(t)} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{rL_b'(r)}{L_b(r)} = \lim_{r\to 0} \frac{r^2L_b''(r)}{L_b(r)} = 0. \tag{1.6}$$

The function $\tilde{h}$ defined by (1.5) can be extended to the rest of the interval $[0, \infty)$ so that $\tilde{h}(t)/t$ is increasing on $(0, \infty)$ and $\tilde{h}$ is continuous on $[0, \infty)$. Moreover, using (1.6) we obtain that

$$\lim_{t\to\infty} \frac{\tilde{h}(t)}{h(t)} = 1, \quad \lim_{t\to\infty} \frac{t\tilde{h}'(t)}{\tilde{h}(t)} = q, \quad \lim_{t\to\infty} \frac{t\tilde{h}''(t)}{\tilde{h}'(t)} = q - 1 > 0. \tag{1.7}$$

1.2.5. A priori estimates. It is useful to introduce the following functions

$$\begin{cases} f(t) := \tilde{h}(t)/t = t^{q-1}L_h(t) & \text{for large } t > 0, \\ \mathcal{J}(r) := r^{\theta+2}L_b(r) \ \text{ and } \ \mathcal{K}(r) := f^{-1}(1/\mathcal{J}(r)) & \text{for small } r > 0, \end{cases} \tag{1.8}$$

where $f^{-1}(t)$ denotes the inverse of f at t, which exists for $t > 0$ large because f is increasing on $(0, \infty)$ with $\lim_{t\to\infty} f(t) = \infty$. Using that $\lim_{r\to 0} 1/\mathcal{J}(r) = \infty$, we can define $\mathcal{K}$ as above. This function plays an important role in the paper such as in the *a priori* estimates of Lemma 4.1: for every $r > 0$ sufficiently small, there exists a positive constant C depending on r such that

$$u(x) \le C\mathcal{K}(|x|) \quad \text{for any } 0 < |x| \le r,$$

where u is an arbitrary positive (sub-)solution of (1.3).

Using (1.6), we readily obtain C^2-smoothness for f and $\mathcal{J}$, whose properties given by Remark A.12 in the Appendix A will be later exploited in the construction of sub-super-solutions of (1.3) in Lemma 5.13.

1.2.6. Solutions with a dominating behaviour at zero. An important common feature for all $\lambda \le (N-2)^2/4$ regards a positive solution u of (1.3) with a *dominating* behaviour at zero (meaning that for any positive solution $\tilde{u}$ of (1.3), we have either $\lim_{|x|\to 0} u(x)/\tilde{u}(x) = 1$ or $\lim_{|x|\to 0} u(x)/\tilde{u}(x) = \infty$). We define

$$\Theta := \frac{\theta+2}{q-1}, \qquad \ell := \lambda + \Theta(\Theta + 2 - N), \quad p := \frac{N-2}{2} - \sqrt{\frac{(N-2)^2}{4} - \lambda}. \tag{1.9}$$

If $\ell > 0$ and $\lambda \le (N-2)^2/4$, then we prove that any positive solution u of (1.3) with a *dominating* behaviour at 0 satisfies

$$u(x) \sim \ell^{\frac{1}{q-1}} \mathcal{K}(|x|) \quad \text{as } |x| \to 0. \tag{1.10}$$

To formulate this more precisely, we need to look at the roots of $\ell = 0$ as an equation in q. If $\lambda \ne 0$, then the equation $\ell(q) = 0$ has two roots q^* and q^{**}:

$$q^* = q^*(N, \lambda, \theta) := \frac{N + \theta - p}{N - 2 - p}, \quad q^{**} = q^{**}(N, \lambda, \theta) := \frac{p + \theta + 2}{p}. \tag{1.11}$$

Note that if $\lambda = 0$, then $\ell(q) = 0$ renders only one root $q = (N+\theta)/(N-2)$, that is $q = q^*$ in which p is replaced by 0. When $\lambda = (N-2)^2/4$, then

$$q^* = q^{**} = \frac{N + 2\theta + 2}{N - 2}$$

since $\ell = [\Theta - (N-2)/2]^2$. It turns out that q^* and q^{**} play a critical role when it comes to classifying the asymptotic behaviour near zero for all positive solutions of (1.3). Assuming (1.5) and excluding for the moment $q = q^*$ and $q = q^{**}$, our main results prove in particular that

- (S_1) If $-\infty < \lambda \le (N-2)^2/4$ and $1 < q < q^*$, then any positive solution of (1.3) with a *dominating* behaviour at zero satisfies (1.10) (see Theorem 2.4(C1) and Theorem 2.7(C1)).
- (S_2) If $0 < \lambda \le (N-2)^2/4$ and $q > q^{**}$, then *all* positive solutions of (1.3) are asymptotic at zero and satisfy (1.10) (we refer to Theorem 2.3(a) for $0 < \lambda < (N-2)^2/4$ and Theorem 2.6(b1) for $\lambda = (N-2)^2/4$).
- (S_3) If $-\infty < \lambda \le 0$ and $q > q^*$, then for every positive solution u of (1.3), $|x|^p u(x)$ converges to some positive number. The same conclusion applies for $0 < \lambda < (N-2)^2/4$ provided that $q^* < q < q^{**}$ (see Theorem 2.2).

The analysis of the critical exponents $q = q^*$ and $q = q^{**}$ would require additional information on L_h and L_b. Our further assumptions are essential only for $q = q^{**}$ in Theorem 2.3 or $q = q^*$ in Theorem 2.4 when $\lambda < (N-2)^2/4$, respectively $q = q^*$ in Theorems 2.6 and 2.7 when $\lambda = (N-2)^2/4$.

1.2.7. Further assumptions. In addition to $L_h(t)$ and $L_b(1/t)$ being slowly varying functions at ∞, it will be convenient to have extra information on their asymptotic behaviour at ∞. The first in a set of additional assumptions involves regular variation theory, being easily verifiable in each particular case:

$$\begin{cases} \text{(a) } t \longmapsto L_h(e^t) & \text{is regularly varying at } \infty \text{ with index } \alpha_1 \in \mathbb{R}, \\ \text{(b) } t \longmapsto L_b(e^{-t}) & \text{is regularly varying at } \infty \text{ with index } \alpha_2 \in \mathbb{R}. \end{cases} \tag{1.12}$$

Remark 1.5. When (1.12)(a) holds in either of the situations (S_1) and (S_2) above, then (1.10) can be refined as follows

$$u(x) \sim \left\{ \frac{\Theta^{\alpha_1}}{\ell} |x|^{\theta+2} L_b(|x|)\, L_h(1/|x|) \right\}^{-\frac{1}{q-1}} \quad \text{as } |x| \to 0. \tag{1.13}$$

Indeed, from (1.10) and (1.8), we have

$$u^{q-1} L_h(u) \sim \frac{\ell}{|x|^{\theta+2} L_b(|x|)} \quad \text{as } |x| \to 0. \tag{1.14}$$

Since $u(x)$ is asymptotically equivalent to a regularly varying function at zero of index $-\Theta$, we have $\log u(x) \sim \Theta \log(1/|x|)$ as $|x| \to 0$. Thus (1.12)(a) implies that

$$L_h(u(x)) \sim \Theta^{\alpha_1} L_h(1/|x|) \quad \text{as } |x| \to 0,$$

which combined with (1.14), proves our claim in (1.13).

Example 1.6. One prototype model to keep in mind for (1.12)(a) and (1.12)(b), respectively is given by

$$\begin{cases} L_h(t) \sim (\log t)^{\alpha_1} & \text{as } t \to \infty, \quad \text{where } \alpha_1 \in \mathbb{R}, \\ L_b(1/t) \sim (\log t)^{\alpha_2} & \text{as } t \to \infty, \quad \text{where } \alpha_2 \in \mathbb{R}. \end{cases} \tag{1.15}$$

In Chapter 7 we summarize the classification of the behaviour near zero of all positive solutions of (1.3) on the example in (1.15).

More generally, (1.12)(a) holds if $L_h(t) \sim \mathcal{L}(t)$ as $t \to \infty$ and

$$\mathcal{L}(t) = \Pi_{i=1}^{j} (\log_{m_i} t)^{\beta_i},$$

where j, m_i are positive integers and $\beta_i \in \mathbb{R}$ for $i = 1, \ldots, j$. Here we use the notation $\log_{m_i} t$ for the m_i–iterated logarithm. Without loss of generality, we can take $1 \leq m_1 < m_2 < \ldots < m_j$. Then $t \longmapsto L_h(e^t)$ is regularly varying at ∞ with index equal to β_1 (respectively, 0) if $m_1 = 1$ (respectively, $m_1 > 1$). Similarly, (1.12)(b) is verified if $L_b(1/t) \sim \mathcal{L}(t)$ as $t \to \infty$.

But not all slowly varying functions $L_h(t)$ and $L_b(1/t)$ can be subsumed under (1.12) as shown by Example 1.3(b), (c). For such cases, we envisage a different set of postulates stated below as either (a) or (b) for L_b, respectively (c) or (d) for L_h:

$$\begin{cases} \text{(a) } L_b(e^{-t}) \sim \Lambda(t) \quad \text{as } t \to \infty, & \text{(c) } L_h(e^{t}) \sim \Lambda(t) \quad \text{as } t \to \infty, \\ \text{(b) } L_b(e^{-t}) \sim 1/\Lambda(t) \quad \text{as } t \to \infty, & \text{(d) } L_h(e^{t}) \sim 1/\Lambda(t) \quad \text{as } t \to \infty. \end{cases} \tag{1.16}$$

The function Λ in (1.16) is defined on some interval $[A, \infty)$ by

$$\Lambda(t) := \exp\left(\int_A^t \frac{d\tau}{S(\tau)} \right), \tag{1.17}$$

where $S \in C^1[A, \infty)$ is a positive function which satisfies

$$\lim_{t\to\infty} S(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} S'(t) = 0.$$

Remark 1.7. Unlike (1.12), the assumptions in (1.16) do not make $L_b(e^{-t})$ and $L_h(e^t)$ regularly varying at ∞. Indeed, we see that Λ in (1.17) is a positive C^2-function on (A, ∞) such that $\lim_{t\to\infty} t\Lambda'(t)/\Lambda(t) = \infty$. Then by Proposition 2.7 in [**12**], it follows that Λ in (1.17) is *rapidly varying* at ∞ with index ∞, that is

$$\lim_{t\to\infty} \frac{\Lambda(\xi t)}{\Lambda(t)} = \xi^{\infty} = \begin{cases} \infty & \text{if } \xi > 1, \\ 0 & \text{if } \xi \in (0,1), \\ 1 & \text{if } \xi = 1. \end{cases}$$

To be more precise, we have that Λ is a Γ-*varying function* at ∞ with *auxiliary function* S (see Lemma 3.4 in [**12**]). We recall the following.

Definition 1.8 (see [**31**]). A non-decreasing function f defined on an interval (A, ∞) is called Γ-*varying* at ∞ if $\lim_{t\to\infty} f(t) = \infty$ and there exists a function $\chi : (A, \infty) \to (0, \infty)$ such that

$$\lim_{t\to\infty} \frac{f(t + \xi\chi(t))}{f(t)} = e^{\xi} \quad \text{for every } \xi \in \mathbb{R}.$$

The function χ is called an *auxiliary function* and is unique up to asymptotic equivalence. The class of Γ-varying functions was introduced by de Haan [**17**] in connection with extreme value theory. For examples and properties of Γ-varying functions, we refer to [**5**], [**17**], [**20**] or [**31**].

Example 1.9. As models for Λ in (1.17) which can be used in (1.16), we have

(a) $\Lambda(t) = \exp(t/\log t)$ when $S(t) \sim \log t$ as $t \to \infty$,
(b) $\Lambda(t) = \exp(t^{\nu})$ with $0 < \nu < 1$ when $S(t) \sim (1/\nu)\, t^{1-\nu}$ as $t \to \infty$.

We refer to Section 7.2 in Chapter 7 for a summary of our results in situations that involve (1.16). Corollaries 7.6–7.8 show how different combinations of hypotheses in (1.12) and (1.16) affect the asymptotic behaviour of solutions of (1.3) for the critical exponents $q = q^*$ and $q = q^{**}$ in (1.5).

1.3. On Vázquez–Véron's open question

We recall that the origin is called a *removable singularity* for a solution u of (1.2) if u can be extended to a C^1–solution of (1.2) in $\mathcal{D}'(\Omega)$. From Theorems 2.1 and 2.2 with $\lambda = 0$ and $b = 1$, we obtain that

$$\int_1^\infty h(t)\, t^{-2(N-1)/(N-2)}\, dt = +\infty \tag{1.18}$$

is the weakest condition on h to resolve Vázquez–Véron's question (when h satisfies (1.5) and Assumption A in §1.2). Assumption A ensures that the strong maximum principle holds for (1.2) so that we need consider only positive solutions of (1.2). We know that the origin is a removable singularity for (1.2) if and only if $\lim_{|x|\to 0} u(x)/|x|^{2-N} = 0$ (cf., [**44**]). Moreover, from [**44**] or [**42**], we have that

$$\int_1^\infty h(t)\, t^{-2(N-1)/(N-2)}\, dt < +\infty \tag{1.19}$$

is a necessary and sufficient condition for the existence of positive solutions of (1.2) with a weak singularity at zero (that is, $0 < \lim_{|x|\to 0} u(x)/|x|^{2-N} < \infty$). Brezis and Bénilan introduced the condition (1.19) for solving equations such as $-\Delta u + h(u) = \nu$, where the right-hand side is a bounded measure (see [**2**]). Although (1.18) is a necessary condition for the removability of all singularities of the positive solutions of (1.2), it is in general not sufficient. The main difficulty of Vázquez–Véron's question amounts to ruling out the solutions with strong singularities at 0 (when $\lim_{|x|\to 0} u(x)/|x|^{2-N} = \infty$). Theorem 2.2 in [**42**] shows that there exist no positive solutions of (1.2) with a weak singularity at zero, but infinitely many with a strong singularity at zero provided that (1.18) holds and

$$\int_1^\infty \frac{dt}{\sqrt{t h(t)}} = \infty. \tag{1.20}$$

By Remark 2.2 in [**42**], there exist continuous non-decreasing functions h satisfying (1.18) and (1.20). In a seminal paper [**9**], Brezis and Véron proved that if h satisfies

$$\liminf_{|t|\to\infty} \frac{|h(t)|}{|t|^{N/(N-2)}} > 0, \tag{1.21}$$

then the origin is a removable singularity for any solution of (1.2). This result was extended by Vázquez and Véron [**42**, Theorem 3.1] under the weaker assumption

$$\liminf_{|t|\to\infty} \frac{|h(t)| \log(|t|)}{|t|^{N/(N-2)}} > 0, \tag{1.22}$$

which, in fact, can be improved as

$$\liminf_{|t|\to\infty} \frac{|h(t)| \log(|t|)\, \log\log(|t|)}{|t|^{N/(N-2)}} > 0, \tag{1.23}$$

where the function log can be further iterated in (1.23) (cf., Remark 3.1 in [**42**]).

Our Theorem 2.2 with $\lambda = 0$ and $b = 1$ improves the above-mentioned removability results. Indeed, let h satisfy Assumption A in §1.2 and $h(t) = t^{N/(N-2)}L(t)$ for large $t > 0$, where $L(t)$ is given by Example 1.3(d) in §1.2, namely

$$h(t) = t^{\frac{N}{N-2}} \exp\{(\log t)^{1/3} \cos((\log t)^{1/3})\} \quad \text{for large } t > 0. \tag{1.24}$$

For such an example we cannot use the removability result in [**42**] since (1.23) fails (even when taking into account the improvements in Remark 3.1 of [**42**]). However, h in (1.24) satisfies (1.18) and (1.5) with $q = N/(N-2)$ so that by Theorem 2.2 with $\lambda = 0$ and $b = 1$, we conclude that the origin is a removable singularity for all positive solutions of (1.2). Remark that in Theorem 2.2 we do not require any monotonicity for h and the integral in (1.20) is *finite* because h satisfies (1.5).

We also contrast Theorem 2.2 with an analogous result (Theorem 1.3) in [**15**]. Although the latter applies to quasilinear elliptic equations of the form

$$\Delta_m u = b(x)h(u) \quad \text{in } \Omega^* \quad \text{with } 1 < m < N, \tag{1.25}$$

for the sake of comparison, we restrict to $m = 2$ in (1.25), corresponding to (1.3) with $\lambda = 0$. Theorem 1.3 in [**15**] proves that if Assumption A and (1.5) hold with $q \geq (N+\theta)/(N-2)$ and, in addition, for $q = (N+\theta)/(N-2)$ we have

$$\liminf_{t\to\infty} L_h(t) > 0 \quad \text{and} \quad \liminf_{r\to 0} L_b(r) > 0, \tag{1.26}$$

then any positive solution of (1.3) with $\lambda = 0$ can be extended as a (distribution) solution of $\Delta u = b(x)h(u)$ in all Ω. This follows easily when $q > (N+\theta)/(N-2)$ since Corollary 4.3 gives that

$$\lim_{|x|\to 0} u(x)/|x|^{2-N} = 0 \quad \text{for any positive solution } u. \tag{1.27}$$

Hence, $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and Theorem 1 of Serrin [**35**] is applicable. However, to derive (1.27) for $q = (N+\theta)/(N-2)$, the argument of Theorem 1.3 in [**15**] requires (1.26). This assumption clearly implies that

$$\lim_{\tau\to 0} \int_\tau^\varpi r^{N+\theta-1-(N-2)q} L_b(r)\, L_h(r^{2-N})\, dr = \infty \tag{1.28}$$

for any small $\varpi > 0$ when we prove that (1.27) is still valid (see Lemma 5.8 in Chapter 5). Moreover, our condition (1.28) is sharp in the sense that if (1.28) fails and h is non-decreasing on $[0,\infty)$, then there exist positive solutions of

$$\Delta u = b(x)h(u) \quad \text{in } B_1(0) \setminus \{0\}$$

satisfying $\lim_{|x|\to 0} u(x)/|x|^{2-N} \in (0,\infty)$ (see Lemma 5.6).

More importantly, in this paper we reveal all the possible behaviour near zero for the positive solutions of more general equations such as (1.3), which introduce an inverse square potential $\lambda|x|^{-2}u$ with $-\infty < \lambda \leq (N-2)^2/4$.

1.4. An outline of our results

For background and motivation of our study, see [**22**], [**11**] and [**15**]. Our findings here incorporate several improvements and extensions over those recently published in [**11**] on (1.3) with $0 < \lambda < (N-2)^2/4$, $b = 1$ and $1 < q < q^*$, as well as in [**15**] on (1.25) with $m = 2$. We confine our details below to $-\infty < \lambda < (N-2)^2/4$. We assume (1.5) and define p as in (1.9). Let q^* and q^{**} be given by (1.11).

Instead of (1.27), we prove that

$$\lim_{|x|\to 0} u(x)/|x|^{2-N+p} = 0 \ \text{ for any positive solution } u \text{ of (1.3)}, \tag{1.29}$$

under a sharp condition that generalizes (1.28), namely

$$\lim_{\tau\to 0} \int_\tau^{\varpi} r^{N+\theta-1-p-(N-2-p)q} L_b(r)\, L_h(r^{2-N+p})\, dr = \infty. \tag{1.30}$$

However, a more delicate task in this paper is to refine the conclusion of (1.29). In Theorem 2.2, we assume (1.30) and establish that $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$ for any positive solution of (1.3) provided that

$$\lim_{\tau\to 0} \int_\tau^{\varpi} r^{\theta+1-p(q-1)} L_b(r)\, L_h(r^{-p})\, dr < \infty \tag{1.31}$$

for some small $\varpi > 0$. This assumption is always verified if $\lambda \le 0$. For $0 < \lambda < (N-2)^2/4$, we prove that condition (1.31) in Theorem 2.2 is sharp.

Theorem 2.3 tackles the case when (1.31) does not hold, which is specific to $0 < \lambda < (N-2)^2/4$, proving that all positive solutions of (1.3) are asymptotic at zero and they satisfy $\lim_{|x|\to 0} |x|^p u(x) = 0$. Theorem 2.3 implies that $q \ge q^{**}$ and applies automatically when $q > q^{**}$ (see (2.11)). We also describe the behaviour of the positive solutions of (1.3) near 0 by differentiating between $q > q^{**}$ (when u satisfies (1.10)) and $q = q^{**}$. The analysis of the critical exponent $q = q^{**}$ is more involved and depends on the additional hypotheses in (1.12) and (1.16).

Theorem 2.4 treats the remaining situation that the limit in (1.30) is finite and proves that any positive solution u of (1.3) with $-\infty < \lambda < (N-2)^2/4$ satisfies exactly one of the following as $|x| \to 0$:

(A) $|x|^p u(x)$ converges to a positive number;
(B) $|x|^{N-2-p} u(x)$ converges to a positive number;
(C) $|x|^{N-2-p} u(x) \to \infty$.

Theorem 2.4 implies that $q \le q^*$ (see (2.9)) and applies whenever $1 < q < q^*$. In Case C above, we give a precise asymptotic behaviour of u at 0 by distinguishing between $q < q^*$ (when u satisfies (1.10)) and $q = q^*$. The examination of $q = q^*$ is here new compared with [**15**] and [**11**]. We find that the solutions in Case (C) of Theorem 2.4 satisfy (2.21) when $q = q^*$ and (1.12)(a) holds.

Our classification results are established under the optimal conditions and the existence of positive solutions as prescribed by each of these theorems is guaranteed if $h(t)/t$ is increasing on $(0,\infty)$ (see Lemmas 5.6 and 5.7). More exactly, in Theorem 2.1 we prove that if $h(t)/t$ is increasing on $(0,\infty)$, then (1.3) admits positive solutions in $B_1(0) \setminus \{0\}$ with $\lim_{|x|\to 0} |x|^{N-2-p} u(x) = 0$ and any such solution satisfies $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$ if and only if (1.31) holds. In turn, there exist positive solutions of (1.3) in $B_1(0) \setminus \{0\}$ with $\lim_{|x|\to 0} |x|^{N-2-p} u(x) \in (0,\infty)$ if and only if the limit in (1.30) is finite.

1.4.1. Some details of our proofs. We now explain the main advances and innovation of our proofs. The crucial ingredient in the proof of Theorems 2.1 and 2.2 is Proposition 5.1, which shows that if (1.31) holds, then $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$ for any positive solution of (1.3) satisfying $\lim_{|x|\to 0} u(x)/|x|^{2-N+p} = 0$. While $\limsup_{|x|\to 0} |x|^p u(x) < \infty$ is a simple consequence of the comparison principle (see Lemma A.9 in Appendix A), the proof of $\liminf_{|x|\to 0} |x|^p u(x) > 0$ is quite intricate. To this end, we show that u is bounded from below by a positive solution v of a suitable ODE. We aim to prove that $\lim_{r\to 0} r^p v(r) \in (0,\infty)$ for $\lambda \le 0$ (when we need to rule out $\lim_{r\to 0} r^p v(r) = 0$) and $\lim_{|x|\to 0} u(x) = \infty$ for $0 < \lambda < (N-2)^2/4$. We proceed by contradiction and perform a suitable change of variable

$$y(s) = r^p v(r) \quad \text{with } s = r^{2p-N+2}. \tag{1.32}$$

This leads to a second order linear ODE for which we can apply Theorem 1.14 in [**28**] to reach a contradiction. To conclude that $\liminf_{|x|\to 0} |x|^p u(x) > 0$ for $0 < \lambda < (N-2)^2/4$, we need an extra step that requires (1.31) and relies on the results in Chapter 3. Having proved that $\limsup_{|x|\to 0} |x|^p u(x) \in (0,\infty)$, we modify a blow-up technique of Friedman–Véron [**19**] (also used in Theorem 5.1 of [**15**]) to complete the proof of Proposition 5.1. Our adaptation in Lemma 5.4 takes into account the inverse square potential, which does not appear in [**19**] or [**15**].

Theorem 2.4 extends and improves through a different approach Theorem 1.1 in [**11**], where (1.3) is considered for $0 < \lambda < (N-2)^2/4$, $b = 1$ and $1 < q < q^*$. More precisely, we prove that Case (B) of Theorem 2.4 holds for a positive solution u of (1.3) satisfying $\limsup_{|x|\to 0} |x|^{N-2-p} u(x) \in (0,\infty)$ by applying a blow-up technique similar to Lemma 5.4. By Proposition 5.1 explained above, we have that Case (A) in Theorem 2.4 occurs whenever $\lim_{|x|\to 0} |x|^{N-2-p} u(x) = 0$. Indeed, (2.8) shows that (1.31) holds if the limit in (1.30) is finite because the integral in (1.30) and (1.31) corresponds to $\mathcal{I}^*$ and $\mathcal{I}^{**}$ in (2.4), respectively. For Theorem 2.4(C) with $q < q^*$, we establish (1.10) using a perturbation technique inspired by Theorem 1.4 in [**15**], but we simplify our construction of sub-super-solutions in Lemma 5.13. The trichotomy of solutions in Theorem 2.4 is also valid for $q = q^*$ provided that the limit in (1.30) is finite. This sharp condition is crucially involved in the asymptotic behaviour of (2.21), which pertains to Theorem 2.4(C) with $q = q^*$. In this case, we have $\log u(x) \sim \log(|x|^{2-N+p})$ as $|x| \to 0$ and, under the extra assumption (1.12)(a), we obtain that

$$h(u(x)) \sim L_h(|x|^{2-N+p})[u(x)]^{q^*} \quad \text{as } |x| \to 0.$$

We are now able to reduce our investigation to the study of positive radial solutions for equations with critical power nonlinearities and apply Corollary 3.3 in Chapter 3.

In relation to Theorem 2.2, we note that $q \ge q^*$ if (1.30) holds. By Proposition 5.1, it remains to prove that (1.30) implies (1.29) and we need to treat separately $q = q^*$ (when $q > q^*$ we can use Corollary 4.3). Our argument in Lemma 5.8 is new and distinct from Theorem 1.3 of [**15**], which was compared with Theorem 2.2 in Section 1.3. By Harnack inequality, it is enough to show that $\liminf_{|x|\to 0} u(x)/|x|^{2-N+p} = 0$ (see Corollary 4.5). We proceed by contradiction and reduce to the case of positive solutions for ODEs studied in Proposition 3.1.

For Theorem 2.3, we adapt the ideas in Lemma 5.8 to prove that when the integral in (1.31) is infinite, then $\lim_{|x|\to 0} |x|^p u(x) = 0$ for every positive solution of (1.3). This enables us to prove here *a priori* that all positive solutions of (1.3) are asymptotic at zero to any positive C^2-function $\mathcal{U}$ satisfying

$$\mathcal{U}''(r) + \frac{N-1}{r}\mathcal{U}'(r) + \frac{\lambda}{r^2}\mathcal{U}(r) \sim r^\theta L_b(r)\tilde{h}(\mathcal{U}(r)) \quad \text{as } r \to 0, \tag{1.33}$$

where $\tilde{h}$ is given by (1.5) (see Lemma 5.10). The proof uses again a suitable reduction to the case of radial solutions v, in conjunction with a change of variable of the type (1.32). For the new equation, we show that any two positive solutions are asymptotic as $s \to \infty$ using an argument inspired by Theorem 1.1 in Taliaferro [**36**]. Each of the explicit asymptotic behaviour of the solutions in Theorem 2.3(a)–(d) is found by constructing adequate C^2-functions $\mathcal{U}$ satisfying (1.33). This is an extremely useful piece of information that was not available a priori in Case (C) of Theorem 2.4, which explains the different approach between these two theorems (although they share (1.10) for non-critical exponents).

1.4.2. Organization of the paper. All the main results summarized above for $-\infty < \lambda < (N-2)^2/4$ will be stated precisely in Section 2.1 of Chapter 2 and proved in Chapter 5. Although analogous to a certain extent, the results for $\lambda = (N-2)^2/4$ are very different from those in the case $-\infty < \lambda < (N-2)^2/4$. In Section 2.2 of Chapter 2, we present our main theorems for $\lambda = (N-2)^2/4$ (see Theorems 2.5–2.7), which will be demonstrated in Chapter 6. Often in the proofs of our main results, we shall try to reduce to the study of positive radial solutions to equations with power nonlinearities $h(t) = t^q$ with $q > 1$, whose asymptotic properties near zero are investigated in Chapter 3. But to extend these properties to all positive solutions of (1.3) in the framework of (1.5), we need many more tools and a number of techniques as outlined in our previous section. In Chapter 4 we include a number of basic ingredients such as *a priori* estimates for the positive solutions of (1.3), a Harnack-type inequality and a regularity lemma. These results and their consequences form the foundation on which we can develop our methods in Chapters 5 and 6 to establish the main results for (1.3) with $\lambda < (N-2)^2/4$ and $\lambda = (N-2)^2/4$, respectively. In Chapter 7, we illustrate our complete classification results for (1.3) in several situations. In particular, in Section 7.1, we show all the possible behaviour near zero of the positive solutions of (1.3) when

$$h(t) \sim t^q (\log t)^{\alpha_1} \quad \text{as } t \to \infty \quad \text{and} \quad b(x) \sim |x|^\theta [\log(1/|x|)]^{\alpha_2} \quad \text{as } |x| \to 0,$$

where $q > 1$, $\theta > -2$ and α_1, α_2 are any real numbers. We distinguish between $\lambda < (N-2)^2/4$ in Corollary 7.1 and $\lambda = (N-2)^2/4$ in Corollary 7.3. Finally, in Appendix A, we include properties of regularly varying functions needed in this paper along with a comparison principle (Lemma A.9) and some asymptotic results.

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CHAPTER 2

Main results

In this paper, we establish a complete classification of all positive solutions of (1.3) for any λ with $-\infty < \lambda \le (N-2)^2/4$ in the presence of (1.5) and any of the following sets of hypotheses:

- (1.12)(a) and (1.12)(b). We later illustrate this complete classification on the example in (1.15) (see Section 7.1 in Chapter 7).
- (1.12)(a) and (1.16)(a) (see Corollary 7.6);
- (1.12)(a) and (1.16)(b) (see Corollary 7.7);
- (1.12)(b), (1.16)(c) and S is regularly varying at ∞ (see Corollary 7.8).

The condition $\lambda \le (N-2)^2/4$ can be seen as related to the Hardy inequality (see, for example, [**8**] and [**1**]). More precisely, if Ω_1 is a bounded open subset of $\mathbb{R}^N$ ($N \ge 3$) and $0 \in \Omega_1$, then for $R > 0$ sufficiently large, there exists a positive constant C, depending on N and R, such that

$$\int_{\Omega_1} |\nabla u|^2\,dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega_1} \frac{u^2}{|x|^2}\,dx \ge C \int_{\Omega_1} \frac{u^2}{|x|^2(\log(R/|x|))^2}\,dx \tag{2.1}$$

for every $u \in H_0^1(\Omega_1)$. The equality holds if and only if $u = 0$ and $(N-2)^2/4$ is the best constant, which is never achieved. As noted in Chapter 1, it is essential to distinguish between λ *less* than $(N-2)^2/4$, referred to as the subcritical parameter, and λ *equal* to $(N-2)^2/4$, the critical parameter. We separate our main results accordingly (see §2.1 for the subcritical parameter and §2.2 for the critical parameter). The reason for this distinction is that the fundamental solutions of

$$-\Delta u - \frac{\lambda}{|x|^2} u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\} \tag{2.2}$$

play a crucial role in understanding the behaviour near zero of the positive solutions for (1.3). These fundamental solutions are given by (2.3) for $\lambda < (N-2)^2/4$ and by (2.24) for $\lambda = (N-2)^2/4$. As remarked in [**22**], Φ_λ^- is a *regular solution* of (2.2) in the sense that $\lambda|\cdot|^{-2}\Phi_\lambda^-(\cdot)$ is locally integrable in $\mathbb{R}^N$ and

$$-\Delta \Phi_\lambda^-(x) - \frac{\lambda}{|x|^2}\Phi_\lambda^-(x) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

The same holds for Φ_λ^+ if and only if $0 < \lambda < (N-2)^2/4$. Similarly, $\Psi^\pm$ given by (2.24) is a regular solution of (2.2) for $\lambda = (N-2)^2/4$.

Another variance in our analysis occurs at the subcritical level between $\lambda \le 0$ and $0 < \lambda < (N-2)^2/4$. A non-positive parameter dissociates from the case of a positive subcritical λ when it comes to the existence of positive solutions of (1.3) satisfying $\lim_{|x|\to 0} u(x)/\Phi_\lambda^-(x) = 0$. There are *no* such solutions when $\lambda \le 0$, which in terms of the behaviour of an arbitrary positive solution u of (1.3) translates as either a single-type behaviour ($u(x)/\Phi_\lambda^-(x)$ converges to a positive number as

$|x| \to 0$ in the framework of Theorem 2.2) or a trichotomy as stated in Theorem 2.4. This classification will give rise to only one critical exponent $q = q^*$ which is defined in (1.11). A more complicated picture emerges for a positive subcritical parameter λ as another possible behaviour must be accommodated, namely positive solutions satisfying $\lim_{|x|\to 0} u(x)/\Phi_\lambda^-(x) = 0$ (see Theorem 2.3). As a result, a second critical exponent $q = q^{**}$ arises from our analysis ($q^{**} > q^*$ as it can be seen from (1.11)). The two critical exponents q^* and q^{**} will merge when $\lambda = (N-2)^2/4$.

The reasons for which our results are separated at the above two levels will be clear once we give the necessary and sufficient conditions for the existence of positive solutions of (1.3) that are comparable with the fundamental solutions of (2.2). Two key players in the formulation of these conditions are f and $\mathcal{J}$ in (1.8).

Before proceeding further, a reader interested in getting some intuition behind our argument might want to look now at Chapter 3. There we study the positive *radially symmetric* solutions of (1.3) in $B_1(0) \setminus \{0\}$ when $h(t) = t^q$ with $q > 1$ and $b(x) = b_0(|x|)$ for $0 < |x| \le 1$. Such knowledge provides the motivation and the guiding principle in the attempt to extend the results in Chapter 3 on the positive solutions of (3.1) to *all* positive solutions of (1.3). All our main results in relation to (1.3) are established under the implicit structural assumption (1.5).

2.1. The subcritical parameter

2.1.1. Preliminaries. In Section 2.1, we assume that $-\infty < \lambda < (N-2)^2/4$. Let $\Phi_\lambda^\pm$ denote the fundamental solutions of (2.2), namely

$$\Phi_\lambda^+(x) = |x|^{2-N+p} \quad \text{and} \quad \Phi_\lambda^-(x) = |x|^{-p} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \tag{2.3}$$

where p is given by (1.9). In Theorem 2.1 we provide sharp conditions for (1.3) to admit positive solutions such that $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x) \in (0,\infty)$, respectively $\lim_{|x|\to 0} u(x)/\Phi_\lambda^-(x) \in (0,\infty)$. To formulate these conditions, we fix $\varpi > 0$ sufficiently small and define

$$\mathcal{I}^*(\tau,\varpi) := \int_\tau^\varpi \frac{\mathcal{J}(r) f(\Phi_\lambda^+(r))}{r}\, dr; \qquad \mathcal{I}^{**}(\tau,\varpi) := \int_\tau^\varpi \frac{\mathcal{J}(r) f(\Phi_\lambda^-(r))}{r}\, dr \tag{2.4}$$

for every $\tau \in (0,\varpi)$. Using (1.8), we see that for any $r > 0$ small, we have

$$\begin{cases} \mathcal{J}(r) f(\Phi_\lambda^+(r)) = r^{N+\theta-p-(N-2-p)q} L_b(r)\, L_h(\Phi_\lambda^+(r)), \\ \mathcal{J}(r) f(\Phi_\lambda^-(r)) = r^{\theta+2-p(q-1)} L_b(r)\, L_h(\Phi_\lambda^-(r)). \end{cases} \tag{2.5}$$

If $\lim_{\tau\to 0} \mathcal{I}^*(\tau,\varpi) < \infty$, then we observe that

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{\mathcal{J}(r) f(\Phi_\lambda^+(r))}{\int_0^r \frac{\mathcal{J}(t) f(\Phi_\lambda^+(t))}{t}\, dt} = N + \theta - p - (N-2-p)q \ge 0, \\ \lim\limits_{r\to 0} \Phi_\lambda^+(r)/\mathcal{K}(r) = 0. \end{cases} \tag{2.6}$$

For the first limit in (2.6), we use Karamata's Theorem at zero (see Proposition A.6 in Appendix A). Thus, by (1.8), we have $\lim_{r\to 0} f(\Phi_\lambda^+(r))/f(\mathcal{K}(r)) = 0$. The second limit in (2.6) follows from $f \in RV_{q-1}(\infty)$ and $\lim_{r\to 0} \Phi_\lambda^+(r) = \lim_{r\to 0} \mathcal{K}(r) = \infty$.

Similarly, if $\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty$, then we find that

$$\begin{cases} \lim\limits_{r\to 0}\dfrac{\mathcal{J}(r)f(\Phi_\lambda^-(r))}{\int_0^r \frac{\mathcal{J}(t)f(\Phi_\lambda^-(t))}{t}\,dt}=\theta+2-p(q-1)\geq 0,\\ \lim\limits_{r\to 0}\Phi_\lambda^-(r)/\mathcal{K}(r)=0. \end{cases} \tag{2.7}$$

Note that the second limit in (2.7) is obvious if $\lambda\leq 0$ since $p\leq 0$.

Since f is an increasing function, we clearly have that

$$\text{if } \lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)<\infty,\quad \text{then } \lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty. \tag{2.8}$$

More generally, when $q\neq q^*$ with q^* given by (1.11), then

$$\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)<\infty \text{ if and only if } q<q^*. \tag{2.9}$$

However, the case $q=q^*$ must be analyzed carefully to see whether $\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)$ is finite or not. On the example of (1.15), we see that for $q=q^*$, we have $\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)=\infty$ if and only if $\alpha_1+\alpha_2\geq -1$.

The case when $\lambda\leq 0$ distinguishes from $\lambda>0$ in that we always have

$$\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty \;\text{ if } \lambda\leq 0. \tag{2.10}$$

For this reason, the classification of the positive solutions of (1.3) for $\lambda\leq 0$ needs to be analyzed only in two cases:

$$\begin{cases} \lim\limits_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)=\infty \;\text{ (see Theorem 2.2)},\\ \lim\limits_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)<\infty \;\text{ (see Theorem 2.4)}. \end{cases}$$

For $0<\lambda<(N-2)^2/4$, we observe the following:

(a) When $q\neq q^{**}$ with q^{**} given by (1.11), then

$$\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty \;\text{ if and only if } q<q^{**}. \tag{2.11}$$

The case $q=q^{**}$ is not clear *a priori* as shown by the example of (1.15) when $\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty$ if and only if $\alpha_1+\alpha_2<-1$.

(b) If $\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)=\infty$, then $\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)=\infty$ (see (2.8)).

2.1.2. Statements of main results.

We shall denote $B^*:=B_1(0)\setminus\{0\}$.

THEOREM 2.1. *Let $-\infty<\lambda<(N-2)^2/4$ and (1.5) hold. Assume that $h(t)/t$ is increasing on $(0,\infty)$. Then there always exist positive solutions u of (1.3) in B^* such that $\lim_{|x|\to 0}u(x)/\Phi_\lambda^+(x)=0$ and, moreover, any such solution satisfies*

$$\lim_{|x|\to 0}\frac{u(x)}{\Phi_\lambda^-(x)}\in(0,\infty) \;\textit{ if and only if }\; \lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty. \tag{2.12}$$

There exist positive solutions u of (1.3) in B^* satisfying

$$\lim_{|x|\to 0}\frac{u(x)}{\Phi_\lambda^+(x)}\in(0,\infty) \;\textit{ if and only if }\; \lim_{\tau\to 0}\mathcal{I}^{*}(\tau,\varpi)<\infty. \tag{2.13}$$

We next investigate the behaviour of all positive solutions of (1.3) under the assumption (2.14), revealing q^* as a critical exponent (and the only one if $\lambda\leq 0$).

THEOREM 2.2. *Let* $-\infty < \lambda < (N-2)^2/4$ *and* (1.5) *hold. If*

$$\lim_{\tau\to 0} \mathcal{I}^*(\tau,\varpi) = \infty \quad \text{and} \quad \lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) < \infty, \tag{2.14}$$

then any positive solution u of (1.3) *can be extended as a solution of* (1.3) *in the whole Ω and the ratio $u(x)/\Phi_\lambda^-(x)$ converges to some positive number as $|x| \to 0$.*

The conclusion of Theorem 2.2 is different if instead of $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) < \infty$ we require $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) = \infty$, which make sense only for $0 < \lambda < (N-2)^2/4$. In this case, our next result unveils q^{**} as the *second critical exponent*.

THEOREM 2.3. *Let* $0 < \lambda < (N-2)^2/4$ *and* (1.5) *hold. If*

$$\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) = \infty,$$

then $q \geq q^{**}$ and all positive solutions u of (1.3) *are asymptotic at* 0 *with*

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^-(x)} = 0.$$

Moreover, for every positive solution u of (1.3)*, we have:*

(a) *If $q > q^{**}$, then u satisfies* (1.10).

(b) *If $q = q^{**}$ and* (1.12)*(a) holds, then $u(x) \sim U^{**}(|x|)$ as $|x| \to 0$, where*

$$U^{**}(|x|) = \Phi_\lambda^-(x)\,[\mathcal{M}\,\mathcal{I}^{**}(|x|,\varpi)]^{-\frac{1}{q-1}} \quad \text{and} \quad \mathcal{M} := \frac{q-1}{N-2-2p}. \tag{2.15}$$

(c) *If $q = q^{**}$ and either* (1.16)*(c) or* (1.16)*(d) holds such that in either case*

$$\mathbf{D} := \lim_{\tau\to 0} \frac{\log \mathcal{I}^{**}(\tau,\varpi)}{S(\log \Phi_\lambda^-(\tau))} < \infty \quad \text{for some } \varpi > 0, \tag{2.16}$$

then u satisfies $u(x) \sim \mathbf{C}\,U^{**}(x)$ as $|x| \to 0$, where U^{**} is given by (2.15)*, while $\mathbf{C} := e^{\mathbf{D}/(q-1)^2}$ (respectively, $e^{-\mathbf{D}/(q-1)^2}$) in case of* (1.16)*(c) (respectively,* (1.16)*(d)).*

(d) *If $q = q^{**}$ and* (1.12)*(b) holds, jointly with* (1.16)*(c) such that S is regularly varying at ∞ with index η, then*

$$u(x) \sim \left(\frac{p^{1-\eta}}{\mathcal{M}}\right)^{\frac{1}{q-1}} f^{-1}\left(\frac{1}{\mathcal{J}(|x|)\,S(\log(1/|x|))}\right) \quad \text{as } |x| \to 0, \tag{2.17}$$

where $f^{-1}(t)$ denotes the inverse of f at t, while f and $\mathcal{J}$ are as in (1.8).

Suppose that we are in the framework of Theorem 2.3. From the viewpoint of calculations, we remark that the asymptotic behaviour of U^{**} in (2.15) can be simplified under some additional assumption on L_b. First, if (1.12)(b) holds in Theorem 2.3(b), then (1.12)(a) and the change of variable $t = \log(1/r)$ in $\mathcal{I}^{**}(\tau,\varpi)$ yield that $t \longmapsto L_b(e^{-t})\,L_h(e^t)$ is regularly varying at ∞ with index $\alpha_1+\alpha_2$ satisfying $\alpha_1+\alpha_2 \geq -1$. If $\alpha_1+\alpha_2 > -1$, then $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) = \infty$ holds automatically and by Proposition A.6 in Appendix A, we find that

$$\frac{u(x)}{\Phi_\lambda^-(x)} \sim \left[\frac{\mathcal{M}\,p^{\alpha_1}}{\alpha_1+\alpha_2+1}\,L_b(|x|)\,L_h\left(\frac{1}{|x|}\right)\log\left(\frac{1}{|x|}\right)\right]^{\frac{-1}{q-1}} \quad \text{as } |x| \to 0, \tag{2.18}$$

where $\mathcal{M}$ is as in (2.15). This situation is relevant for (1.15) in Example 1.6, which is treated completely in Corollary 7.1. Second, if (1.16)(a) holds in Theorem 2.3(b), then $\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)=\infty$ since Λ varies at ∞ faster than any power function. Moreover, $u(x)\sim U^{**}(|x|)$ as $|x|\to 0$ gives that

$$\frac{u(x)}{\Phi_\lambda^-(x)}\sim\left[\mathcal{M}p^{\alpha_1}\,L_b(|x|)\,L_h\left(\frac{1}{|x|}\right)\,S\left(\log\left(\frac{1}{|x|}\right)\right)\right]^{\frac{-1}{q-1}}\quad\text{as } |x|\to 0. \tag{2.19}$$

The latter scenario will be analyzed fully in Corollary 7.6 for any $\lambda\le(N-2)^2/4$ and $q>1$. We also refer to Corollary 7.8 for a preview of the classification results complementing Theorem 2.3(d) for every $q>1$ and every $\lambda\le(N-2)^2/4$.

Our next aim is to classify the behaviour near zero of the positive solutions of (1.3) when $\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)<\infty$.

THEOREM 2.4. *Let $-\infty<\lambda<(N-2)^2/4$ and* (1.5) *hold. If*

$$\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)<\infty,$$

then $q\le q^*$ and for any positive solution u of (1.3), *one of the following occurs:*

A. *$u(x)/\Phi_\lambda^-(x)$ converges to a positive number as $|x|\to 0$ and u can be extended to a solution of* (1.3) *in the whole Ω;*

B. *$u(x)/\Phi_\lambda^+(x)$ converges to a positive number γ^+ as $|x|\to 0$. Moreover, u extends to a solution of* (1.3) *in Ω provided that $0<\lambda<(N-2)^2/4$, whereas for $\lambda=0$ we have*

$$-\Delta u+b(x)\,h(u)=(N-2)N\omega_N\gamma^+\delta_0\quad\text{in } \mathcal{D}'(\Omega), \tag{2.20}$$

where ω_N and δ_0 denote the volume of the unit ball in $\mathbb{R}^N$ and the Dirac mass at 0, respectively.

C. *$u(x)/\Phi_\lambda^+(x)\to\infty$ as $|x|\to 0$, in which case we find that:*
   (1) *If $q<q^*$, then u satisfies* (1.10).
   (2) *If $q=q^*$ and* (1.12)*(a) holds, then defining $\mathcal{M}$ as in* (2.15), *we have*

$$\frac{u(x)}{\Phi_\lambda^+(x)}\sim\left[\mathcal{M}\,\mathcal{I}^*(|x|)\right]^{-\frac{1}{q-1}}\quad\text{as } |x|\to 0,\ \text{with } \mathcal{I}^*(|x|):=\lim_{\tau\to 0}\mathcal{I}^*(\tau,|x|). \tag{2.21}$$

The asymptotic behaviour in (2.21) found in Case (C2) of Theorem 2.4 can be fine-tuned in some cases. We illustrate two frameworks. First, if (1.12)(b) is satisfied, then using (1.12)(a) and the change of variable $t=\log(1/r)$ in $\mathcal{I}^*(\tau,\varpi)$, we see that $t\longmapsto L_b(e^{-t})\,L_h(e^t)$ is regularly varying at ∞ with index $\alpha_1+\alpha_2$, where $\alpha_1+\alpha_2\le -1$. Furthermore, when $\alpha_1+\alpha_2<-1$, then $\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)<\infty$ and using Karamata's Theorem in Appendix A, we refine (2.21) by

$$\frac{u(x)}{\Phi_\lambda^+(x)}\sim\left[\frac{\mathcal{M}\,(N-2-p)^{\alpha_1}}{-(\alpha_1+\alpha_2+1)}\,L_b(|x|)\,L_h\left(\frac{1}{|x|}\right)\log\left(\frac{1}{|x|}\right)\right]^{\frac{-1}{q-1}}\quad\text{as } |x|\to 0. \tag{2.22}$$

Second, if in Case (C2) of Theorem 2.4, we assume (1.16)(b) besides (1.12)(a), then $\lim_{\tau\to 0}\mathcal{I}^*(\tau,\varpi)<\infty$ is validated and from (2.21), we conclude that

$$\frac{u(x)}{\Phi_\lambda^+(x)}\sim\left[\mathcal{M}\,(N-2-p)^{\alpha_1}\,L_b(|x|)\,L_h\left(\frac{1}{|x|}\right)\,S\left(\log\frac{1}{|x|}\right)\right]^{\frac{-1}{q-1}}\quad\text{as } |x|\to 0. \tag{2.23}$$

In relation to the second framework, we refer to Corollary 7.7 for a display of the classification results we obtain for every $q>1$ and all $\lambda\le(N-2)^2/4$.

2.2. The critical parameter

2.2.1. Preparation. In Section 2.2, we always take $\lambda = (N-2)^2/4$ when the fundamental solutions $\Psi^{\pm}$ of (2.2) are given by

$$\Psi^+(x) = |x|^{-\left(\frac{N-2}{2}\right)} \log(1/|x|) \quad \text{and} \quad \Psi^-(x) = |x|^{-\left(\frac{N-2}{2}\right)}. \tag{2.24}$$

Analogous to Theorem 2.1, we provide in Theorem 2.5 necessary and sufficient conditions for (1.3) to possess positive solutions with $\lim_{|x|\to 0} u(x)/\Psi^+(x) \in (0,\infty)$ (respectively, $\lim_{|x|\to 0} u(x)/\Psi^-(x) \in (0,\infty)$). However, these conditions involve two kinds of integrals different from those in Theorem 2.1.

We fix $\varpi > 0$ sufficiently small. For every $\tau \in (0,\varpi)$, we define

$$\begin{cases} \mathcal{F}_*(\tau,\varpi) := \displaystyle\int_\tau^\varpi \left(\log\frac{1}{r}\right) \frac{\mathcal{J}(r) f(\Psi^+(r))}{r}\, dr; \\ \mathcal{F}^*(\tau,\varpi) := \displaystyle\int_\tau^\varpi \left(\log\frac{1}{r}\right) \frac{\mathcal{J}(r) f(\Psi^-(r))}{r}\, dr. \end{cases} \tag{2.25}$$

Using (1.8), we see that for any $r > 0$ small, we have

$$\begin{cases} \mathcal{J}(r) f(\Psi^+(r)) = r^{\frac{N+2\theta+2-(N-2)q}{2}}\, L_b(r)\, L_h(\Psi^+(r)) \left(\log\frac{1}{r}\right)^{q-1}, \\ \mathcal{J}(r) f(\Psi^-(r)) = r^{\frac{N+2\theta+2-(N-2)q}{2}}\, L_b(r)\, L_h(\Psi^-(r)). \end{cases}$$

Defining ℓ as in (1.9), we observe that $\ell(q) = 0$ yields a double root $q = q^*$ with

$$q^* = \frac{N+2\theta+2}{N-2} \quad \text{since} \quad \ell = \left[\Theta - \frac{N-2}{2}\right]^2. \tag{2.26}$$

The behaviour of the positive solutions of (1.3) at zero depends on $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi)$ (respectively, $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi)$) being finite or infinite. Since f is increasing on $(0,\infty)$, we observe that

$$\text{if } \lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty, \text{ then } \lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty. \tag{2.27}$$

When $q \neq q^*$, then $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty$ ($\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty$) if and only if $q < q^*$. If $q = q^*$, then $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi)$ may be finite in some cases and infinite in others and the same goes for $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi)$. This fact can be observed on the example in (1.15) with $q = q^*$ for which we have the following:

- $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) = \infty$ and $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty$ if $-q-1 \le \alpha_1+\alpha_2 < -2$,
- $\mathcal{F}^*(\tau,\varpi)$ tends to ∞ as $\tau \to 0$ if $\alpha_1 + \alpha_2 \ge -2$,
- both $\mathcal{F}^*(\tau,\varpi)$ and $\mathcal{F}_*(\tau,\varpi)$ have finite limits as $\tau \to 0$ if $\alpha_1+\alpha_2 < -q-1$.

In view of (2.27), the following three cases exhaust all the possibilities:

$$\begin{cases} \text{(a)} \ \lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) = \infty \ \text{ and } \ \lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty; \\ \text{(b)} \ \lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) = \infty; \\ \text{(c)} \ \lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty. \end{cases} \tag{2.28}$$

We reveal the dynamics near zero of all positive solutions of (1.3) by subsuming under Theorem 2.6 the first two cases in (2.28). For the last one, see Theorem 2.7.

2.2.2. Statements of main results. We start by giving a result corresponding to Theorem 2.1 for the critical parameter $\lambda = (N-2)^2/4$.

THEOREM 2.5. *Let $\lambda = (N-2)^2/4$ and (1.5) hold. Assume that $h(t)/t$ is increasing on $(0,\infty)$. Then there always exist positive solutions u of (1.3) in B^* such that $\lim_{|x|\to 0} u(x)/\Psi^+(x) = 0$. Moreover, any such solution satisfies*

$$\lim_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} \in (0,\infty) \quad \textit{if and only if} \quad \lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty. \tag{2.29}$$

There exist positive solutions u of (1.3) in B^* satisfying

$$\lim_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} \in (0,\infty) \quad \textit{if and only if} \quad \lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty. \tag{2.30}$$

In Theorem 2.6 we study the asymptotic behaviour near zero of the positive solutions of (1.3) in the case $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) = \infty$ under an additional hypothesis:

$$\begin{cases} [\log(1/r)]^{q+3} L_b(r)\, L_h(\Psi^+(r)) \text{ be asymptotic to a non-increasing} \\ C^1\text{-function at } 0^+ \text{ whenever } q = q^*. \end{cases} \tag{2.31}$$

We cannot expect to dispense of (2.31) since it is essentially used to rule out solutions u of (1.3) satisfying $\lim_{|x|\to 0} u(x)/\Psi^+(x) = \infty$ (see Remark 3.6).

THEOREM 2.6. *Let $\lambda = (N-2)^2/4$ and (1.5) hold. When $q = q^*$, we also require that (2.31) be satisfied.*

(a) *If (2.28)(a) holds, then $q = q^*$ and $\lim_{|x|\to 0} u(x)/\Psi^-(x) \in (0,\infty)$ for every positive solution u of (1.3).*

(b) *If (2.28)(b) holds, then $q \geq q^*$ and all positive solutions u of (1.3) are asymptotic at 0 with $u(x)/\Psi^-(x) \to 0$ as $|x| \to 0$. Furthermore, the precise asymptotic behaviour at zero for any such solution u is as follows:*
   (1) *When $q > q^*$, then u satisfies (1.10).*
   (2) *When $q = q^*$ and (1.12) holds, then we have $\alpha_1 + \alpha_2 \geq -2$ and*

$$\frac{u(x)}{\Psi^-(x)} \sim \left[\frac{(q-1)^2}{q+\alpha_1+\alpha_2+1}\, \mathcal{F}^*(|x|,\varpi)\right]^{-\frac{1}{q-1}} \quad \textit{as } |x| \to 0. \tag{2.32}$$

   (3) *When $q = q^*$ and (1.12)(a), (1.16)(a) hold, then as $|x| \to 0$*

$$\frac{u(x)}{\Psi^-(x)} \sim \left\{ \left(\frac{N-2}{2}\right)^{\alpha_1} \left[(q-1) S\left(\log\frac{1}{|x|}\right)\right]^2 L_b(|x|)\, L_h\left(\frac{1}{|x|}\right) \right\}^{\frac{-1}{q-1}}. \tag{2.33}$$

   (4) *When $q = q^*$ and (1.12)(b) holds, jointly with (1.16)(c) such that S is regularly varying at ∞ of index η, then as $|x| \to 0$*

$$u(x) \sim \left\{ \frac{[(N-2)/2]^{1-\eta}}{q-1} \right\}^{\frac{2}{q-1}} f^{-1}\left(\frac{1}{\mathcal{J}(|x|)\,[S(\log(1/|x|))]^2}\right), \tag{2.34}$$

where $f^{-1}(t)$ is the inverse of f at t, while f and $\mathcal{J}$ are as in (1.8).

From (1.5), the function in (2.31) is slowly varying at zero, which may not necessarily have a limit at 0. However, in Case (b2) of Theorem 2.6, the hypothesis (2.31) is automatically satisfied (since by letting $t = \log(1/r)$, the function in (2.31) is regularly varying at $t = \infty$ with *positive* index $q + 3 + \alpha_1 + \alpha_2$). Furthermore, if $\alpha_1 + \alpha_2 > -2$, then we always have $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) = \infty$ and using the change of variable $t = \log(1/r)$, jointly with Karamata's Theorem in Appendix A, the asymptotics in (2.32) becomes as $|x| \to 0$

$$\frac{u(x)}{\Psi^-(x)} \sim \left[\frac{(q-1)^2[(N-2)/2]^{\alpha_1} \left(\log \frac{1}{|x|}\right)^2 L_b(|x|)\, L_h(1/|x|)}{(2+\alpha_1+\alpha_2)(1+\alpha_1+\alpha_2+q)} \right]^{\frac{-1}{q-1}} . \tag{2.35}$$

In relation to Corollaries 7.7 and 7.8, we see that for Theorem 2.6(b3) and (b4), the assumptions (2.28)(b) and (2.31) are easily verified (using Remark 1.7).

Finally, we assume that $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) < \infty$ and establish the asymptotic behaviour near zero of the positive solutions of (1.3). When (1.12) holds, then unless $q + \alpha_1 + \alpha_2 = -3$, the following monotonicity property is always fulfilled

$$\begin{cases} [\log(1/r)]^{q+3}\, L_b(r)\, L_h(1/r) \quad \text{is asymptotic as } r \to 0 \text{ to either} \\ \text{an increasing continuous function or a non-increasing } C^1 \text{ function.} \end{cases} \tag{2.36}$$

Indeed, the function in (2.36) is asymptotic to an increasing (respectively, decreasing) C^1 function at 0^+ if $q + \alpha_1 + \alpha_2 < -3$ (respectively, $q + \alpha_1 + \alpha_2 > -3$).

THEOREM 2.7. *Let $\lambda = (N-2)^2/4$ and* (1.5) *hold. If $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) < \infty$, then $q \le q^*$ and for any positive solution u of* (1.3)*, we have one of the following:*

A. $u(x)/\Psi^-(x)$ *converges to a positive number as* $|x| \to 0$*;*
B. $u(x)/\Psi^+(x)$ *converges to a positive number as* $|x| \to 0$*;*
C. $u(x)/\Psi^+(x) \to \infty$ *as* $|x| \to 0$ *in which case we precisely describe the asymptotic behaviour of u at 0 as follows:*
   (1) *If $q < q^*$, then u satisfies* (1.10).
   (2) *If $q = q^*$ and* (1.12), (2.36) *hold, then $q + \alpha_1 + \alpha_2 \le -1$ and*

$$\frac{u(x)}{\Psi^+(x)} \sim \left[\frac{(q-1)^2}{-2-\alpha_1-\alpha_2}\, \mathcal{F}_*(|x|) \right]^{\frac{-1}{q-1}} \quad \text{as } |x| \to 0, \tag{2.37}$$

   where $\mathcal{F}_*(|x|)$ is defined by $\mathcal{F}_*(|x|) := \lim_{\tau\to 0} \mathcal{F}_*(\tau, |x|)$.
   (3) *If $q = q^*$ and* (1.12)*(a),* (1.16)*(b) hold, then u satisfies* (2.33).

In Case (C3) of Theorem 2.7, we have $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) < \infty$ and, moreover, (2.36) holds (the function in (2.36) is asymptotic to a C^1-increasing function at 0).

For Theorem 2.7(C) with $q = q^*$, we have $\log u(x) \sim [(N-2)/2] \log(1/|x|)$ as $|x| \to 0$. By requiring (1.12)(a) in both (C2) and (C3) of Theorem 2.7, we find that $h(u(x)) \sim [(N-2)/2]^{\alpha_1} L_h(1/|x|)[u(x)]^{q^*}$ as $|x| \to 0$. We then need to decipher the profile near zero of the positive solutions of (3.1) with $\lim_{r\to 0} u(r)/\Psi^+(r) = \infty$ when $b_0(r) = [(N-2)/2]^{\alpha_1} r^\theta L_b(r) L_h(1/r)$ (see Proposition 3.4 in Chapter 3). The property (2.36) bears the same role as the monotonicity assumption in Proposition 3.4(d) or (e1), which guarantees that all positive solutions of (3.1) with $\lim_{r\to 0} u(r)/\Psi^+(r) = \infty$ are asymptotically equivalent at zero (see Remark 3.5). We need only check (2.36) for Theorem 2.7(C2) when $q + \alpha_1 + \alpha_2 = -3$.

If $q + \alpha_1 + \alpha_2 < -1$ in Theorem 2.7(C2), then $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) < \infty$ holds and by (2.37) and Karamata's Theorem in Appendix A, we conclude (2.35). This agrees (asymptotically) with the behaviour prescribed by Theorem 3.9 in [**36**] (using regular variation theory after applying the change of variable $y(s) = r^{(N-2)/2} u(r)$ with $s = \log(1/r)$). However, the asymptotic behaviour in (2.37) cannot be inferred from [**36**] when $q + \alpha_1 + \alpha_2 = -1$, which is a borderline case emphasizing the role of the sharp condition $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) < \infty$.

CHAPTER 3

Radial solutions in the power case

In this chapter, λ is a real parameter with $-\infty < \lambda \le (N-2)^2/4$, the nonlinearity $h(t) = t^q$ with $q > 1$ and $b_0(r)$ is a positive continuous function on $(0,1]$. We focus on the positive $C^2(0,1]$-solutions of the following equation

$$-u''(r) - \frac{N-1}{r}\,u'(r) - \frac{\lambda}{r^2}u(r) + b_0(r)\,u^q = 0 \quad \text{for } 0 < r < 1. \tag{3.1}$$

Our objective is to investigate the asymptotic properties as $r \to 0$ of the positive solutions of (3.1). In this context, we shift the singularity from 0 to ∞ by making a suitable change of variable which depends upon the parameter λ. More precisely, we define p as in (1.9). If $u(r)$ denotes a positive solution of (3.1), then we apply the change of variable (3.6) if $\lambda < (N-2)^2/4$ and (3.18) if $\lambda = (N-2)^2/4$. For the differential equation satisfied by $y(s)$, we can now invoke results already existing in the literature ([**36**], [**28**]) to understand the profile of the positive solutions of (3.1) near the origin. In Section 3.1 we present the results for the subcritical parameter $\lambda < (N-2)^2/4$, while in Section 3.2 we treat the critical case $\lambda = (N-2)^2/4$.

3.1. The subcritical parameter

We consider the case $-\infty < \lambda < (N-2)^2/4$. Let $\varpi > 0$ be fixed sufficiently small. For every $\tau \in (0, \varpi)$, we define

$$\begin{cases} \mathcal{I}_1(\tau,\varpi) := \displaystyle\int_\tau^\varpi r^{1-(N-2-p)(q-1)} b_0(r)\,dr, \\ \mathcal{I}_2(\tau,\varpi) := \displaystyle\int_\tau^\varpi r^{1-p(q-1)} b_0(r)\,dr. \end{cases} \tag{3.2}$$

We show that there exist positive solutions of (3.1) with $\lim_{r\to 0} u(r)/\Phi_\lambda^+(r) \in (0,\infty)$ if and only if $\lim_{\tau\to 0} \mathcal{I}_1(\tau,\varpi) < \infty$. For each $\gamma > 0$, Eq. (3.1) subject to $u(1) = \gamma$, admits a *unique* positive solution u such that $\lim_{r\to 0} u(r)/\Phi_\lambda^+(r) = 0$. However, this solution satisfies $\lim_{r\to 0} u(r)/\Phi_\lambda^-(r) \in (0,\infty)$ if and only if $\lim_{\tau\to 0} \mathcal{I}_2(\tau,\varpi) < \infty$. More precisely, we prove the following result.

PROPOSITION 3.1. *Let $-\infty < \lambda < (N-2)^2/4$ and $q > 1$. We have:*

(a) *There exist positive solutions of* (3.1) *with* $\lim_{r\to 0} u(r)/\Phi_\lambda^+(r) \in (0,\infty)$ *if and only if* $\lim_{\tau\to 0} \mathcal{I}_1(\tau,\varpi) < \infty$. *Moreover, if we also have*

$$\lim_{\tau\to 0} \int_\tau^\varpi r^{p+1-(N-2-p)q} b_0(r)\,dr < \infty, \tag{3.3}$$

then there exist some constants c and d with $c > 0$ such that

$$u(r) = c\,\Phi_\lambda^+(r) + d\,\Phi_\lambda^-(r) + o(\Phi_\lambda^-(r)) \quad \textit{as } r \to 0.$$

(b) *For any constant* $\gamma > 0$, *Eq.* (3.1), *subject to* $u(1) = \gamma$, *has a unique positive solution with* $\lim_{r\to 0} u(r)/\Phi_\lambda^+(r) = 0$. *This solution satisfies* $\lim_{r\to 0} u(r)/\Phi_\lambda^-(r) \in (0,\infty)$ *if and only if* $\lim_{\tau\to 0} \mathcal{I}_2(\tau,\varpi) < \infty$ *and, in this case, there exists a positive constant* c *such that*

$$\frac{u(r)}{\Phi_\lambda^-(r)} \sim c + \frac{c^q + o(1)}{N-2-2p}\int_0^r b_0(\zeta)\zeta^{N-1-(q+1)p}(\zeta^{2p-N+2} - r^{2p-N+2})\,d\zeta,$$

as $r \to 0$. *Furthermore, if* $\lim_{\tau\to 0} \mathcal{I}_2(\tau,\varpi) = \infty$, *then any two positive solutions* u_i ($i = 1, 2$) *of* (3.1) *with* $\lim_{r\to 0} u_i(r)/\Phi_\lambda^+(r) = 0$ *will satisfy* $\lim_{r\to 0} u_i(r)/\Phi_\lambda^-(r) = 0$ *and* $\lim_{r\to 0} u_1(r)/u_2(r) = 1$.

(c) *Assume that* $\lim_{\tau\to 0} \mathcal{I}_1(\tau,\varpi) < \infty$. *Then for any* $\gamma > 0$, *there exists a positive solution of* (3.1) *with* $u(1) = \gamma$ *and* $\lim_{r\to 0} u(r)/\Phi_\lambda^+(r) \in (0,\infty)$. *Moreover, there also exist positive solutions for the problem*

$$\begin{cases} -u''(r) - \dfrac{N-1}{r}u'(r) - \dfrac{\lambda}{r^2}u(r) + b_0(r)u^q = 0 & \text{for } 0 < r < 1, \\ \lim\limits_{r\to 0} u(r)/\Phi_\lambda^+(r) = \infty \quad \text{and} \quad u(1) = \gamma. \end{cases} \tag{3.4}$$

(d) *If* $r^{(q+3)(p-N+2)+2N-2}b_0(r)$ *is asymptotic as* $r \to 0$ *to an increasing continuous function, then* (3.4) *has a unique positive solution for every* $\gamma > 0$.

(e) *Assume that* $r^{-p(q+3)+2N-2}b_0(r) \sim b_1(r)$ *as* $r \to 0$, *where* $b_1(r)$ *is a positive* C^1*-function on* $(0,1]$ *such that*

$$\left[r^{(q+3)(2p-N+2)}b_1(r)\right]' \le 0 \quad \text{for } 0 < r \le 1. \tag{3.5}$$

1. *If* $\lim_{\tau\to 0} \mathcal{I}_1(\tau,\varpi) < \infty$, *then for every* $\gamma > 0$ *there exists exactly one positive solution of* (3.4).
2. *If* $\lim_{\tau\to 0} \mathcal{I}_1(\tau,\varpi) = \infty$, *then for each* $\gamma > 0$ *there exists exactly one positive solution of* (3.1) *with* $u(1) = \gamma$, *namely the positive solution whose behaviour is given in (b) above.*

PROOF. (a) We apply the change of variable

$$y(s) = u(r)/\Phi_\lambda^-(r) \quad \text{with } s = (N-2)r^{2p-N+2}. \tag{3.6}$$

Hence, $y(s)$ satisfies the following differential equation

$$y''(s) = \phi(s)\,[y(s)]^q \quad \text{for } s \in (N-2,\infty), \tag{3.7}$$

where ϕ is defined by

$$\phi(s) = \frac{(N-2)^{\frac{p(q-1)-2}{2p-N+2}}}{(2p-N+2)^2} s^{\frac{2(N-1)-p(q+3)}{2p-N+2}} b_0\left(\left(\frac{s}{N-2}\right)^{1/(2p-N+2)}\right).$$

Writing $Y(\bar{s}) = y(s)$ for $\bar{s} = s - N + 2$, then we have

$$Y''(\bar{s}) = \phi(\bar{s}+N-2)[Y(\bar{s})]^q \quad \text{for } \bar{s} \in (0,\infty). \tag{3.8}$$

As in [**36**], a positive solution Y which is twice continuously differentiable on $[0,\infty)$ is called a *proper* solution. If $Y(\bar{s})$ is a proper solution of (3.8), then $Y''(\bar{s}) \ge 0$ so that $\lim_{\bar{s}\to\infty} Y'(\bar{s})$ exists and is non-negative. We see that ϕ in (3.7) satisfies $\int_0^\infty \bar{s}^q\phi(\bar{s}+N-2)\,d\bar{s} < \infty$ (respectively, $\int_0^\infty \bar{s}^{q+1}\phi(\bar{s}+N-2)\,d\bar{s} < \infty$) if and only if $\lim_{\tau\to 0} \mathcal{I}_1(\tau,\varpi) < \infty$ (respectively, (3.3) holds). By Theorem 2.4 of [**36**], we know that (3.8) possesses positive proper solutions with $\lim_{\bar{s}\to\infty} Y'(\bar{s}) \in (0,\infty)$ if and only if $\int_0^\infty \bar{s}^q \phi(\bar{s}+N-2)\,d\bar{s}<\infty$. If $Y(\bar{s})$ is such a solution, then $\lim_{\bar{s}\to\infty} Y(\bar{s})/\bar{s}=c_0$ for some $c_0>0$. Furthermore, if $\int_0^\infty \bar{s}^{q+1}\phi(\bar{s}+N-2)\,d\bar{s}<\infty$, then we have

$$Y(\bar{s})=c_0\bar{s}+c_1+(c_0)^q[1+o(1)]\int_{\bar{s}}^\infty(\xi-\bar{s})\xi^q\phi(\xi+N-2)\,d\xi\quad\text{as } \bar{s}\to\infty, \tag{3.9}$$

for some constants c_0 and c_1 with $c_0>0$. Hence, (3.1) has positive solutions with $\lim_{r\to 0} u(r)/\Phi_\lambda^+(r)\in(0,\infty)$ if and only if $\lim_{\tau\to 0}\mathcal{I}_1(\tau,\varpi)<\infty$. Moreover, if (3.3) also holds, then (3.9) yields that

$$u(r)=c_0(N-2)\Phi_\lambda^+(r)+\Phi_\lambda^-(r)\left[c_1+c_0(2-N)+c_0^q(1+o(1))\frac{(N-2)^q}{N-2-2p}V(r)\right]$$

as $r\to 0$, where we define $V(r)$ by

$$V(r):=\int_0^r\left(\zeta^{2p-N+2}-r^{2p-N+2}\right)\left(\zeta^{2p-N+2}-1\right)^q\zeta^{N-1-p(q+1)}b_0(\zeta)\,d\zeta.$$

Using (3.3), we see that $V(r)\to 0$ as $r\to 0$, which concludes the proof of (a).

(b) Since $\int_0^\infty \bar{s}\phi(\bar{s}+N-2)\,d\bar{s}<\infty$ is equivalent to $\lim_{\tau\to 0}\mathcal{I}_2(\tau,\varpi)<\infty$, our claim follows by applying Theorem 1.1 in [**36**] to the proper solutions of (3.8), then returning to $u(r)$ via (3.6).

(c) We need to show that for any $\gamma>0$ there exist positive solutions of (3.8) with $Y(0)=\gamma$ and $\lim_{\bar{s}\to\infty}Y(\bar{s})/\bar{s}\in(0,\infty)$ (respectively, $\lim_{\bar{s}\to\infty}Y(\bar{s})/\bar{s}=\infty$). This follows by applying Corollary 2.5 in [**36**] (respectively, Theorem 3.2 of [**36**]) to (3.8) since $\lim_{\tau\to 0}\mathcal{I}_1(\tau,\varpi)<\infty$ yields that $\int_0^\infty \bar{s}^q\phi(\bar{s}+N-2)\,d\bar{s}<\infty$.

(d) Since $r^{(q+3)(p-N+2)+2(N-1)}b_0(r)$ is asymptotic to an increasing function as $r\to 0$, we have $\lim_{s\to\infty}\phi(s)/\phi_1(s)=1$ with $s^{q+3}\phi_1(s)$ decreasing for $s\geq s_0>0$. The claim follows from Theorem 3.10 in [**36**].

(e) We conclude (e1) and (e2) by using Theorem 3.12 in [**36**] for (3.8), jointly with (3.6). This finishes the proof of Proposition 3.1. □

Remark 3.2. In Proposition 3.1(d), we have $\lim_{\tau\to 0}\mathcal{I}_1(\tau,\varpi)<\infty$. In the settings of either Proposition 3.1(d) or (e1), all positive solutions of (3.1) such that $\lim_{r\to 0}u(r)/\Phi_\lambda^+(r)=\infty$ are asymptotically equivalent as $r\to 0$ to any positive C^2-function $\mathcal{U}$ satisfying

$$\begin{cases}\mathcal{U}''(r)+\dfrac{N-1}{r}\mathcal{U}'(r)+\dfrac{\lambda}{r^2}\mathcal{U}(r)\sim b_0(r)[\mathcal{U}(r)]^q\quad\text{as } r\to 0,\\ \lim\limits_{r\to 0}\mathcal{U}(r)/\Phi_\lambda^+(r)=\infty.\end{cases}\tag{3.10}$$

This follows via the change of variable (3.6) by applying [**36**, Theorem 3.7] to (3.7).

Our next result will be very useful in the proof of Lemma 5.14.

Corollary 3.3. *Let $-\infty<\lambda<(N-2)^2/4$ and b_0 be a regularly varying function at 0 with index θ. Let $\theta>-2$ and $q=q^*$ with q^* defined by* (1.11)*. If $\mathcal{I}_1(\tau,\varpi)$ in* (3.2) *satisfies $\lim_{\tau\to 0}\mathcal{I}_1(\tau,\varpi)<\infty$, then for every $\gamma>0$ there exists a unique positive solution $u_{\gamma,\infty}$ of* (3.4)*. Moreover, denoting $\mathcal{M}$ as in* (2.15)*, the solution $u_{\gamma,\infty}$ satisfies*

$$u_{\gamma,\infty}(r)\sim\Phi_\lambda^+(r)\left[\mathcal{M}\,\mathcal{I}_1(r)\right]^{\frac{-1}{q-1}}\quad\text{as } r\to 0,\quad\text{where } \mathcal{I}_1(r):=\lim_{\tau\to 0}\mathcal{I}_1(\tau,r).\tag{3.11}$$

PROOF. By Remark A.3 in Appendix A, there exists a $C^1(0,1]$-function $\tilde{b}_0$ with $\lim_{r\to 0} b_0(r)/\tilde{b}_0(r) = 1$ and $\lim_{r\to 0} r\tilde{b}_0'(r)/\tilde{b}_0(r) = \theta$. Since $q = q^*$, we see that

$$\mathcal{B}(r) := r^{(q+3)(p-N+2)+2N-2}\tilde{b}_0(r)$$

is a C^1-function on $(0,1]$ which varies regularly at zero with negative index, namely $-2(N-2-2p)$. Moreover, we have $\lim_{r\to 0} r\mathcal{B}'(r)/\mathcal{B}(r) = -2(N-2-2p)$. Hence, by Proposition 3.1(e), we infer that (3.4) has a unique positive solution for each $\gamma > 0$. To prove (3.11), we need only show that (3.10) holds for

$$\mathcal{U}(r) = \Phi_\lambda^+(r)\,[\mathcal{M}\,\mathcal{I}_1(r)]^{-1/(q-1)}\,.$$

This is a simple calculation, which is left to the reader. Since $\lim_{r\to 0}\mathcal{I}_1(r) = 0$, we clearly have $\lim_{r\to 0}\mathcal{U}(r)/\Phi_\lambda^+(r) = \infty$. $\square$

3.2. The critical parameter

In Proposition 3.4 below, we give an analogue of Proposition 3.1 corresponding to $\lambda = (N-2)^2/4$, when the fundamental solutions of (2.2) are given by $\Psi^\pm$ in (2.24). Let $\varpi \in (0,1]$ be fixed sufficiently small. For any $\tau \in (0,\varpi)$, we define

$$\begin{cases} \mathcal{F}_1(\tau,\varpi) := \displaystyle\int_\tau^\varpi r^{\frac{N-q(N-2)}{2}} b_0(r)\,[\log(1/r)]^q\,dr, \\ \mathcal{F}_2(\tau,\varpi) := \displaystyle\int_\tau^\varpi r^{\frac{N-q(N-2)}{2}} b_0(r)\log(1/r)\,dr. \end{cases} \tag{3.12}$$

We prove that (3.1) admits positive solutions with $\lim_{r\to 0} u(r)/\Psi^+(r) \in (0,\infty)$ if and only if $\lim_{\tau\to 0}\mathcal{F}_1(\tau,\varpi) < \infty$. Moreover, for any $\gamma > 0$, Eq. (3.1) with $u(1) = \gamma$ has a *unique* positive solution u such that $\lim_{r\to 0} u(r)/\Psi^+(r) = 0$. However, this solution satisfies $\lim_{r\to 0} u(r)/\Psi^-(r) \in (0,\infty)$ if and only if $\lim_{\tau\to 0}\mathcal{F}_2(\tau,\varpi) < \infty$.

PROPOSITION 3.4. *Let $\lambda = (N-2)^2/4$ and $q > 1$.*

(a) *Eq.* (3.1) *admits positive solutions satisfying* $\lim_{r\to 0} u(r)/\Psi^+(r) \in (0,\infty)$ *if and only if* $\lim_{\tau\to 0}\mathcal{F}_1(\tau,\varpi) < \infty$. *Furthermore, if we also have*

$$\lim_{\tau\to 0}\int_\tau^\varpi r^{\frac{N-q(N-2)}{2}} b_0(r)\,[\log(1/r)]^{q+1}\,dr < \infty, \tag{3.13}$$

then there exist some constants c and d with $c > 0$ such that

$$u(r) = c\,\Psi^+(r) + d\,\Psi^-(r) + o(\Psi^-(r)) \quad \textit{as } r \to 0. \tag{3.14}$$

(b) *For any number $\gamma > 0$, Eq.* (3.1) *with $u(1) = \gamma$ admits a unique positive solution with* $\lim_{r\to 0} u(r)/\Psi^+(r) = 0$. *This solution satisfies*

$$\lim_{r\to 0} u(r)/\Psi^-(r) := \alpha_0 \in (0,\infty)$$

if and only if $\lim_{\tau\to 0}\mathcal{F}_2(\tau,\varpi) < \infty$ *and in this case*

$$\frac{u(r)}{\Psi^-(r)} = \alpha_0 + [(\alpha_0)^q + o(1)]\int_0^r b_0(\zeta)\zeta^{\frac{N-q(N-2)}{2}}\log(r/\zeta)\,d\zeta \tag{3.15}$$

as $r \to 0^+$. Furthermore, if $\lim_{\tau\to 0}\mathcal{F}_2(\tau,\varpi) = \infty$, *then any positive solutions u_i ($i = 1,2$) of* (3.1) *with* $\lim_{r\to 0} u_i(r)/\Psi^+(r) = 0$ *will satisfy* $\lim_{r\to 0} u_1(r)/u_2(r) = 1$ *and* $\lim_{r\to 0} u_i(r)/\Psi^-(r) = 0$.

(c) *Assume that* $\lim_{\tau\to 0} \mathcal{F}_1(\tau, \varpi) < \infty$. *Then for any* $\gamma > 0$, *there exists a positive solution of* (3.1) *with* $u(1) = \gamma$ *and* $\lim_{r\to 0} u(r)/\Psi^+(r) \in (0, \infty)$. *Moreover, there also exist positive solutions for the problem*

$$\begin{cases} -u''(r) - \dfrac{N-1}{r}u'(r) - \dfrac{\lambda}{r^2}u(r) + b_0(r)u^q = 0 \quad \textit{for } 0 < r < 1, \\ \lim\limits_{r\to 0} u(r)/\Psi^+(r) = \infty \quad \textit{and} \quad u(1) = \gamma. \end{cases} \tag{3.16}$$

(d) *If* $[\log(1/r)]^{q+3}\, r^{\frac{N+2-q(N-2)}{2}} b_0(r)$ *is asymptotic to an increasing continuous function as* $r \to 0^+$, *then for every* $\gamma > 0$ *there exists exactly one positive solution of* (3.16).

(e) *Assume that* $r^{\frac{N+2-q(N-2)}{2}} b_0(r) \sim b_1(r)$ *as* $r \to 0^+$, *where* $b_1(r)$ *is a positive* C^1*-function on* $(0, 1]$ *such that*

$$\left[(\log(1/r))^{q+3}\, b_1(r)\right]' \leq 0 \quad \textit{for } 0 < r \leq 1. \tag{3.17}$$

1. *Let* $\lim_{\tau\to 0} \mathcal{F}_1(\tau, \varpi) < \infty$. *Then for every* $\gamma > 0$, *there exists exactly one positive solution of* (3.16).
2. *Let* $\lim_{\tau\to 0} \mathcal{F}_1(\tau, \varpi) = \infty$. *Then for each* $\gamma > 0$, *there exists exactly one positive* $C^2(0, 1]$*-solution of* (3.1) *with* $u(1) = \gamma$. *This solution has the property that* $u(r)/\Psi^-(r)$ *is increasing and its behaviour is given in (b) above.*

Proof. By applying the change of variable

$$y(s) = r^{\frac{N-2}{2}} u(r) \quad \text{with } s = \log(1/r), \tag{3.18}$$

we see that $y(s)$ satisfies the differential equation

$$y''(s) = \phi(s)[y(s)]^q \text{ for } s \in (0, \infty) \text{ with } \phi(s) := b_0(e^{-s})e^{-s[N+2-q(N-2)]/2}. \tag{3.19}$$

(a) Since $\lim_{s\to\infty} y'(s)$ exists, we have $\lim_{s\to\infty} y(s)/s \in (0, \infty)$ if and only if $\lim_{s\to\infty} y'(s) \in (0, \infty)$. By Theorem 2.4 in [**36**], we conclude that (3.19) has positive solutions satisfying $\lim_{s\to\infty} y(s)/s \in (0, \infty)$ if and only if

$$\int_0^\infty s^q \phi(s)\, ds < \infty,$$

which is equivalent to $\lim_{\tau\to 0} \mathcal{F}_1(\tau, \varpi) < \infty$. Moreover, if (3.13) holds, that is

$$\int_0^\infty s^{q+1} \phi(s)\, ds < \infty,$$

then $y(s) = cs + d + o(1)$ as $s \to \infty$ for some constants c and d with $c > 0$. This proves the assertion of (a).

(b) We apply Theorem 1.1 in [**36**], observing that $\int_0^\infty s\phi(s)\, ds = \infty$ is equivalent to $\lim_{\tau\to 0} \mathcal{F}_2(\tau, \varpi) = \infty$ since $s = \log(1/r)$.

(c) We use Theorem 3.2 in [**36**] and Corollary 2.5 in [**36**].

(d) The assertion follows immediately from Theorem 3.10 in [**36**].

(e) Using Theorem 3.12 and Theorem 1.1 in [**36**], we conclude the claim of (e1) and (e2), respectively. This completes the proof. □

Similar to Remark 3.2, we observe the following.

REMARK 3.5. In Proposition 3.4(d), the condition $\lim_{\tau\to 0}\mathcal{F}_1(\tau,\varpi)<\infty$ is always satisfied. In either Proposition 3.4(d) or (e1), all positive solutions of (3.1) with $\lim_{r\to 0}u(r)/\Psi^+(r)=\infty$ are asymptotic as $r\to 0$ to any positive C^2–function $\mathcal{U}$ satisfying

$$\begin{cases} \mathcal{U}''(r)+\dfrac{N-1}{r}\,\mathcal{U}'(r)+\dfrac{(N-2)^2}{4}\,\dfrac{\mathcal{U}(r)}{r^2}\sim b_0(r)[\mathcal{U}(r)]^q \quad \text{as } r\to 0,\\ \lim\limits_{r\to 0}\mathcal{U}(r)/\Psi^+(r)=\infty. \end{cases} \tag{3.20}$$

We use the change of variable (3.18) and apply Theorem 3.7 in [**36**] to (3.19). Moreover, under additional assumptions on b_0, we can obtain the precise asymptotic behaviour of $\mathcal{U}$ in (3.20) (apply Theorem 3.9 in [**36**] to (3.19)).

REMARK 3.6. If $\lim_{\tau\to 0}\mathcal{F}_1(\tau,\varpi)=\infty$ for $\lambda=(N-2)^2/4$ (respectively, $\lim_{\tau\to 0}\mathcal{I}_1(\tau,\varpi)=\infty$ for $\lambda<(N-2)^2/4$), then by Theorem 3.2 in [**36**], we have

(A) **either** for each $\gamma>0$, there are infinitely many positive solutions of (3.16) (respectively, (3.4));

(B) **or** there are no positive solutions of (3.1) with $\lim_{r\to 0}u(r)/\Psi^+(r)=\infty$ (respectively, $\lim_{r\to 0}u(r)/\Phi^+_\lambda(r)=\infty$).

Condition (3.17) in Proposition 3.4(e2) (respectively, (3.5) in Proposition 3.1(e2)) is sufficient to ensure Case (B). However, Taliaferro [**36**] shows that Case (A) above may occur (see Example 3.14 in [**36**]), implying that the claim of Proposition 3.4(e2) (respectively, Proposition 3.1(e2)) does not hold without imposing some additional requirement on b_0.

CHAPTER 4

Basic ingredients

Our aim here is to prove results analogous to Lemma 3.1 and Lemma 4.1 in [**15**]. We establish a priori estimates, a Harnack-type inequality and a regularity result for (1.3) by modifying the ideas in [**15**] to take into account the inverse square potential. We rely on regular variation theory and use standard techniques going back to works of Friedman–Véron [**19**] and Véron [**43**]. The results included here will be applied many times in Chapters 5 and 6. Throughout this chapter, we assume that (1.5) holds. Unless otherwise stated, the parameter λ is any real number (except for Corollaries 4.5 and 4.7). Let f, $\mathcal{J}$ and $\mathcal{K}$ be given by (1.8).

4.1. A priori estimates

The following result is the appropriate extension of Lemma 3.1(a) in [**15**], whose proof uses some ideas of Vázquez [**39**]. Our construction of the super-solution $\mathcal{P}$ in (4.3) is slightly different here compared with [**15**].

LEMMA 4.1. *Fix $r_0 > 0$ such that $\overline{B_{2r_0}(0)} \subset \Omega$. There exists a constant $C_1 > 0$, which depends on r_0, such that for every positive sub-solution u of* (1.3)*, we have*

$$u(x) \le C_1 \mathcal{K}(|x|) \quad \text{for every } 0 < |x| \le r_0. \tag{4.1}$$

REMARK 4.2. The function $\mathcal{K}$ defined by (1.8) is regularly varying at 0 with index $-(\theta+2)/(q-1)$. Hence, by Proposition A.5, we have $\lim_{r\to 0} \mathcal{K}(r)/\mathcal{R}(r) = 0$ for every $\mathcal{R} \in RV_j(0+)$ with $j < -(\theta+2)/(q-1)$.

PROOF. We Lemma 3.1 in [**15**], Without any loss of generality, we can take $h(t)/t$ to be increasing for $t > 0$ (in view of Lemma A.10 and Remark A.11 in Appendix A). Fix $x_0 \in \mathbb{R}^N$ with $0 < |x_0| \le r_0$. We define ζ on $B_{|x_0|/2}(x_0)$ as follows

$$\zeta(x) := 1 - \left(\frac{2|x - x_0|}{|x_0|}\right)^2 \quad \text{for } x \in B_{\frac{|x_0|}{2}}(x_0).$$

We have $\zeta(x_0) = 1$ and $0 < \zeta \le 1$ in $B_{|x_0|/2}(x_0)$. Using the functions f and $\mathcal{J}$ in (1.8), we define a function $\mathcal{P}$ on $B_{|x_0|/2}(x_0)$ by

$$\frac{1}{f(\mathcal{P}(x))} = C\mathcal{J}(|x_0|)\,[\zeta(x)]^2 \quad \text{for } x \in B_{\frac{|x_0|}{2}}(x_0). \tag{4.2}$$

Claim: *In* (4.2) *we can take a constant $C > 0$ that is independent of x_0 such that*

$$-\Delta\mathcal{P} - \frac{\lambda}{|x|^2}\mathcal{P} + b(x)\,h(\mathcal{P}) \ge 0 \quad \text{in } B_{|x_0|/2}(x_0). \tag{4.3}$$

The right-hand side of (4.2) equals zero for $x \in \partial B_{|x_0|/2}(x_0)$. Hence $\mathcal{P} = \infty$ on $\partial B_{|x_0|/2}(x_0)$. Suppose that the claim has been verified. We conclude the proof using the comparison principle (Lemma A.9 in Appendix A). Indeed, we have

$$u(x) \le \mathcal{P}(x) \quad \text{for every } x \in B_{|x_0|/2}(x_0). \tag{4.4}$$

Using $x = x_0$ in (4.4) and (4.2), we find that

$$f(u(x_0)) \le \frac{1}{C} f(\mathcal{K}(|x_0|)) \quad \text{for every } x_0 \text{ satisfying } 0 < |x_0| \le r_0.$$

On the other hand, since $\lim_{r\to 0} \mathcal{K}(r) = \infty$ and $f \in RV_{q-1}(\infty)$, we infer that there exists a constant $C_1 > 0$ such that

$$f(\mathcal{K}(r)) \le C f(C_1 \mathcal{K}(r)) \quad \text{for every } r \in (0, r_0].$$

Consequently, we have

$$f(u(x_0)) \le f(C_1 \mathcal{K}(|x_0|)) \quad \text{for every } x_0 \text{ satisfying } 0 < |x_0| \le r_0.$$

Using that f is an increasing function, we conclude (4.1).

Proof of Claim. A simple calculation shows that for $x \in B_{|x_0|/2}(x_0)$, we have

$$\Delta \mathcal{P}(x) = 16\, C \frac{\mathcal{J}(|x_0|)}{|x_0|^2} \frac{f^2(\mathcal{P})}{f'(\mathcal{P})} \left[N\zeta(x) + 16 \frac{|x-x_0|^2}{|x_0|^2} \left(\frac{3}{2} - \frac{f(\mathcal{P}) f''(\mathcal{P})}{[f'(\mathcal{P})]^2} \right) \right].$$

From (1.5), we have $\lim_{r\to 0} \mathcal{J}(r) = 0$ so that

$$M := \sup_{0<r\le r_0} \mathcal{J}(r) < \infty.$$

Hence, the right-hand side of (4.2) is bounded above by CM. Therefore, for any $T > 0$, we can choose a sufficiently small constant $C > 0$, which is independent of x_0, such that $\mathcal{P}(x) \ge T$ in $B_{|x_0|/2}(x_0)$. In particular, using (A.7) in Appendix A, we can find $T > 0$ large such that for every $t \ge T$, we have

$$\frac{t}{h(t)} \le \frac{2}{f(t)}, \qquad \frac{f^2(t)}{f'(t) h(t)} \le \frac{2}{q-1}, \qquad \frac{f(t) f''(t)}{[f'(t)]^2} \ge \frac{q-2}{q-1} - \frac{N}{4}. \tag{4.5}$$

Using (1.5) and Proposition A.4, we can find a positive constant c such that

$$\frac{\mathcal{J}(|x_0|)}{|x_0|^2} \le c\, b(x) \tag{4.6}$$

for every x, x_0 such that $0 < |x_0| \le r_0$ and $|x_0|/2 \le |x| \le 3|x_0|/2$. Since $\mathcal{P}(x) \ge T$ and $0 < \zeta \le 1$ in $B_{|x_0|/2}(x_0)$, using (4.5) and (4.6), it follows that

$$\begin{cases} \Delta \mathcal{P}(x) \le \dfrac{64\, c\, C[N(q-1)+q+1]}{(q-1)^2} b(x)\, h(\mathcal{P}) & \text{in } B_{|x_0|/2}(x_0), \\ \dfrac{\lambda}{|x|^2} \mathcal{P} \le \dfrac{2|\lambda| h(\mathcal{P})}{|x|^2 f(\mathcal{P})} \le 8|\lambda| c\, C\, b(x)\, h(\mathcal{P}) & \text{in } B_{|x_0|/2}(x_0). \end{cases}$$

Therefore, we obtain (4.3) by diminishing $C > 0$ such that

$$8c\, C \left[|\lambda| + \frac{8[N(q-1)+q+1]}{(q-1)^2} \right] \le 1.$$

This proves our claim, which completes the proof of Lemma 4.1. $\square$

From Lemma 4.1 (see also Remark 4.2), we obtain the following.

COROLLARY 4.3. *Any positive sub-solution u of* (1.3) *satisfies*

$$\lim_{|x|\to 0} \frac{u(x)}{\mathcal{R}(|x|)} = 0 \quad \textit{for every function } \mathcal{R} \in RV_j(0+) \textit{ with } j < -\frac{\theta+2}{q-1}. \tag{4.7}$$

4.2. A Harnack-type inequality

In Lemma 4.4 we show that the Harnack inequality in Proposition 2.4 in [**14**] or Lemma 3.1(b) in [**15**] can be extended to equations of the form (1.3). Then we give two consequences of Lemma 4.4. The first one, Corollary 4.5, will be used in Chapter 5 (for proving Lemma 5.8, Lemma 5.10 and Theorem 2.4), as well as in Chapter 6 (for Lemma 6.6 and Theorem 2.7). The second consequence, Corollary 4.7, will be relevant for Lemma 5.2.

LEMMA 4.4 (Harnack-type inequality). *Fix $r_0 > 0$ such that $\overline{B_{4r_0}(0)} \subset \Omega$. There exists a positive constant C_2, which depends on r_0, such that for every positive solution u of* (1.3)*, it holds*

$$\max_{|x|=r} u(x) \leq C_2 \min_{|x|=r} u(x) \quad \textit{for every } r \in (0, r_0]. \tag{4.8}$$

PROOF. The argument is essentially the same as for the case $\lambda = 0$ in Proposition 2.4 of [**14**] or Lemma 1.5 in [**43**]. We give it here for the sake of completeness. Let $y \in \mathbb{R}^N$ be such that $0 < |y| \leq r_0$. Hence $B_{2|y|/3}(y) \subset B^*_{2r_0} := B_{2r_0}(0) \setminus \{0\}$. We apply the Harnack inequality of Theorem 8.20 in [**21**] for the operator

$$Lu = \Delta u + d(x)u \quad \text{in } \Omega_y := B_{2|y|/3}(y), \tag{4.9}$$

where $d(x)$ is defined by

$$d(x) := \frac{\lambda}{|x|^2} - \frac{b(x)\, h(u(x))}{u(x)} \text{ for } x \in B^*_{2r_0}.$$

We see that the hypotheses (8.5) and (8.6) in [**21**] are satisfied with $\lambda = 1$, $\Lambda = \sqrt{N}$ and $\nu(y) = \sup_{x\in\Omega_y} \sqrt{|d(x)|}$. Since $B_{4\eta}(y) \subset \Omega_y$ for $\eta(y) := |y|/8$, by the Harnack inequality there exists a constant $\mathcal{C} > 0$ such that

$$\sup_{B_\eta(y)} u \leq \mathcal{C} \inf_{B_\eta(y)} (y), \tag{4.10}$$

where $\mathcal{C} = \mathcal{C}(N, \eta\nu)$ can be estimated by

$$\mathcal{C} \leq \mathcal{C}_0^{\sqrt{N}+\eta(y)\nu(y)} \quad \text{with } \mathcal{C}_0 = \mathcal{C}_0(N). \tag{4.11}$$

We next show that $\eta(y)\,\nu(y)$ is bounded above by a constant independent of y and u. Since $x \in \Omega_y$ implies that $|x| > |y|/3$, it follows that $\sqrt{|d(x)|}$ is bounded above on Ω_y by $3\sqrt{|\lambda|}/|y| + b_2(x)$, where $b_2(x)$ is a positive function defined by

$$b_2(x) := \sqrt{\frac{b(x)\, h(u(x))}{u(x)}} \quad \text{for every } x \in \mathbb{R}^N \text{ with } 0 < |x| \leq 2r_0. \tag{4.12}$$

By the definition of $\nu(y)$ and $\eta(y) = |y|/8$, we infer that

$$\begin{aligned} 8\eta(y)\nu(y) &= |y| \sup_{x\in\Omega_y} \sqrt{|d(x)|} \leq 3\sqrt{|\lambda|} + |y| \sup_{x\in\Omega_y} b_2(x) \\ &\leq 3\sqrt{|\lambda|} + 3 \sup_{0<|x|\leq 2r_0} [|x| b_2(x)]. \end{aligned} \tag{4.13}$$

Using (4.1), (4.12) and (A.4) in Lemma A.10, we find

$$|x|^2 [b_2(x)]^2 \leq |x|^2 b(x) \frac{h_2(C_1\mathcal{K}(|x|))}{C_1\mathcal{K}(|x|)} \quad \text{for every } 0 < |x| \leq 2r_0. \tag{4.14}$$

By (1.5), (1.8) and Remark A.11, we find that the right-hand side of (4.14) converges to $(C_1)^{q-1}$ as $|x| \to 0$. Hence, for some constant $A > 0$, we have

$$\sup_{0<|x|\leq 2r_0} [|x|b_2(x)] \leq A.$$

This, jointly with (4.13), shows that $\eta(y)\nu(y)$ is bounded above by a constant independent of u and y with $0 < |y| \leq r_0$. In view of (4.10) and (4.11), we conclude (4.8) using a standard covering argument. This ends the proof of Lemma 4.4. $\square$

COROLLARY 4.5. *Assume that* $-\infty < \lambda < (N-2)^2/4$. *Let* u *be a positive solution of* (1.3). *The following hold:*

(a) $\lim_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = \infty$ *if and only if* $\limsup_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = \infty$;

(b) $\lim_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = 0$ *if and only if* $\liminf_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = 0$.

REMARK 4.6. If $\lambda = (N-2)^2/4$, then the statements of Corollary 4.5 hold with $\Psi^{\pm}$ instead of $\Phi_\lambda^{\pm}$.

PROOF. (a) We need only prove that $\limsup_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = \infty$ implies that $\lim_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = \infty$. Suppose by contradiction that

$$l^{\pm} := \liminf_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^{\pm}(x)} < \infty.$$

Then there exists a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^N$ which converges to zero such that $\lim_{n\to\infty} u(x_n)/\Phi_\lambda^{\pm}(x_n) = l^{\pm}$. We can assume that $|x_n|$ decreases to zero with $0 < |x_n| \leq r_0$ and $\overline{B_{4r_0}(0} \subset \Omega$. By Lemma 4.4, there exists a positive constant C_2 such that (4.8) holds. Let n_0 be a large positive integer such that

$$\frac{u(x_n)}{\Phi_\lambda^{\pm}(x_n)} \leq l^{\pm} + 1 \quad \text{for every } n \geq n_0.$$

It follows that

$$\begin{aligned} \max_{|x|=|x_n|} u(x) &\leq C_2 \min_{|x|=|x_n|} u(x) \leq C_2 u(x_n) \\ &\leq C_2(l^{\pm}+1)\Phi_\lambda^{\pm}(x_n) \quad \text{for every } n \geq n_0. \end{aligned} \tag{4.15}$$

We see that u is a positive sub-solution of (1.3) if h is replaced by the function h_1 in Lemma A.10. Applying the comparison principle (Lemma A.9) with $u = u_1$ and $u_2 = C_2(l^{\pm}+1)\Phi_\lambda^{\pm}$ on each annulus $\{x \in \mathbb{R}^N : \ |x_n| < |x| < |x_{n_0}|\}$ with $n > n_0$, we conclude that

$$u(x) \leq C_2(l^{\pm}+1)\Phi_\lambda^{\pm}(x) \quad \text{for all } 0 < |x| \leq |x_{n_0}|.$$

This is a contradiction with the hypothesis that $\limsup_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = \infty$. Hence, the assertion of (a) is proved.

(b) We slightly modify the above ideas to show that $\liminf_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = 0$ yields that $\lim_{|x|\to 0} u(x)/\Phi_\lambda^{\pm}(x) = 0$. We assume by contradiction that

$$\limsup_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^{\pm}(x)} = j^{\pm} \in (0, \infty]. \tag{4.16}$$

Let $(x_n)_{n\geq 1}$ be a sequence in $\mathbb{R}^N$ such that $\lim_{n\to\infty} |x_n| = 0$ and

$$\lim_{n\to\infty} \frac{u(x_n)}{\Phi_\lambda^{\pm}(x_n)} = 0. \tag{4.17}$$

As before, we can assume that $|x_n|$ decreases to zero with $0 < |x_n| \le r_0$ and $\overline{B_{4r_0}(0)} \subset \Omega$. Let c be a positive number such that $c\,C_2 < j^{\pm}$, where C_2 is the positive constant in (4.8). From (4.17), we have

$$\frac{u(x_n)}{\Phi_\lambda^{\pm}(x_n)} \le c \quad \text{for every } n \ge n_0,$$

provided that $n_0 \ge 1$ is a large integer. Hence, (4.15) and the argument that follows remain valid if $(l^{\pm}+1)$ is replaced by c. So, we arrive at

$$u(x) \le c\,C_2 \Phi_\lambda^{\pm}(x) \quad \text{for all } 0 < |x| \le |x_{n_0}|,$$

which contradicts (4.16). This finishes the proof of Corollary 4.5. □

Corollary 4.7. *Let* $0 < \lambda \le (N-2)^2/4$. *If* u *is a positive solution of* (1.3), *then* $\lim_{|x|\to 0} u(x) = \infty$ *if and only if* $\limsup_{|x|\to 0} u(x) = \infty$.

Remark 4.8. We prove that if $0 < \lambda \le (N-2)^2/4$, then $\lim_{|x|\to 0} u(x) = \infty$ for every positive solution u of (1.3) (see Remark 5.3 in Chapter 5). This statement does *not* remain true if $\lambda \le 0$ since in this case there exist solutions u of (1.3) with the property that $\lim_{|x|\to 0} u(x)/\Phi_\lambda^{-}(x) \in (0,\infty)$.

Proof. Let u be a positive solution of (1.3) such that $\limsup_{|x|\to 0} u(x) = \infty$. We reason by contradiction and assume that $\liminf_{|x|\to 0} u(x) < \infty$. Then there exists a constant C such that

$$\liminf_{|x|\to 0} u(x) < C. \tag{4.18}$$

Since $|x|^2 b(x) \to 0$ as $|x| \to 0$ and $0 < \lambda \le (N-2)^2/4$, then we can find $r_0 > 0$ sufficiently small such that $B_{4r_0}(0) \subset\subset \Omega$ and

$$|x|^2 b(x) h_2(C) \le \lambda C \quad \text{for every } 0 < |x| \le r_0,$$

where h_2 is provided by Lemma A.10 in the Appendix A. This yields that C is a sub-solution of (1.3) in $B_{r_0}(0)\setminus\{0\}$, where h is replaced by h_2. From Harnack inequality, there exists a constant C_2 such that (4.8) holds. Using that $\limsup_{|x|\to 0} u(x) = \infty$, there exists a sequence $(r_n)_{n\ge 1}$ which decreases to zero such that

$$\max_{|x|=r_n} u(x) \ge CC_2 \quad \text{for every } n \ge 1.$$

Without loss of generality, we assume that $0 < r_n < r_0$ for all $n \ge 1$. Hence, from (4.8), we have $u(x) \ge C$ for $|x| = r_n$ and all $n \ge 1$. By (A.4), we see that u is a super-solution of (1.3) in $B_{r_0}(0) \setminus \{0\}$ with h_2 instead of h. Therefore, by the comparison principle in Lemma A.9 on each annulus $\{x \in \mathbb{R}^N : \; r_n < |x| < r_1\}$ with $n \ge 2$, we conclude that $u(x) \ge C$ for all $0 < |x| \le r_1$. This is a contradiction with (4.18). Hence, we find $\lim_{|x|\to 0} u(x) = \infty$. □

4.3. A regularity lemma

In this section, we give a suitable extension of the regularity result (Lemma 4.1 with $p = 2$) in [**15**] to equations of the form (1.3) (see Lemma 4.9). As an application of Lemma 4.9, we include here Lemma 4.12.

LEMMA 4.9 (Regularity). *Fix* $r_0 > 0$ *small such that* $\overline{B_{4r_0}(0)} \subset \Omega$. *Assume that* $0 \leq \delta \leq (\theta+2)/(q-1)$ *and* $g \in RV_{-\delta}(0+)$ *is a positive continuous function defined on* $(0, 4r_0]$ *such that* $\limsup_{r\to 0} g(r)/\mathcal{K}(r) < \infty$*, where* $\mathcal{K}$ *is given by* (1.8).

If u *is a positive solution of* (1.3) *such that, for some constant* $C_1 > 0$*, we have*

$$0 < u(x) \leq C_1 g(|x|) \quad \text{for } 0 < |x| < 2r_0, \tag{4.19}$$

then there exist constants $C > 0$ *and* $\alpha \in (0,1)$ *such that*

$$|\nabla u(x)| \leq C\frac{g(|x|)}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \leq C\frac{g(|x|)}{|x|^{1+\alpha}}|x-x'|^{\alpha}, \tag{4.20}$$

for any x*,* x' *in* $\mathbb{R}^N$ *satisfying* $0 < |x| \leq |x'| < r_0$.

REMARK 4.10. (i) Lemma 4.1 shows that there exists a constant $C_1 > 0$ such that (4.19) holds with $g \equiv \mathcal{K}$ for *every* positive solution u of (1.3).

(ii) If $g \in RV_{-\delta}(0+)$ with $-\infty < \delta < (\theta+2)/(q-1)$, then $\lim_{r\to 0} g(r)/\mathcal{K}(r) = 0$ since $\mathcal{K} \in RV_{-\frac{\theta+2}{q-1}}(0+)$.

PROOF. We use a standard method as in [**19**], [**15**], which relies on a $C^{1,\alpha}$-regularity result of Tolksdorf [**37**] applied here for nonlinear degenerate elliptic equations of the form

$$-\Delta v + D = 0 \quad \text{in } \mathcal{A} := \{y \in \mathbb{R}^N : \ 1 < |y| < 7\}, \tag{4.21}$$

where $D \in L^\infty(\mathcal{A})$. If $v \in L^\infty(\mathcal{A}) \cap W^{1,2}(\mathcal{A})$ is a weak solution of (4.21), then there exist constants $\alpha = \alpha(N) \in (0,1)$ and $C^* = C^*(N, \|v\|_{L^\infty(\mathcal{A})}, \|D\|_{L^\infty(\mathcal{A})})$ such that

$$\|\nabla v\|_{C^{0,\alpha}(\mathcal{A}^*)} \leq C^*, \quad \text{where } \mathcal{A}^* := \{y \in \mathbb{R}^N : \ 2 < |y| < 6\}. \tag{4.22}$$

For every $\beta \in (0, r_0/6)$, we define v_β on $\mathcal{A}$ as in [**15**], namely

$$v_\beta(\xi) := \frac{u(\beta\xi)}{g(\beta)} \quad \text{for } \xi \in \mathcal{A}. \tag{4.23}$$

Using (4.19) and the argument of Lemma 4.1 in [**15**], we find that $v_\beta \in L^\infty(\mathcal{A})$ with its L^∞-norm bounded above by a constant independent of $\beta \in (0, r_0/6)$. A simple computation yields that v_β is a solution of (4.21) with $D = D_\beta$ given by

$$D_\beta(\xi) := \frac{\beta^2}{g(\beta)} b(\beta\xi) h(u(\beta\xi)) - \frac{\lambda}{|\xi|^2} v_\beta(\xi) \quad \text{for } \xi \in \mathcal{A}. \tag{4.24}$$

The first term in the right-hand side of (4.24) is B_β in (4.7) with $p = 2$ in the proof of [**15**, Lemma 4.1], which shows that $B_\beta \in L^\infty(\mathcal{A})$ with its L^∞-norm bounded above by a constant independent of $\beta \in (0, r_0/6)$. Thus D_β is also in $L^\infty(\mathcal{A})$ and

$$\|D_\beta\|_{L^\infty(\mathcal{A})} \leq \|B_\beta\|_{L^\infty(\mathcal{A})} + |\lambda| \|v_\beta\|_{L^\infty(\mathcal{A})},$$

which is bounded above by a constant independent of β. We can thus apply the regularity result of Tolksdorf [**37**] to obtain (4.22) for each $v = v_\beta$, where the constant $C^* > 0$ is independent of $\beta \in (0, r_0/6)$. To prove (4.20), we can now follow the same argument as in [**15**] so that we skip the details. □

REMARK 4.11. If in Lemma 4.9 we assume that (4.19) holds for $g \in RV_{-\delta}(0+)$ with $\delta < 0$, then the assertion of (4.20) is still valid subject to a small modification only in the second inequality, which now reads as follows

$$|\nabla u(x) - \nabla u(x')| \leq C\frac{g(|x'|)}{|x|^{1+\alpha}}|x-x'|^{\alpha} \tag{4.25}$$

for any x, x' in $\mathbb{R}^N$ satisfying $0 < |x| \le |x'| < r_0$. Here (and in the first inequality of (4.20), the constant C will depend on $|\delta|$ (only when $\delta < 0$, since the constant $\widehat{C}$ appearing in (4.10) of [**15**] will depend on $-\delta$).

The reason for the change in (4.25) is that g being now a regularly varying function at zero with *positive* index will behave near zero as a *non-decreasing* function (see Proposition A.8). To prove (4.25), we proceed as in [**15**] by taking $0 < |x| < r_0/2$ and considering two cases:

(a) $0 < |x| \le |x'| < 2|x|$;
(b) $2|x| \le |x'| < r_0$.

In case (a), similar to Lemma 4.1 in [**15**], we use (4.22) with $v = v_\beta$ to deduce the second inequality in (4.20). This implies (4.25) (by possibly enlarging C). In case (b) above, we use the first inequality in (4.20), jointly with the inequality

$$|x' - x| \ge |x'| - |x| \ge |x|.$$

More precisely, we have

$$\begin{aligned}|\nabla u(x) - \nabla u(x')| &\le C\left(\frac{g(|x|)}{|x|} + \frac{g(|x'|)}{|x'|}\right) \le C\frac{g(|x|)+g(|x'|)}{|x|} \\ &\le C'\frac{g(|x'|)}{|x|} \le C'\frac{g(|x'|)}{|x|^{\alpha+1}}|x-x'|^\alpha,\end{aligned}$$

where $C' > 0$ denotes a large constant. This establishes (4.25).

Lemma 4.12. *Let $-\infty < \lambda \le (N-2)^2/4$. Assume that u is a positive solution of* (1.3) *such that* $\lim_{|x|\to 0} u(x) = \infty$. *Then, for every $\epsilon \in (0,1)$, there exists $r_\epsilon > 0$ such that the equation*

$$-\Delta v - \lambda |x|^{-2} v + |x|^\theta L_b(|x|)\,\tilde{h}(v) = 0 \quad \text{in } B^*_{r_\epsilon} := B_{r_\epsilon}(0)\setminus\{0\} \tag{4.26}$$

has a positive solution v_* with the property that

$$(1-\epsilon)u \le v_* \le (1+\epsilon)u \quad \text{in } B^*_{r_\epsilon}. \tag{4.27}$$

Remark 4.13. If $0 < \lambda \le (N-2)^2/4$, then Lemma 4.12 can be applied to any positive solution of (1.3) (cf., Remark 5.3).

Proof. Since $\lim_{|x|\to 0} u(x) = \infty$, from (1.5) we find that

$$\tilde{h}((1\pm\epsilon)u) \sim (1\pm\epsilon)^q h(u) \quad \text{as } |x| \to 0.$$

Moreover, we infer that there exists $r_\epsilon > 0$ small such that $\overline{B_{r_\epsilon}(0)} \subset \Omega$ and

$$\begin{cases} (1+\epsilon)\,b(x)\,h(u(x)) \le |x|^\theta L_b(|x|)\,\tilde{h}((1+\epsilon)u(x)) & \text{in } B^*_{r_\epsilon}. \\ (1-\epsilon)\,b(x)\,h(u(x)) \ge |x|^\theta L_b(|x|)\,\tilde{h}((1-\epsilon)u(x)) & \text{in } B^*_{r_\epsilon}. \end{cases} \tag{4.28}$$

For any integer $n \ge 1$ such that $n > 1/r_\epsilon$, we define

$$\mathcal{A}_{n,\epsilon} := \left\{x \in \mathbb{R}^N : \ \frac{1}{n} < |x| < r_\epsilon\right\}$$

and consider the boundary value problem

$$\begin{cases} -\Delta v - \dfrac{\lambda}{|x|^2}v + |x|^\theta L_b(|x|)\,\tilde{h}(v) = 0 & \text{in } \mathcal{A}_{n,\epsilon}, \\ v = (1+\epsilon)u & \text{on } \partial\mathcal{A}_{n,\epsilon}. \end{cases} \tag{4.29}$$

Using (4.28), we infer that $(1+\epsilon)u$ is a super-solution of (4.29) and $(1-\epsilon)u$ is a sub-solution of (4.29). Let v_n denote the unique positive solution of (4.29) (the uniqueness follows from Lemma A.9). Hence, we must have

$$(1-\epsilon)u \leq v_n \leq (1+\epsilon)u \quad \text{and} \quad v_m \leq v_n \ \text{ in } \mathcal{A}_{n,\epsilon} \text{ for } m > n.$$

By Lemma 4.1 and Lemma 4.9, we have that (up to a sub-sequence) v_n converges to some v_* in $C^1_{\rm loc}(B^*_{r_\epsilon})$ and v_* is a positive solution of (4.26) satisfying (4.27). $\square$

CHAPTER 5

The analysis for the subcritical parameter

Throughout this chapter, we assume implicitly that (1.5) holds. Unless otherwise specified, we understand that $-\infty < \lambda < (N-2)^2/4$. Our aim here is to prove the main results on (1.3) corresponding to the subcritical parameter, that is Theorems 2.1–Theorem 2.4 stated in Section 2.1 of Chapter 2. We establish consecutively these results, each being treated in their respective section.

5.1. Proof of Theorem 2.1

There are several different lines of thought in this proof, which is quite involved. Thus we first explain the structure of this section. We prove the converse implication "$\Leftarrow$" of (2.12) in Proposition 5.1 below.

Proposition 5.1. *Assume that* $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) < \infty$ *for some small* $\varpi > 0$. *If* u *is a positive solution of* (1.3) *such that* $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = 0$*, then*

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^-(x)} \in (0,\infty).$$

Moreover, u *can be extended as a solution of* (1.3) *in* Ω.

In Section 5.1.1 we give the proof of Proposition 5.1, which follows by combining Lemma 5.2 with Lemma 5.4. We complete the proof of Theorem 2.1 in Section 5.1.2 as follows. The direct implication "$\Rightarrow$" in (2.12), as well as in (2.13), is demonstrated by Lemma 5.5. The existence assertions of Theorem 2.1 are incorporated into Lemma 5.6 and Lemma 5.7, which require $h(t)/t$ be increasing on $(0,\infty)$. More precisely, we prove in Lemma 5.6 that if $\vartheta \in C^1(\partial B_1(0))$ is a non-negative function and $\lim_{\tau\to 0} \mathcal{I}^*(\tau,\varpi) < \infty$, then for every number $\gamma > 0$, the problem

$$\begin{cases} -\Delta u - \lambda |x|^{-2} u + b(x)h(u) = 0 \quad \text{in } B^* := B_1(0)\setminus\{0\}, \\ \lim\limits_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = \gamma, \quad u = \vartheta \quad \text{on } \partial B_1(0), \\ u > 0 \ \text{ in } B^*, \end{cases} \tag{5.1}$$

admits a unique solution u_γ, which is in $C^{1,\alpha}_{\mathrm{loc}}(B^*)$ for some $\alpha \in (0,1)$. This assertion also holds for $\gamma = \infty$ provided we are in either of the cases of Theorem 2.4(C) (when all solutions are asymptotic at zero irrespective of the value of ϑ on $\partial B_1(0)$). Furthermore, in Lemma 5.7 we show that when $\vartheta \in C^1(\partial B_1(0))$ is a non-negative and non-trivial function, then (5.1) with $\gamma = 0$ has always a unique solution, which is in $C^{1,\alpha}_{\mathrm{loc}}(B^*)$ for some $\alpha \in (0,1)$. Our proof of Lemmas 5.6 and 5.7 will adapt the argument of Theorem 1.2 in [**15**].

5.1.1. Proof of Proposition 5.1. It consists of Lemmas 5.2 and 5.4.

LEMMA 5.2. *Assume that* $\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty$ *for some small* $\varpi>0$. *If* u *is a positive solution of* (1.3) *such that* $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x)=0$, *then*

$$0<\liminf_{|x|\to 0}\frac{u(x)}{\Phi_\lambda^-(x)}\le\limsup_{|x|\to 0}\frac{u(x)}{\Phi_\lambda^-(x)}<\infty. \tag{5.2}$$

PROOF. Let u be a positive solution of (1.3) with $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x)=0$. Without any loss of generality, we can assume that $\overline{B_1(0)}\subset\Omega$. We split the proof of (5.2) into four steps. The last inequality in (5.2) follows easily from the comparison principle in Lemma A.9, as shown in Step 1. However, the proof of the first inequality in (5.2) is more difficult, being achieved in Steps 2–4. From (1.5) and Lemma A.10, there exists a positive constant C such that

$$b(x)\,h(u)\le C|x|^\theta L_b(|x|)\,(h_2(u)+u)\quad\text{for } 0<|x|\le 1. \tag{5.3}$$

This inequality will be relevant for Step 2, which shows that $u\ge v_\infty$ in $B_1(0)\setminus\{0\}$ for some positive radial solution v_∞ of

$$-\Delta v-\frac{\lambda}{|x|^2}v+C|x|^\theta L_b(|x|)\,(h_2(v)+v)=0\quad\text{for } 0<|x|<1. \tag{5.4}$$

If $\lambda\le 0$, then the proof of (5.2) will be finished by showing in Step 3 that $\lim_{r\to 0} v_\infty(r)/\Phi_\lambda^-(r)>0$. This is done by contradiction. Using a change of variable $z(s)=v_\infty(r)/\Phi_\lambda^-(r)$ with $s=r^{2p-N+2}$ we are led to an ODE without the inverse square potential for which we can apply Theorem 1.14 in [**28**] to reach a contradiction. The argument of Step 3 will also be useful for $0<\lambda<(N-2)^2/4$ when we show that $\limsup_{|x|\to 0} u(x)=\infty$ without using that $\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty$. This assumption does come into play in Step 4, which is needed to complete the proof of $\liminf_{|x|\to 0} u(x)/\Phi_\lambda^-(x)>0$ for $0<\lambda<(N-2)^2/4$. We next show the details.

Step 1. *We have* $\gamma^-<\infty$, *where* $\gamma^-:=\limsup_{|x|\to 0} u(x)/\Phi_\lambda^-(x)$.

For any $\epsilon>0$, there exists $r_\epsilon\in(0,1)$ small such that

$$u(x)\le\epsilon\Phi_\lambda^+(x)\ \text{ for every } x\in\mathbb{R}^N\ \text{ with } 0<|x|\le r_\epsilon.$$

Let $M=\max_{|x|=1} u(x)$. By Lemmas A.9 and A.10 in Appendix, it follows that

$$u(x)\le\epsilon\Phi_\lambda^+(x)+M\Phi_\lambda^-(x)\quad\text{for every } x\in\mathbb{R}^N\ \text{ with } 0<|x|<1.$$

Passing to the limit with $\epsilon\to 0$, we conclude that $\gamma^-<\infty$. This finishes Step 1.

Step 2. *The positive solution* u *of* (1.3) *is bounded from below by a positive* radial *solution* v_∞ *of* (5.4), *which satisfies* $\lim_{r\to 0} v_\infty(r)/\Phi_\lambda^-(r)\in[0,\infty)$.

To construct v_∞, we first set

$$\mathcal{A}_n:=\{x\in\mathbb{R}^N:\ 1/n<|x|<1\}\quad\text{for any integer } n\ge 2. \tag{5.5}$$

We consider the boundary value problem

$$\begin{cases} -\Delta v-\dfrac{\lambda}{|x|^2}v+C|x|^\theta L_b(|x|)\,(h_2(v)+v)=0 & \text{in } \mathcal{A}_n,\\ v(x)=\min\limits_{|y|=|x|} u(y) & \text{for } |x|=1/n \ \text{ and }\ |x|=1. \end{cases} \tag{5.6}$$

We denote by v_n the unique positive C^2-solution of (5.6). The uniqueness follows from Lemma A.9. We have that v_n is radially symmetric (by the invariance of the operator $v \longmapsto -\Delta v - \lambda |x|^{-2} v$ under rotation, the symmetry of the domain $\mathcal{A}_n$ and the boundary data). From (5.3), we infer that u is a super-solution of (5.6). Therefore, we have

$$v_{n+1} \leq v_n \leq u \quad \text{in } \mathcal{A}_n \quad \text{for every } n \geq 2.$$

By Lemma 4.9, we conclude that for a sequence $n_k \to \infty$, we have $v_{n_k} \to v_\infty$ in $C^1_{\text{loc}}(B^*)$ with $B^* := B_1(0) \setminus \{0\}$ and v_∞ is a non-negative radial solution of (5.4) satisfying $v_\infty \leq u$ for $0 < |x| \leq 1$. Moreover, since $v_\infty(1) = \min_{|y|=1} u(y) > 0$ and $h_2(t)/t$ is bounded from above by a positive constant for $t \in [0, \delta]$ and $\delta > 0$ small (by Lemma A.10), from the strong maximum principle (see [**30**]), we must have $v_\infty(r) > 0$ for each $r \in (0, 1)$.

We now show that $v_\infty(r)/\Phi^-_\lambda(r)$ admits a limit as $r \to 0$, which is in $[0, \infty)$ by Step 1. If we assume that $v_\infty(r)/\Phi^-_\lambda(r)$ does not have a limit as $r \to 0^+$, then there exists a positive constant c such that

$$\liminf_{r \to 0} \frac{v_\infty(r)}{\Phi^-_\lambda(r)} < c < \limsup_{r \to 0} \frac{v_\infty(r)}{\Phi^-_\lambda(r)}. \tag{5.7}$$

Let $(R_n)_{n \geq 1}$ be a sequence that decreases to zero and $v_\infty(R_n)/\Phi^-_\lambda(R_n)$ tends to $\liminf_{r \to 0} v_\infty(r)/\Phi^-_\lambda(r)$ as $n \to \infty$. We can assume that $R_n < 1$ and

$$v_\infty(R_n)/\Phi^-_\lambda(R_n) \leq c \quad \text{for every } n \geq 1.$$

Therefore, by the comparison principle (Lemma A.9), we infer that

$$v_\infty(r) \leq c\Phi^-_\lambda(r) \quad \text{for every } r \in (R_n, R_1)$$

and every integer $n \geq 2$. Since $\lim_{n \to \infty} R_n = 0$, we find that $v_\infty(r) \leq c\Phi^-_\lambda(r)$ for every $r \in (0, R_1)$. This implies that $\limsup_{r \to 0} v_\infty(r)/\Phi^-_\lambda(r) \leq c$, which is a contradiction with (5.7). This completes Step 2.

Step 3. *We have* $\lim_{r \to 0} v_\infty(r)/\Phi^-_\lambda(r) > 0$ *when* $\lambda \leq 0$ *and*

$$\limsup_{|x| \to 0} u(x) = \infty \quad \textit{when } 0 < \lambda < (N-2)^2/4. \tag{5.8}$$

Our proof of (5.8) does not use the assumption that $\lim_{\tau \to 0} \mathcal{I}^{**}(\tau, \varpi) < \infty$. This, jointly with Corollary 4.7, yields that $\lim_{|x| \to 0} u(x) = \infty$ for every positive solution of (1.3) when $0 < \lambda < (N-2)^2/4$, which will be relevant for later.

To prove the assertion of Step 3, we argue by contradiction and assume that

$$\begin{cases} \lim\limits_{r \to 0} v_\infty(r)/\Phi^-_\lambda(r) = 0 \text{ when } \lambda \leq 0 \\ \limsup\limits_{|x| \to 0} u(x) < \infty \quad \text{when } 0 < \lambda < (N-2)^2/4. \end{cases} \tag{5.9}$$

Since $v_\infty \leq u$ for $0 < |x| \leq 1$ and $\Phi^-_\lambda(r)$ tends to ∞ (respectively, 0) as $r \to 0$ if $0 < \lambda < (N-2)^2/4$ (respectively, $\lambda < 0$), from (5.9) and Lemma A.10, we find that

$$h_2(v_\infty(x)) \leq \widehat{C} v_\infty(x) \quad \text{for all } 0 < |x| \leq 1,$$

where $\widehat{C} > 0$ is a large constant. By defining

$$a(|x|) := C \left(\frac{h_2(v_\infty(|x|))}{v_\infty(|x|)} + 1 \right), \tag{5.10}$$

we observe that

$$C \leq a(|x|) \leq C(\widehat{C} + 1) \quad \text{for } 0 < |x| \leq 1. \tag{5.11}$$

Hence, v_∞ is a positive radial solution of the following equation

$$-\Delta v - \frac{\lambda}{|x|^2} v + a(|x|)\,|x|^\theta L_b(|x|)\,v = 0 \quad \text{for } 0<|x|<1. \tag{5.12}$$

Let p be given by (1.9). We introduce the function

$$z(s) = \frac{v_\infty(r)}{\Phi_\lambda^-(r)} \quad \text{with } s = r^{2p-N+2}.$$

Since v_∞ is a radial solution of (5.12), it follows that z satisfies the following ODE

$$z''(s) = \frac{a(s^{\frac{1}{2p-N+2}})}{(2p-N+2)^2} s^{\frac{2+\theta}{2p-N+2}-2} L_b(s^{\frac{1}{2p-N+2}})\,z(s) \quad \text{for } s>1. \tag{5.13}$$

Given that $2p-N+2<0$ and $2+\theta>0$, by (5.11) and Proposition A.5, we have

$$\lim_{s\to\infty} a(s^{\frac{1}{2p-N+2}}) s^{\frac{2+\theta}{2p-N+2}} L_b(s^{\frac{1}{2p-N+2}}) = 0. \tag{5.14}$$

From (5.14) and L'Hôpital's rule, we deduce that

$$\lim_{s\to\infty} s\int_s^\infty a(t^{\frac{1}{2p-N+2}}) t^{\frac{2+\theta}{2p-N+2}-2} L_b(t^{\frac{1}{2p-N+2}})\,dt = 0. \tag{5.15}$$

In view of (5.15), by applying Theorem 1.14 in [**28**], we have that there exist two linearly independent regularly varying solutions z_1 and z_2 of (5.13) of the form

$$z_1(s) = L_1(s) \quad \text{and} \quad z_2(s) = sL_2(s),$$

where L_1 and L_2 are some slowly varying functions at ∞. Our assumption (5.9) implies that $\lim_{s\to\infty} z(s) = 0$ for any $-\infty<\lambda<(N-2)^2/4$. Hence, $\lim_{s\to\infty} z_1(s) = 0$ (due to $\lim_{s\to\infty} z_2(s) = \infty$). Since $z_1''(s)>0$, we infer that $\lim_{s\to\infty} z_1'(s) = 0$ and

$$z_1(s) = \frac{1}{(2p-N+2)^2}\int_s^\infty \left(\int_t^\infty \frac{a(\xi^{\frac{1}{2p-N+2}})}{\xi^{2-\frac{2+\theta}{2p-N+2}}} L_b(\xi^{\frac{1}{2p-N+2}})\,z_1(\xi)\,d\xi\right) dt \tag{5.16}$$

for any $s>1$. Since $s \longmapsto z_1(s)$ is slowly varying at ∞, we obtain that

$$F(s) := \int_s^\infty \left(\int_t^\infty \xi^{\frac{2+\theta}{2p-N+2}-2} L_b(\xi^{\frac{1}{2p-N+2}})\,z_1(\xi)\,d\xi\right) dt$$

is regularly varying at ∞ with index $(2+\theta)/(2p-N+2)$, which is a *negative* number. Using (5.11), we get that the right-hand side of (5.16), say (RHS), satisfies $C_1F(s) \le (RHS) \le C_2F(s)$ for some positive constants C_1 and C_2 with $C_1<C_2$. This leads to a contradiction because the left-hand side of (5.16), namely $z_1(s)$, is slowly varying at ∞, while the right-hand side of (5.16) is not. This contradiction proves that (5.9) cannot hold, that is $\lim_{r\to 0} v_\infty(r)/\Phi_\lambda^-(r) > 0$ when $\lambda \le 0$, which concludes the proof of Lemma 5.2 in this case (using Step 2 as well).

Step 4. If $0<\lambda<(N-2)^2/4$ and $\lim_{\tau\to 0}\mathcal{I}^{**}(\tau,\varpi)<\infty$ for some small $\varpi>0$, then $\liminf_{|x|\to 0} u(x)/\Phi_\lambda^-(x) > 0$.

Using (1.5), there exists a positive constant C such that

$$b(x)\,h(u) \le C|x|^\theta L_b(|x|)\,u^q L_h(u) \quad \text{for every } 0<|x|\le 1. \tag{5.17}$$

We can proceed as in Step 2 (replacing (5.3) by (5.17)) to deduce that the positive solution u of (1.3) is bounded from below by a positive radial solution w_∞ of

$$-\Delta w - \frac{\lambda}{|x|^2} w + C|x|^\theta L_b(|x|)\,w^q L_h(w) = 0 \quad \text{for } 0<|x|<1. \tag{5.18}$$

Clearly, $\lim_{r\to 0} w_\infty(r) = \infty$ (we can use w_∞ instead of u in the argument of Step 3). Since $w_\infty(r)/\Phi_\lambda^-(r)$ converges to a non-negative number as $r \to 0$, there exists a constant $d_0 > 0$ such that $w_\infty(r) \le d_0 \Phi_\lambda^-(r)$ for every $r \in (0,1]$. If $1 < m < q$, then $t^{q-m} L_h(t)$ is increasing for large $t > 0$ (using (1.6)). Hence, there exists a constant $d_1 > 0$ such that

$$[w_\infty(r)]^{q-m} L_h(w_\infty(r)) \le d_1 [\Phi_\lambda^-(r)]^{q-m} L_h(\Phi_\lambda^-(r)) \quad \text{for every } r \in (0,1]. \tag{5.19}$$

Notice that $U = w_\infty$ is a positive solution of

$$U''(r) + \frac{N-1}{r} U'(r) + \frac{\lambda}{r^2} U(r) = C r^\theta L_b(r) (w_\infty)^{q-m} L_h(w_\infty) U^m \quad \text{in } (0,1). \tag{5.20}$$

To conclude the proof, it suffices to rule out $\lim_{r\to 0} w_\infty(r)/\Phi_\lambda^-(r) = 0$. If we assume that $\lim_{r\to 0} w_\infty(r)/\Phi_\lambda^-(r) = 0$, then by Proposition 3.1(b), we have that

$$\lim_{\tau\to 0} \int_\tau^\varpi r^{1-p(m-1)+\theta} L_b(r) [w_\infty(r)]^{q-m} L_h(w_\infty(r))\, dr = \infty. \tag{5.21}$$

This fact, combined with (5.19), leads to

$$\lim_{\tau\to 0} \int_\tau^\varpi r^{p+\theta+1-pq} L_b(r) L_h(\Phi_\lambda^-(r))\, dr = \infty,$$

which contradicts our assumption $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) < \infty$ (see (2.4) and (2.5)). Hence, we conclude that $\lim_{r\to 0} w_\infty(r)/\Phi_\lambda^-(r) \in (0,\infty)$, which yields that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^-(x)} > 0$$

also for the case $0 < \lambda < (N-2)^2/4$. This completes the proof of Lemma 5.2. □

REMARK 5.3. If $0 < \lambda \le (N-2)^2/4$, then *every* positive solution u of (1.3) satisfies $\lim_{|x|\to 0} u(x) = \infty$. Indeed, if $\limsup_{|x|\to 0} u(x)/\Phi_\lambda^+(x) > 0$ then we apply Corollary 4.7. When $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = 0$ and $0 < \lambda < (N-2)^2/4$, then we refer to Step 3 in the proof of Lemma 5.2. Finally, if $\lambda = (N-2)^2/4$ then $u \ge w_\infty$ in $B_1(0) \setminus \{0\}$, where w_∞ is a positive radial solution of (5.18) for some $\lambda = \lambda_1$ with $0 < \lambda_1 < (N-2)^2/4$.

LEMMA 5.4. *Assume that* $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) < \infty$ *for some small* $\varpi > 0$. *If* u *is a positive solution of* (1.3) *such that*

$$\gamma^- := \limsup_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^-(x)} \in (0,\infty),$$

then we have

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^-(x)} = \gamma^- \quad \textit{and} \quad \lim_{|x|\to 0} \frac{x\cdot \nabla u(x)}{\Phi_\lambda^-(x)} = -p\,\gamma^-. \tag{5.22}$$

Moreover, u *can be extended as a solution of* (1.3) *in* Ω.

PROOF. We modify a technique used for proving Theorem 5.1 in [**15**], which followed the line of argument of Friedman and Véron [**19**, Theorem 1.1]. Both [**15**] and [**19**] treat quasilinear elliptic equations without a singular potential. We provide all the details since our proof here brings in new distinctions due to the inverse square potential. We fix $r_0 > 0$ small such that $\overline{B_{2r_0}(0)} \subset \Omega$. Since $\gamma^- \in (0,\infty)$, we can find a positive constant C_1, which depends on r_0, such that

$$u(x) \le C_1 |x|^{-p} \quad \text{for every } 0 < |x| \le 2r_0. \tag{5.23}$$

By (2.7), we have $\lim_{r\to 0}\Phi_\lambda^-(r)/\mathcal{K}(r)=0$. Thus by Lemma 4.9 when $0\le\lambda<(N-2)^2/4$ and Remark 4.11 when $\lambda<0$, there exist positive constants C and $\alpha\in(0,1)$ such that

$$\tag{5.24}\begin{cases}|\nabla u(x)|\le C|x|^{-p-1}\\ |\nabla u(x)-\nabla u(x')|\le C|x|^{-p-1-\alpha}|x-x'|^\alpha\left(|x|/|x'|\right)^{\min\{0,p\}}\end{cases}$$

for any x, x' in $\mathbb{R}^N$ with $0<|x|\le|x'|<r_0$. For $r\in(0,r_0)$ fixed, we define the following function

$$\tag{5.25} V_{(r)}(\xi):=\frac{u(r\xi)}{\Phi_\lambda^-(r)\Phi_\lambda^+(\xi)}=\frac{u(r\xi)\,r^p}{|\xi|^{2-N+p}}\quad\text{for } 0<|\xi|<\frac{r_0}{r}.$$

Claim 1. If $(\overline{r}_n)$ is a sequence decreasing to zero as $n\to\infty$, then $V_{(\overline{r}_n)}$ contains a subsequence $V_{(r_n)}$ which converges in $C^1_{\rm loc}(\mathbb{R}^N\setminus\{0\})$.

This assertion follows from the Arzela–Ascoli Theorem once we prove that there exist positive constants C_2 and C_3 such that for every fixed $r\in(0,r_0)$, we have

$$\tag{5.26}\begin{cases}0<V_{(r)}(\xi)\le C_1|\xi|^{N-2-2p},\qquad |\nabla V_{(r)}(\xi)|\le C_2|\xi|^{N-3-2p},\\ |\nabla V_{(r)}(\xi)-\nabla V_{(r)}(\xi')|\le C_3\,\mathcal{B}(\xi,\xi'),\quad\text{where } \mathcal{B}(\xi,\xi')\ \text{ is defined by}\\ \mathcal{B}(\xi,\xi'):=\dfrac{|\xi'|^{N-2-p}}{|\xi|^{p+2}}\left(\dfrac{|\xi|}{|\xi'|}\right)^{\min\{0,p\}}\left[|\xi-\xi'|^{1-\alpha}+|\xi|^{1-\alpha}\right]|\xi-\xi'|^\alpha,\end{cases}$$

for every ξ, ξ' in $\mathbb{R}^N$ satisfying $0<|\xi|\le|\xi'|<r_0/r$. The first two inequalities are immediate by using (5.23) and (5.24). Therefore, we only check the last inequality in (5.26). From (5.25), we derive that

$$\tag{5.27}\nabla V_{(r)}(\xi)-\nabla V_{(r)}(\xi')=r^{p+1}S_1(r,\xi,\xi')+(N-2-p)r^pS_2(r,\xi,\xi'),$$

where we define $S_i(r,\xi,\xi')$ with $i=1,2$ as follows

$$\tag{5.28}\begin{cases}S_1(r,\xi,\xi'):=|\xi|^{N-2-p}(\nabla u)(r\xi)-|\xi'|^{N-2-p}(\nabla u)(r\xi');\\ S_2(r,\xi,\xi'):=u(r\xi)\,|\xi|^{N-4-p}\xi-u(r\xi')\,|\xi'|^{N-4-p}\xi'.\end{cases}$$

We prove that there exist positive constants A_1 and A_2 such that

$$\tag{5.29}\begin{cases}|S_1(r,\xi,\xi')|\le A_1r^{-p-1}\mathcal{B}(\xi,\xi'),\\ |S_2(r,\xi,\xi')|\le A_2r^{-p}|\xi-\xi'||\xi|^{-p-2}|\xi'|^{N-2-p}\left(|\xi|/|\xi'|\right)^{\min\{0,p\}}\end{cases}$$

for every ξ, ξ' in $\mathbb{R}^N$ with $0<|\xi|\le|\xi'|<r_0/r$. Indeed, from (5.28) we find that

$$\begin{cases}|S_1(r,\xi,\xi')|\le|(\nabla u)(r\xi)|\ \left|\,|\xi|^{N-2-p}-|\xi'|^{N-2-p}\,\right|\\ \qquad\qquad+|\xi'|^{N-2-p}\left|\,(\nabla u)(r\xi)-(\nabla u)(r\xi')\,\right|,\\ |S_2(r,\xi,\xi')|\le u(r\xi)\ \left|\,|\xi|^{N-4-p}\xi-|\xi'|^{N-4-p}\xi'\,\right|+|\xi'|^{N-3-p}\left|\,u(r\xi)-u(r\xi')\,\right|.\end{cases}$$

These inequalities, jointly with (5.23) and (5.24), yield that

$$\tag{5.30}\begin{aligned}|S_1(r,\xi,\xi')|\le Cr^{-p-1}\Big[&|\xi|^{-p-1}\left|\,|\xi|^{N-2-p}-|\xi'|^{N-2-p}\,\right|\\ &+|\xi'|^{N-2-p}|\xi|^{-p-1-\alpha}|\xi-\xi'|^\alpha\left(|\xi|/|\xi'|\right)^{\min\{0,p\}}\Big],\end{aligned}$$

as well as

$$
\begin{aligned}
|S_2(r,\xi,\xi')| \le r^{-p} \Big[& C_1 |\xi|^{-p} \, \big| \, |\xi|^{N-4-p}\xi - |\xi'|^{N-4-p}\xi' \, \big| \\
& + C|\xi-\xi'| \, |\xi|^{-p-2} \, |\xi'|^{N-2-p} \left(|\xi|/|\xi'|\right)^{\min\{0,p\}} \Big].
\end{aligned} \tag{5.31}
$$

We see that there exist positive constants a_1 and a_2 such that for every ξ and ξ' in $\mathbb{R}^N$ with $0 < |\xi| \le |\xi'|$, we have

$$
\begin{cases}
\big| \, |\xi|^{N-2-p} - |\xi'|^{N-2-p} \, \big| \le a_1 |\xi-\xi'| \dfrac{|\xi'|^{N-2-p}}{|\xi|}, \\[2ex]
\big| \, |\xi|^{N-4-p}\xi - |\xi'|^{N-4-p}\xi' \, \big| \le a_2 |\xi-\xi'| \dfrac{|\xi'|^{N-2-p}}{|\xi|^2}.
\end{cases} \tag{5.32}
$$

Using (5.32) in (5.30) and (5.31), we conclude the inequalities in (5.29). The last inequality of (5.26) follows from (5.27) and (5.29). This proves Claim 1.

Claim 2. For every $\xi \in \mathbb{R}^N \setminus \{0\}$, the following holds

$$
\lim_{r\to 0^+} V_{(r)}(\xi) = \gamma^- |\xi|^{N-2-2p} \quad \text{and} \quad \lim_{r\to 0^+} \nabla V_{(r)}(\xi) = \gamma^- \nabla \left(|\xi|^{N-2-2p}\right). \tag{5.33}
$$

Then we conclude (5.22) by taking $|\xi| = 1$ and $x = r\xi$ in (5.33). We now prove (5.33). Using the definition of $V_{(r)}$ in (5.25) and that u is a solution of (1.3), we obtain the following equation

$$
\Delta V_{(r)}(\xi) + 2(2-N+p)\nabla V_{(r)}(\xi) \cdot \frac{\xi}{|\xi|^2} = r^{2+p} |\xi|^{N-2-p} b(r\xi)\, h(u(r\xi)) \tag{5.34}
$$

for $0 < |\xi| < r_0/r$. For every fixed $\xi \in \mathbb{R}^N \setminus \{0\}$, we have $0 < |\xi| < r_0/r$ provided that $r > 0$ is sufficiently small. We now observe that

$$
\lim_{r\to 0} r^{2+p} b(r\xi)\, h(u(r\xi)) = 0 \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.35}
$$

From (1.5), Lemma A.10 and (5.23), it suffices to see that

$$
\mathcal{H}(r) := r^{2+\theta+p} L_b(r)\, h_2(C_1 \Phi_\lambda^-(r)) \to 0 \quad \text{as } r \to 0.
$$

This follows immediately if $\lambda \le 0$, since $p \le 0$ and $h_2(t)/t$ is bounded for $t > 0$ small so that $\mathcal{H}$ is bounded from above by a regularly varying function at zero with positive index $(\theta+2)$. If $0 < \lambda < (N-2)^2/4$, then from $h_2(t) \sim tf(t)$ as $t \to \infty$ and (2.7), we conclude that $\mathcal{H}(r) \to 0$ as $r \to 0$. This proves (5.35).

From Claim 1, (5.34) and (5.35), we find that any sequence $\overline{r}_n$ decreasing to zero contains a subsequence r_n so that

$$
V_{(r_n)} \to V \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \quad \text{as } n \to \infty, \tag{5.36}
$$

where V satisfies the following equation

$$
\Delta V(x) + 2(2-N+p)\nabla V(x) \cdot \frac{x}{|x|^2} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}). \tag{5.37}
$$

We use the strong maximum principle to show that

$$
V(\xi) = \gamma^- |\xi|^{N-2-2p} \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.38}
$$

To this aim, we define

$$
F^-(r) := \sup_{|x|=r} \frac{u(x)}{\Phi_\lambda^-(r)} \quad \text{for } r \in (0, r_0). \tag{5.39}
$$

Let ξ_{r_n} be on the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ such that

$$F^-(r_n) = \frac{u(\xi_{r_n} r_n)}{\Phi_\lambda^-(r_n)}.$$

We may assume that $\xi_{r_n} \to \xi_0$ as $n \to \infty$. Using (5.25) and (5.39), we obtain that

$$\text{(5.40)} \quad \frac{V_{(r_n)}(\xi)}{|\xi|^{N-2-2p}} \le F^-(r_n|\xi|) \quad \text{for } 0 < |\xi| < \frac{r_0}{r_n} \quad \text{and} \quad \frac{V_{(r_n)}(\xi_{r_n})}{|\xi_{r_n}|^{N-2-2p}} = F^-(r_n).$$

By the strong maximum principle (in the form of Lemma 1.3 in [**19**]), we find that $\lim_{r\to 0^+} F^-(r) = \gamma^-$. This uses an argument similar to $\lim_{r\to 0} \tilde{\gamma}(r) = \gamma$ in the proof of Theorem 5.1 of [**15**]. Letting $n \to \infty$ in (5.40) and using (5.36), we get

$$\frac{V(\xi)}{|\xi|^{N-2-2p}} \le \gamma^- \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\} \quad \text{and} \quad \frac{V(\xi_0)}{|\xi_0|^{N-2-2p}} = \gamma^-.$$

Hence, by the strong maximum principle (see Theorem 8.19 in [**21**]) applied to $V(\xi) - \gamma^-|\xi|^{N-2-2p}$ satisfying (5.37), we conclude (5.38). Using (5.36), we have

$$\begin{cases} \lim_{n\to\infty} V_{(r_n)}(\xi) = \gamma^-|\xi|^{N-2-2p} \\ \lim_{n\to\infty} \nabla V_{(r_n)}(\xi) = (N-2-2p)\gamma^-|\xi|^{N-4-2p}\xi \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \end{cases}$$

Since $(\bar{r}_n)$ is an arbitrary sequence decreasing to zero, we obtain (5.33). Hence, when $|\xi| = 1$ and $x = r\xi$ in (5.33), we conclude (5.22).

We now show that u extends to a solution of (1.3) in Ω, that is

$$\text{(5.41)} \quad \int_\Omega \nabla u \cdot \nabla \phi \, dx - \int_\Omega \frac{\lambda}{|x|^2} u\phi \, dx + \int_\Omega b(x)\, h(u)\phi \, dx = 0 \quad \text{for every } \phi \in C_c^1(\Omega).$$

From (5.22) and $p < (N-2)/2$, we easily check that $u \in W^{1,2}_{\rm loc}(\Omega)$ and all the integrals in (5.41) are well defined. Let $\phi \in C_c^1(\Omega)$ be fixed arbitrarily. For every $\epsilon > 0$ small, we let $w_\epsilon(r)$ be a non-decreasing and smooth function on $(0,\infty)$ such that $0 < w_\epsilon(r) < 1$ for $r \in (\epsilon, 2\epsilon)$ and

$$w_\epsilon(r) = \begin{cases} 1 & \text{for } r \ge 2\epsilon, \\ 0 & \text{for } r \in (0, \epsilon]. \end{cases}$$

Using $\phi w_\epsilon \in C_c^1(\Omega^*)$ as a test function in Definition 1.1, we obtain that

$$\int_\Omega w_\epsilon \nabla u \cdot \nabla \phi \, dx - \int_\Omega \frac{\lambda}{|x|^2} u\phi w_\epsilon \, dx + \int_\Omega b(x)\, h(u)\phi w_\epsilon \, dx = -\int_\Omega \phi \nabla u \cdot \nabla w_\epsilon \, dx.$$

Letting $\epsilon \to 0$ in the above equality, we conclude (5.41) based on

$$\text{(5.42)} \quad \lim_{\epsilon\to 0} \int_\Omega \phi \nabla u \cdot \nabla w_\epsilon \, dx = \lim_{\epsilon \to 0} \int_{\{\epsilon<|x|<2\epsilon\}} \phi\, w_\epsilon'(|x|) \nabla u \cdot \frac{x}{|x|} \, dx = 0.$$

Indeed, using $\phi(x)|x|^p \nabla u \cdot x \to -p\gamma^- \phi(0)$ as $|x| \to 0$ (from (5.22)) and

$$\int_{\{\epsilon<|x|<2\epsilon\}} |x|^{-p-1} w_\epsilon'(|x|)\, dx = N\omega_N \int_\epsilon^{2\epsilon} r^{-p+N-2} w_\epsilon'(r)\, dr \to 0 \quad \text{as } \epsilon \to 0,$$

we arrive at (5.42). This completes the proof of Lemma 5.4. □

5.1.2. Proof of Theorem 2.1 completed. Here we prove Lemmas 5.5–5.7.

LEMMA 5.5. *If* (1.3) *admits positive solutions such that*

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^-(x)} \in (0,\infty) \text{ and } \lim_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^+(x)} \in (0,\infty), \text{ respectively} \tag{5.43}$$

then we have

$$\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) < \infty \text{ and } \lim_{\tau\to 0} \mathcal{I}^{*}(\tau,\varpi) < \infty, \text{ respectively.} \tag{5.44}$$

PROOF. Let u be a positive solution of (1.3). To show that the first limit in (5.43) implies $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) < \infty$, we need only consider $0 < \lambda < (N-2)^2/4$ for which $\lim_{|x|\to 0} \Phi_\lambda^-(x) = \infty$ (see (2.10)). The proof that (5.43) implies (5.44) follows the same pattern as in Lemma 6.4 in Chapter 6. We need only replace $\Psi^\pm$ by $\Phi_\lambda^\pm$ and use Proposition 3.1 instead of Proposition 3.4. We omit the details. □

LEMMA 5.6. *Let $h(t)/t$ be increasing on $(0,\infty)$ and $\vartheta \in C^1(\partial B_1(0))$ be a non-negative function. If $\lim_{\tau\to 0} \mathcal{I}^*(\tau,\varpi) < \infty$ and γ is any positive number, then* (5.1) *has a unique solution u_γ, which is in $C^{1,\alpha}_{\rm loc}(B^*)$ for some $\alpha \in (0,1)$. The same assertion holds for $\gamma = \infty$ provided that* (1.12)*(a) is satisfied when $q = q^*$.*

PROOF. Remark that when $\lambda = 0$, it suffices to have h non-decreasing on $[0,\infty)$ to have the comparison principle. We adapt the argument of Theorem 1.2 in [**15**]. We first assume that $0 < \gamma < \infty$ and prove the existence of a solution u_γ for (5.1). By Proposition 3.1(c) and Lemma A.9, there exists a unique positive solution $u_* \in C^2(0,1]$ satisfying

$$\begin{cases} u_*''(r) + \dfrac{N-1}{r} u_*'(r) + \dfrac{\lambda}{r^2} u_*(r) = 2r^\theta L_b(r)\, L_h(\Phi_\lambda^+(r))[u_*(r)]^q \text{ in } (0,1), \\ \lim\limits_{r\to 0} \dfrac{u_*(r)}{\Phi_\lambda^+(r)} = \gamma \quad \text{and } u_*(1) = 1. \end{cases} \tag{5.45}$$

Hence, we have $L_h(u_*(r)) \sim L_h(\Phi_\lambda^+(r))$ as $r \to 0$. From (1.5), we can find $r_0 \in (0,1)$ small such that

$$b(x)\, h(u_*(|x|)) \le 2|x|^\theta L_b(|x|)\, L_h(\Phi_\lambda^+(x))\, [u_*(|x|)]^q \quad \text{for every } 0 < |x| \le r_0. \tag{5.46}$$

Let $C_0 > 0$ be a large constant such that

$$C_0 \Phi_\lambda^-(r_0) \ge u_*(r_0) \quad \text{and} \quad C_0 \ge \max_{x\in\partial B_1(0)} \vartheta(x). \tag{5.47}$$

Then, by the comparison principle (Lemma A.9) and (5.46), we obtain that

$$\begin{cases} u_*(r) \le \gamma \Phi_\lambda^+(r) + C_0 \Phi_\lambda^-(r) \quad \text{for every } r \in (0,r_0], \\ -\Delta u_* - \lambda |x|^{-2} u_* + b(x) h(u_*) \le 0 \quad \text{for } 0 < |x| < r_0. \end{cases} \tag{5.48}$$

For every integer $n \ge 1$ with $n > 1/r_0$, we denote by v_n the unique solution of

$$\begin{cases} -\Delta v - \lambda |x|^{-2} v + b(x)\, h(v) = 0 \quad \text{for } x \in B_1(0) \setminus \overline{B_{1/n}(0)}, \\ v(x) = \gamma \Phi_\lambda^+(x) + C_0 \Phi_\lambda^-(x) \quad \text{for } |x| = 1/n \quad \text{and} \quad v = \vartheta \quad \text{on } \partial B_1(0), \\ v > 0 \quad \text{in } B_1(0) \setminus \overline{B_{1/n}(0)}. \end{cases} \tag{5.49}$$

By Lemma A.9 and the method of sub-super-solutions, jointly with (5.47) and (5.48), we deduce that

$$\begin{cases} v_{n+1} \le v_n \le \gamma \Phi_\lambda^+ + C_0 \Phi_\lambda^- & \text{in } B_1(0) \setminus \overline{B_{1/n}(0)}, \\ u_*(|x|) \le v_n(x) + C_0 \Phi_\lambda^-(x) & \text{for } 1/n < |x| < r_0. \end{cases} \tag{5.50}$$

By Lemma 4.9, we can find a sequence $n_k \to \infty$ such that $v_{n_k} \to v_\infty$ in $C^1_{\text{loc}}(B^*)$ and v_∞ is a positive solution of (1.3) in B^* such that $v_\infty = \vartheta$ on $\partial B_1(0)$. Furthermore, from (5.45) and (5.50), we conclude that $\lim_{|x|\to 0} v_\infty(x)/\Phi_\lambda^+(x) = \gamma$, that is v_∞ is a solution of (5.1). The uniqueness of the solution of (5.1) with $0 < \gamma < \infty$ follows easily from the comparison principle. Indeed, if u_1 and u_2 are two solutions of (5.1), then $\lim_{|x|\to 0} u_1(x)/u_2(x) = 1$ so that we can apply Lemma A.9 to deduce that $u_1(x) \le (1+\epsilon) u_2(x)$ in B^* and $u_2(x) \le (1+\epsilon) u_1(x)$ in B^* for any $\epsilon > 0$. Letting $\epsilon \to 0$, we obtain that $u_1 \equiv u_2$ in B^*. Consequently, for every $\gamma \in (0,\infty)$, there exists a unique solution $u_\gamma \in C^1(B^*)$ of (5.1). By Lemma 4.9, we obtain that $u_\gamma \in C^{1,\alpha}_{\text{loc}}(B^*)$ for some $\alpha \in (0,1)$.

To prove the assertion of Lemma 5.6 when $\gamma = \infty$, we let u_n be the unique solution of (5.1) corresponding to $\gamma = n \ge 1$. By the comparison principle, we have $u_n \le u_{n+1}$ in B^*. Lemma 4.9 shows that, up to a subsequence, u_n converges to u_∞ in $C^1_{\text{loc}}(B^*)$, where u_∞ is a solution of (5.1) with $\gamma = \infty$. Moreover, u_∞ is in $C^{1,\alpha}_{\text{loc}}(B^*)$ for some $\alpha \in (0,1)$. From the assumption $\lim_{\tau\to 0} \mathcal{I}^*(\tau, \varpi) < \infty$, we have $q \le q^*$. If $q < q^*$ (respectively, $q = q^*$ and (1.12)(a) holds), then the uniqueness of the solution of (5.1) with $\gamma = \infty$ follows from Theorem 2.4(C). $\square$

Lemma 5.7. *Let $h(t)/t$ be increasing on $(0,\infty)$ and $\vartheta \in C^1(\partial B_1(0))$ be a non-negative and non-trivial function. If $\gamma = 0$, then* (5.1) *has a unique solution, which is in $C^{1,\alpha}_{\text{loc}}(B^*)$ for some $\alpha \in (0,1)$.*

Proof. We first prove the existence. Let $C_0 > 0$ be a large constant such that the second inequality in (5.47) holds. Let v_n be the unique solution of (5.49) with $\gamma = 0$. It follows that $v_{n+1} \le v_n \le C_0 \Phi_\lambda^-$ in $B_1(0) \setminus \overline{B_{1/n}(0)}$. By Lemma 4.9, we conclude that, up to a sub-sequence, v_n converges to v_∞ in $C^1_{\text{loc}}(B^*)$ and v_∞ is a non-negative $C^{1,\alpha}_{\text{loc}}(B^*)$–solution of (1.3) in B^* for some $0 < \alpha < 1$ such that $v_\infty = \vartheta$ on $\partial B_1(0)$. Since $\vartheta \not\equiv 0$ on $\partial B_1(0)$, by the strong maximum principle (see [**30**]), we have $v_\infty > 0$ in B^*. Hence, v_∞ is a $C^{1,\alpha}_{\text{loc}}(B^*)$–solution of (5.1) with $\gamma = 0$.

We now prove the uniqueness assertion. Let u_1 and u_2 be two solutions of (5.1) with $\gamma = 0$. We distinguish two cases depending on $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau, \varpi)$ being finite or infinite (for some small $\varpi > 0$).

Case A: $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau, \varpi) < \infty$. By Proposition 5.1, we have $u_i \in W^{1,2}_{\text{loc}}(B_1(0))$ and u_i $(i = 1,2)$ can be extended as a solution of (1.3) in $B_1(0)$. Hence, we can proceed as in the uniqueness proof of Theorem 1.2 in [**15**]. Indeed, for every $\phi \in C^1_c(B_1(0))$, we have

$$\begin{aligned} &\int_{B_1(0)} \nabla u_i(x) \cdot \nabla \phi(x)\, dx - \lambda \int_{B_1(0)} \frac{u_i(x)\, \phi(x)}{|x|^2}\, dx \\ &\qquad + \int_{B_1(0)} b(x)\, h(u_i)\, \phi(x)\, dx = 0. \end{aligned} \tag{5.51}$$

In fact, (5.51) holds for every $\phi \in H_0^1(B_1(0))$ (using Hardy's inequality). Taking $\phi = (u_1 - u_2)$ in (5.51) for $i = 1, 2$ and subtracting them, we obtain that

$$\int_{B_1(0)} |\nabla(u_1 - u_2)|^2\,dx - \lambda \int_{B_1(0)} \frac{(u_1 - u_2)^2}{|x|^2}\,dx + \int_{B_1(0)} b(x)\,(h(u_1) - h(u_2))\,(u_1 - u_2)\,dx = 0.$$

Since $b > 0$ in B^* and h is increasing, by Hardy's inequality (see (2.1)) and the above equality, we must have $u_1 \equiv u_2$ in B^*.

Case B: $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau, \varpi) = \infty$. In this case, we shall prove later in Lemma 5.10 that $\lim_{|x|\to 0} u_1(x)/u_2(x) = 1$. Since $u_1 = u_2$ on $\partial B_1(0)$, we conclude that $u_1 \equiv u_2$ in B^*. This completes the proof of Lemma 5.7. □

5.2. Proof of Theorem 2.2

By virtue of Proposition 5.1, we conclude the proof of Theorem 2.2 once we show the following.

LEMMA 5.8. *If* $\lim_{\tau\to 0} \mathcal{I}^*(\tau, \varpi) = \infty$, *then* $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = 0$ *for every positive solution* u *of* (1.3).

REMARK 5.9. Recall that when $q \neq q^*$, then $\lim_{\tau\to 0} \mathcal{I}^*(\tau, \varpi) = \infty$ if and only if $q > q^*$ (see (2.9)). Whilst the case $q > q^*$ in Lemma 5.8 can be concluded from Corollary 4.3 since $\Phi_\lambda^+ \in RV_{2-N+p}(0+)$ and $2 - N + p < -(\theta + 2)/(q - 1)$, the critical case $q = q^*$ requires a different reasoning. It is important to see that we could have $\lim_{\tau\to 0} \mathcal{I}^*(\tau, \varpi) = \infty$ when $q = q^*$. Take, for instance, $h(t) = t^{q^*}$ and

$$\text{(i) } L_b(r) \sim 1 \text{ as } r \to 0 \quad \text{or} \quad \text{(ii) } L_b(r) \sim -\frac{1}{\log r} \text{ as } r \to 0.$$

For (i), we have $\lim_{r\to 0} \mathcal{K}(r)/\Phi_\lambda^+(r) = 1$, whereas $\lim_{r\to 0} \mathcal{K}(r)/\Phi_\lambda^+(r) = \infty$ for (ii). Hence, from Lemma 4.1 we cannot infer anymore that $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = 0$.

PROOF. In view of Remark 5.9, we would need to treat only the case $q = q^*$. Our argument is new and it also applies when $q > q^*$. Thus we understand here that $q \geq q^*$. For simplicity of notation, we assume that $\overline{B_1(0)} \subset \Omega$. Let u be an arbitrary positive solution of (1.3). By Corollary 4.5 we need only show that $\liminf_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = 0$. Suppose by contradiction that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^+(x)} > 0.$$

Then we have

$$u(x) \geq a_0 \Phi_\lambda^+(x) \quad \text{for every } 0 < |x| \leq 1 \tag{5.52}$$

for some small positive constant a_0. We choose $m \in \mathbb{R}$ such that

$$q - \frac{\theta + 2}{N - 2 - p} < m < q. \tag{5.53}$$

In particular, we have $m > 1$. We set $\chi(t) := t^{q-m} L_h(t)$ for $t > 0$. Notice that χ is a positive C^1-function in $RV_{q-m}(\infty)$, which satisfies

$$\lim_{t\to\infty} \frac{h(t)}{t^m \chi(t)} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{t\chi'(t)}{\chi(t)} = q - m > 0. \tag{5.54}$$

Therefore, using also (5.52) and (1.5), we infer that there exist positive constants c_0 and c_1 such that

$$\begin{aligned} b(x)\,h(u) &\geq c_0\,|x|^\theta L_b(|x|)\,\chi(a_0\Phi_\lambda^+(x))\,u^m \\ &\geq c_1\,|x|^\theta L_b(|x|)\,\chi(\Phi_\lambda^+(x))\,u^m \quad \text{for every } 0<|x|\leq 1. \end{aligned} \tag{5.55}$$

We define a function $b_0(r)$ as follows

$$b_0(r) := c_1 r^\theta L_b(r)\,\chi(\Phi_\lambda^+(r)) \quad \text{for } r\in(0,1].$$

Hence, b_0 varies regularly at zero of index $\theta-(q-m)(N-2-p)$, which is greater than -2 from our choice of m in (5.53). We conclude the proof once we construct a positive solution U_∞ of

$$-U''(r)-\frac{N-1}{r}U'(r)-\frac{\lambda}{r^2}U(r)+b_0(r)[U(r)]^m=0 \quad \text{for } 0<r<1. \tag{5.56}$$

with the property that $u\leq U_\infty$ in $B_1(0)\setminus\{0\}$. To this end, we see that the assumptions of Proposition 3.1(e2) regarding (3.1) are satisfied here for (5.56) (replacing q in Chapter 3 by m). Indeed, the $C^1(0,1]$–function

$$r^{(m+3)(p-N+2)+2N-2}b_0(r)$$

is regularly varying at 0 with index

$$(q+3)(p-N+2)+2N-2+\theta$$

which is *negative* (since $q\geq q^*$ and $p<(N-2)/2$). From our assumption that $\lim_{\tau\to0}\mathcal{I}^*(\tau,\varpi)=\infty$, we obtain that

$$\lim_{\tau\to0}\int_\tau^\varpi r^{1-(N-2-p)(m-1)}b_0(r)\,dr=\infty. \tag{5.57}$$

By applying Proposition 3.1(e2) to the positive solution U_∞ of (5.56), we would get $\lim_{r\to0}U_\infty(r)/\Phi_\lambda^+(r)=0$. Hence, $\lim_{|x|\to0}u(x)/\Phi_\lambda^+(x)=0$, which would contradict (5.52). We end the proof by constructing U_∞. We let $n\geq2$ be an arbitrary integer and $\mathcal{A}_n$ be given by (5.5). Let U_n be the unique positive solution of the boundary value problem

$$\begin{cases} -\Delta U-\dfrac{\lambda}{|x|^2}U+b_0(|x|)\,U^m=0 & \text{in } \mathcal{A}_n, \\ U(x)=\max\limits_{|y|=|x|}u(y) & \text{for } |x|=1/n \text{ and } |x|=1. \end{cases} \tag{5.58}$$

Clearly, U_n must be radially symmetric. Since u is a positive solution of (1.3), by (5.55) it follows that u is a sub-solution of (5.58). Thus by the comparison principle in Lemma A.9, we get

$$u\leq U_n\leq U_{n+1} \quad \text{in } \mathcal{A}_n \ \text{ for every } n\geq2.$$

By Lemma 4.9, we conclude that, up to a subsequence (re-labelled U_n), we have $U_n\to U_\infty$ in $C^1_{\text{loc}}(B^*)$ with $B^*:=B_1(0)\setminus\{0\}$ and U_∞ is a positive solution of (5.56) satisfying $u\leq U_\infty$ in B^*. This finishes the proof of Lemma 5.8. $\square$

5.3. Proof of Theorem 2.3

In this section, we always assume that the parameter λ is *positive and subcritical* (i.e., $0 < \lambda < (N-2)^2/4$). The assertion of Theorem 2.3 follows by combining Lemma 5.10 with Lemma 5.11 and Lemma A.13 in Appendix A.

LEMMA 5.10. *If* $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) = \infty$, *then* $\lim_{|x|\to 0} u(x)/\Phi_\lambda^-(x) = 0$ *for every positive solution* u *of* (1.3). *Moreover, all positive solutions of* (1.3) *are asymptotically equivalent at zero to any positive* C^2*–function* $\mathcal{U}$ *satisfying*

$$\mathcal{U}''(r) + \frac{N-1}{r}\mathcal{U}'(r) + \frac{\lambda}{r^2}\mathcal{U}(r) \sim r^\theta L_b(r)\, \tilde{h}(\mathcal{U}(r)) \quad \text{as } r\to 0, \tag{5.59}$$

where $\tilde{h}$ *appears in* (1.5).

PROOF. We divide the proof of Lemma 5.10 into two steps.

Step 1. *Any positive solution* u *of* (1.3) *satisfies* $\lim_{|x|\to 0} u(x)/\Phi_\lambda^-(x) = 0$.

From $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) = \infty$, we have $q \geq q^{**}$. If $q > q^{**}$, then our claim follows from Corollary 4.3 since $\Phi_\lambda^- \in RV_{-p}(0+)$ and $-p < -(\theta+2)/(q-1)$. The case $q = q^{**}$ cannot be treated in the same way (see the explanation for $q = q^*$ in Remark 5.9). We give a unitary treatment for $q \geq q^{**}$ (without using Lemma 4.1). By Corollary 4.5, it remains to establish that $\liminf_{|x|\to 0} u(x)/\Phi_\lambda^-(x) = 0$. We shall slightly modify the argument of Lemma 5.8.

Suppose by contradiction that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^-(x)} \neq 0. \tag{5.60}$$

Then (5.52) holds with Φ_λ^- instead of Φ_λ^+. Here, we choose $m \in \mathbb{R}$ such that

$$q - \frac{\theta+2}{p} < m < q. \tag{5.61}$$

We have $m > 1$ since $q \geq q^{**}$. We imitate the proof of Lemma 5.8 replacing Φ_λ^+ by Φ_λ^-. In particular, we define

$$b_0(r) := c_1 r^\theta L_b(r)\, \chi(\Phi_\lambda^-(r)) \quad \text{for } 0 < r \leq 1,$$

which is a regularly varying function at zero with index $\theta - p(q-m)$ that is greater than -2. Hence, similar to Lemma 5.8, we obtain that $u \leq U_\infty$ in B^*, where U_∞ is a positive solution of (5.56). Using (5.54) and $\lim_{\tau\to 0} \mathcal{I}^{**}(\tau,\varpi) = \infty$, we infer that

$$\lim_{\tau\to 0} \int_\tau^\varpi r^{1-p(m-1)} b_0(r)\, dr = \infty,$$

which corresponds to the condition $\lim_{\tau\to 0} \mathcal{I}_2(\tau,\varpi) = \infty$ in Proposition 3.1 in which q is being replaced by m. Moreover,

$$r^{(m+3)(p-N+2)+2(N-1)} b_0(r)$$

is a C^1 regularly varying function at zero with index

$$m(2p-N+2) - pq + 3p - N + 4 + \theta,$$

which is negative using the first inequality in (5.61), jointly with $0 < p < (N-2)/2$ and $q \geq q^{**}$. Thus from Proposition 3.1(e2), (b) applied to the solution U_∞ of (5.56), we conclude that $\lim_{r\to 0} U_\infty(r)/\Phi_\lambda^-(r) = 0$. Since $u(x) \leq U_\infty(|x|)$ for $0 < |x| < 1$, we obtain a contradiction with the assumption (5.60).

Step 2. *All positive solutions of* (1.3) *are asymptotic as* $|x| \to 0$ *to any positive* C^2*–function* $\mathcal{U}$ *satisfying* (5.59).

Let $R > 0$ be arbitrary such that $\overline{B_R(0)} \subset \Omega$. By Lemma 4.12 and Remark 5.3, it suffices to prove that all positive solutions of

$$-\Delta v - \frac{\lambda}{|x|^2} v + |x|^\theta L_b(|x|)\, \tilde{h}(v) = 0 \quad \text{in } B_R(0) \setminus \{0\} \tag{5.62}$$

are asymptotic as $|x| \to 0$. Similar to the proof of Lemma 5.2 or Lemma 5.8, for every positive solution v of (5.62), there exist positive radial solutions of (5.62), say v_∞ and V_∞, such that $v_\infty \le v \le V_\infty$ in $B_R(0) \setminus \{0\}$. Therefore, it suffices to prove that all positive solutions of

$$-v''(r) - \frac{N-1}{r} v'(r) - \frac{\lambda}{r^2} v(r) + r^\theta L_b(r)\, \tilde{h}(v) = 0 \quad \text{for } 0 < r < R \tag{5.63}$$

are asymptotic as $r \to 0^+$. For simplicity, we take $R = 1$. We apply a change of variable as in Proposition 3.1, namely

$$y(s) = v(r)/\Phi_\lambda^-(r) \quad \text{with } s = (N-2) r^{2p-N+2}. \tag{5.64}$$

Set $\omega := (N-2)^{-\frac{p+\theta+2}{2p-N+2}}/(2p-N+2)^2$ and $c := (N-2)^{\frac{p}{2p-N+2}}$. Using (5.63), we find that $y(s)$ satisfies

$$y''(s) = \varphi(s)\, \tilde{h}\left(c s^{-\frac{p}{2p-N+2}} y(s)\right) \quad \text{for } s > N-2, \tag{5.65}$$

where we define

$$\varphi(s) := \omega s^{\frac{-3p+2N-2+\theta}{2p-N+2}} L_b\left(\left(\frac{s}{N-2}\right)^{\frac{1}{2p-N+2}}\right).$$

Notice that $y''(s) > 0$ for $s > N-2$ so that $y'(s)$ is an increasing function. From Step 1 above, we have $\lim_{s\to\infty} y(s) = 0$ for every positive solution $y(s)$ of (5.65). It follows that $\lim_{s\to\infty} y'(s) = 0$ and $y'(s) < 0$ for every $s > N-2$.

We want to prove that any positive solutions y_1 and y_2 of (5.65) are asymptotic as $s \to \infty$, which will conclude our proof. To this end, we prove that

$$\begin{cases} \text{if } y_1(s_0) = y_2(s_0) \text{ for some point } s_0 \in [N-2, \infty), \\ \text{then } y_1(s) = y_2(s) \text{ for any } s \in [s_0, \infty). \end{cases} \tag{5.66}$$

Indeed, for every $\epsilon > 0$ there exists $s_\epsilon > 0$ sufficiently large such that $|y_1 - y_2| \le \epsilon$ for every $s \in [s_\epsilon, \infty)$. Now if there exists a point $s_1 \in (s_0, s_\epsilon)$ such that $y_1 - y_2 > \epsilon$, then $\sup_{s\in[s_0,\infty)} (y_1 - y_2)(s) = \max_{[s_0,s_\epsilon]} (y_1 - y_2)$. Without loss of generality, we assume that $\max_{[s_0,s_\epsilon]}(y_1 - y_2) = y_1(s_1) - y_2(s_1) > \epsilon$. Hence, $y_1'(s_1) = y_2'(s_1)$ and $y_1(\xi) - y_2(\xi) > 0$ for every $\xi \in (s_1, s_1 + \delta)$ provided that $\delta > 0$ is small enough. Since y_i $(i = 1, 2)$ satisfies (5.65), we obtain that

$$(y_1 - y_2)'(s) = \int_{s_1}^{s} \varphi(\xi) \left[\tilde{h}(c\xi^{-\frac{p}{2p-N+2}} y_1(\xi)) - \tilde{h}(c\xi^{-\frac{p}{2p-N+2}} y_2(\xi))\right] d\xi \tag{5.67}$$

for every $s \in (s_1, s_1 + \delta)$. Using that $\tilde{h}$ is increasing, we conclude that

$$s \longmapsto (y_1 - y_2)(s) \quad \text{is increasing on } (s_1, s_1 + \delta),$$

which contradicts that $\max_{[s_0,s_\epsilon]}(y_1 - y_2)$ is achieved at s_1. This proves that

$$y_1 - y_2 \le \epsilon \quad \text{for every } s \in [s_0, \infty).$$

Changing y_1 with y_2, we infer that $|y_1 - y_2| \leq \epsilon$ on $[s_0, \infty)$. Since $\epsilon > 0$ is arbitrary, we conclude that $y_1(s) = y_2(s)$ for every $s \in [s_0, \infty)$.

In view of (5.66), to prove that any two positive solutions y_1 and y_2 of (5.65) are asymptotic as $s \to \infty$, we need only consider the case when $y_1(s) < y_2(s)$ for every $s \in [s_0, \infty)$, where $s_0 > N-2$ is large. We make the change of variable

$$z(t) = \frac{y_1(s)}{y_2(s)}, \quad \text{where } t = \int_{s_0}^{s} \frac{d\xi}{[y_2(\xi)]^2}. \tag{5.68}$$

This change of variable and the argument to follow is inspired by Taliaferro [**36**, Theorem 1.1], who investigated proper positive solutions for equations of the form $y''(s) = \phi(s)s^q$ with $q > 1$. Notice that $t \to \infty$ as $s \to \infty$ and $0 < z(t) < 1$ for every $t \in [0, \infty)$. By differentiating (5.68) and using (5.65) with $y = y_i$ for $i = 1, 2$, we arrive at

$$\begin{aligned} \frac{d^2 z}{dt^2} &= [y_2(s)]^2 \left[y_2(s) \frac{d^2 y_1}{ds^2} - y_1(s) \frac{d^2 y_2}{ds^2} \right] \\ &= \varphi(s)[y_2(s)]^3 \left[\tilde{h}\left(cs^{-\frac{p}{2p-N+2}} y_2(s) z(t) \right) - z(t)\, \tilde{h}\left(cs^{-\frac{p}{2p-N+2}} y_2(s) \right) \right]. \end{aligned} \tag{5.69}$$

However, $\tilde{h}(t)/t$ is increasing on $(0, \infty)$. Thus, (5.69) implies that $z''(t) < 0$ for $t \in (0, \infty)$. Hence, $z'(t)$ is decreasing on $(0, \infty)$. Since $z(t) > 0$ is bounded at ∞, we deduce that $\lim_{t \to \infty} z'(t) = 0$. Consequently, we have $z'(t) \geq 0$ for $t \geq 0$. So, there exists $\lim_{t \to \infty} z(t) = \beta$ and $0 < \beta \leq 1$. From the change of variable in (5.64) and Remark 5.3 applied to (5.62), we have

$$v_i(r) = cs^{-p/(2p-N+2)} y_i(s) \to \infty \quad \text{as } r \to 0, \text{ where } i = 1, 2. \tag{5.70}$$

It follows that

$$\lim_{s \to \infty} \frac{y_1(s)}{y_2(s)} = \lim_{r \to 0} \frac{v_1(r)}{v_2(r)} = \beta \in (0, 1].$$

Hence, using (5.65) and (5.70), we obtain that

$$\lim_{s \to \infty} \frac{y_1''(s)}{y_2''(s)} = \lim_{r \to 0} \frac{\tilde{h}(v_1(r))}{\tilde{h}(v_2(r))} = \beta^{\,q}. \tag{5.71}$$

Since $\lim_{s \to \infty} y_i(s) = \lim_{s \to \infty} y_i'(s) = 0$, by L'Hôpital's rule and (5.71), we conclude that $\beta = \beta^{\,q}$, that is $\beta = 1$. This completes Step 2 and the proof of Lemma 5.10. □

LEMMA 5.11. *Let $q = q^{**}$ in* (1.5) *and* $\lim_{\tau \to 0} \mathcal{I}^{**}(\tau, \varpi) = \infty$. *Then for every positive solution u of* (1.3)*, we have:*

(i) *If* (1.12)*(a) holds, then $u(x) \sim U^{**}(|x|)$ as $|x| \to 0$, where U^{**} is defined by* (2.15).
(ii) *If either* (1.16)*(c) or* (1.16)*(d) holds such that in either case* (2.16) *is verified, then $u(x) \sim \mathbf{C}U^{**}(|x|)$ as $|x| \to 0$, where $\mathbf{C} := e^{\mathbf{D}/(q-1)^2}$ (respectively, $e^{-\mathbf{D}/(q-1)^2}$) when* (1.16)*(c) (respectively,* (1.16)*(d)) holds.*
(iii) *If* (1.12)*(b) holds, jointly with* (1.16)*(c) such that S is regularly varying at ∞ with index η, then* (2.17) *holds.*

PROOF. For simplicity of notation, we use $I(\tau)$ instead of $\mathcal{I}^{**}(\tau, \varpi)$, where $\varpi > 0$ is fixed sufficiently small. Since $q = q^{**}$ in (1.5), from (2.4) and (2.5), it follows that

$$I(r) := \int_r^{\varpi} \frac{L_b(y)\, L_h(\Phi_\lambda^-(y))}{y}\, dy \quad \text{for } 0 < r < \varpi,$$

where $r \longmapsto r^{-1} L_b(r)\, L_h(\Phi_\lambda^-(r))$ is regularly varying at zero of index -1. Since $\lim_{r\to 0} I(r) = \infty$, by Karamata's Theorem (adapt Proposition A.6 in Appendix A for regular variation at 0), we have $\lim_{r\to 0} rI'(r)/I(r) = 0$. We also see that

$$I''(r) = \frac{I'(r)}{r}\left[-1 + \frac{rL_b'(r)}{L_b(r)} - p\frac{\Phi_\lambda^-(r)\, L_h'(\Phi_\lambda^-(r))}{L_h(\Phi_\lambda^-(r))}\right] \quad \text{for every } r \in (0, \varpi).$$

Using (1.6), we obtain that $\lim_{r\to 0} rI''(r)/I'(r) = -1$.

Proof of (i) and (ii). Suppose that we are in the settings of either (i) or (ii) above. For a constant $\mathbf{C} > 0$ to be specified later, we define $\mathcal{U}(r)$ for $r \in (0, \varpi)$ by

$$\mathcal{U}(r) = \mathbf{C} U^{**}(r) = \mathbf{C}\Phi_\lambda^-(r)\, [\mathcal{M}\, I(r)]^{\frac{-1}{q-1}}, \quad \text{where } \mathcal{M} := \frac{q-1}{N-2-2p}. \tag{5.72}$$

Using a simple calculation, we arrive at

$$\mathcal{U}''(r) + \frac{N-1}{r}\mathcal{U}'(r) + \frac{\lambda}{r^2}\mathcal{U}(r) = -\frac{\mathbf{C}[(q-1)I(r)]^{\frac{-q}{q-1}} r^\theta L_b(r)\, \tilde{h}(\Phi_\lambda^-(r))\mathcal{W}(r)}{(N-2-2p)^{\frac{-1}{q-1}}}, \tag{5.73}$$

where we define

$$\mathcal{W}(r) := 2p - N + 1 - \frac{rI''(r)}{I'(r)} + \frac{q}{(q-1)}\frac{rI'(r)}{I(r)}.$$

As $r \to 0$, the right-hand side of (5.73) is asymptotically equivalent to

$$\mathbf{C}[\mathcal{M}\, I(r)]^{-\frac{q}{q-1}} r^\theta L_b(r)\, \tilde{h}(\Phi_\lambda^-(r)),$$

where $\tilde{h}(t) = t^q L_h(t)$. Using Lemma 5.10, it remains to show that

$$\mathbf{C}^{q-1} L_h(\mathcal{U}(r)) \sim L_h(\Phi_\lambda^-(r)) \quad \text{as } r \to 0 \tag{5.74}$$

for some constant $\mathbf{C}$. Indeed, we have $\log \mathcal{U}(r) \sim \log \Phi_\lambda^-(r)$ as $r \to 0$ since

$$\lim_{r\to 0} \frac{\log I(r)}{\log r} = \lim_{r\to 0} \frac{rI'(r)}{I(r)} = 0.$$

- **(i)** If (1.12)(a) holds, then $L_h(\mathcal{U}(r)) \sim L_h(\Phi_\lambda^-(r))$ as $r \to 0$, proving the assertion of (i) by taking $\mathbf{C} = 1$ in (5.72).
- **(ii)** If (1.16)(c) or (1.16)(d) holds such that (2.16) is satisfied in either case, then we find that

$$\log \mathcal{U}(r) = \log \Phi_\lambda^-(r) - \frac{(\mathbf{D} + o(1))}{q-1} S(\log \Phi_\lambda^-(r)) \quad \text{as } r \to 0.$$

Since Λ in (1.17) is Γ–varying at ∞ with auxiliary function S (see Remark 1.7 in Chapter 1), we obtain that

$$\Lambda(\log \mathcal{U}(r)) \sim e^{-\frac{\mathbf{D}}{q-1}} \Lambda(\log \Phi_\lambda^-(r)) \quad \text{as } r \to 0.$$

This proves (5.74) with $\mathbf{C} := e^{\mathbf{D}/(q-1)^2}$ (respectively, $\mathbf{C} := e^{-\mathbf{D}/(q-1)^2}$) if (1.16)(c) (respectively, (1.16)(d)) holds.

Proof of (iii). Let (1.12)(b) hold, jointly with (1.16)(c) such that S is regularly varying at ∞ with index η. Hence, $r \longmapsto S(\log \Phi_\lambda^-(r))$ is slowly varying at 0 and

$$S(\log \Phi_\lambda^-(r)) \sim p^\eta S(\log(1/|x|)) \quad \text{as } |x| \to 0.$$

By Remark A.3 in Appendix A, there exists a C^1 function $\widehat{S}$ that is regularly varying at ∞ such that $\widehat{S}(t) \sim S(t)$ as $t \to \infty$ and $\lim_{t\to\infty} t\widehat{S}'(t)/\widehat{S}(t) = \eta$. We can assume that $\lim_{t\to\infty} tS'(t)/S(t) = \eta$, otherwise one should use $\widehat{S}$ instead of S

in the argument to follow. From (1.8), the map $t \longmapsto f^{-1}(t)$ is regularly varying at ∞ of index $1/(q-1)$. Thus to prove (2.17), it is enough to show that

$$u(x) \sim V(|x|) \quad \text{as } |x| \to 0, \tag{5.75}$$

where $V(r)$ is defined for $r > 0$ small by

$$V(r) := f^{-1}\left(\frac{p}{\mathcal{M}\,\mathcal{J}(r)\,S(\log \Phi_\lambda^-(r))}\right), \tag{5.76}$$

that is

$$V(r) := \Phi_\lambda^-(r)\left(\frac{\mathcal{M}}{p}\,L_b(r)\,L_h(V(r))\,S(\log \Phi_\lambda^-(r))\right)^{-1/(q-1)}.$$

Since $q = q^{**}$, it follows that $r \longmapsto V(r)$ is regularly varying at 0 with index $-p$. Moreover, we have

$$\lim_{r\to 0}\frac{rV'(r)}{V(r)} = -p, \qquad \log V(r) \sim \log \Phi_\lambda^-(r) \quad \text{as } r \to 0. \tag{5.77}$$

Using $V(r)$ in (5.76), we introduce another function $\mathcal{U}(r)$ for $r > 0$ small by

$$\mathcal{U}(r) := \Phi_\lambda^-(r)\,[\mathcal{G}(r)]^{\frac{-1}{q-1}}, \text{ where } \mathcal{G}(r) := \mathcal{M}\int_r^{\varpi}\frac{L_b(y)\,L_h(V(y))}{y}\,dy. \tag{5.78}$$

To complete the proof of (iii), the idea is to show that

$$\mathcal{U}(r) \sim V(r) \quad \text{as } r \to 0 \quad \text{and} \quad \mathcal{U}(r) \text{ satisfies (5.59)}. \tag{5.79}$$

Then by Lemma 5.10, we conclude the proof of (5.75). Note that $\Lambda(t)$ in (1.16) dominates at ∞ any power function of t (see Remark 1.7). From (1.16)(c), we have

$$L_h(V(r)) \sim \Lambda(\log V(r)) \quad \text{and} \quad \frac{\Lambda(\log V(r))}{\Lambda'(\log V(r))\,S(\log \Phi_\lambda^-(r))} \sim 1 \text{ as } r \to 0. \tag{5.80}$$

Using (1.12)(b) and the assumption on S, we see that $L_b(r) = L_b(e^{-(1/p)\log \Phi_\lambda^-(r)})$ and $S(\log \Phi_\lambda^-(r))$ are both regularly varying functions in the variable $t = \log \Phi_\lambda^-(r)$ (and also in $\log V(r)$ in light of (5.77)). Thus $\Lambda(\log V(r))$ dominates $L_b(r)$, as well as $S(\log \Phi_\lambda^-(r))$, as $r \to 0$. Hence, by L'Hôpital's rule and (5.80), we find that

$$\lim_{r\to 0}\frac{\mathcal{G}(r)}{L_b(r)\,L_h(V(r))\,S(\log \Phi_\lambda^-(r))} = \frac{\mathcal{M}}{p}. \tag{5.81}$$

Thus we have $\mathcal{U}(r) \sim V(r)$ as $r \to 0$. We next show that $\mathcal{U}(r)$ given by (5.78) satisfies (5.59). Indeed, from (5.78) it follows that left-hand side of (5.59) equals

$$\frac{\mathcal{U}(r)\mathcal{G}'(r)}{r\mathcal{G}(r)}\left[\frac{2p-N+1}{q-1} - \frac{1}{(q-1)}\frac{r\mathcal{G}''(r)}{\mathcal{G}'(r)} + \frac{q}{(q-1)^2}\frac{r\mathcal{G}'(r)}{\mathcal{G}(r)}\right]. \tag{5.82}$$

Moreover, using (5.81) we have

$$\lim_{r\to 0}\frac{r\mathcal{G}'(r)}{\mathcal{G}(r)} = 0 \quad \text{and} \quad \lim_{r\to 0}\frac{r\mathcal{G}''(r)}{\mathcal{G}'(r)} = -1.$$

Thus, as $r \to 0$ the quantity in (5.82) is asymptotically equivalent to

$$\frac{p\,\mathcal{U}(r)}{\mathcal{M}r^2 S(\log \Phi_\lambda^-(r))}$$

which, in turn, is asymptotically equivalent to the right-hand side of (5.59) (in light of (5.76)). This completes the proof of (5.79). $\square$

5.4. Proof of Theorem 2.4

Throughout this section, we let $-\infty < \lambda < (N-2)^2/4$ and assume that (1.5) is satisfied. Our first result here proves the first two cases in Theorem 2.4. The last case in Theorem 2.4 corresponds to the positive solutions u of (1.3) with $\limsup_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = \infty$. Then by Corollary 4.5, we have

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^+(x)} = \infty.$$

For such solutions, we establish the precise asymptotic behaviour near zero by differentiating between $q \neq q^*$ in Lemma 5.13 and $q = q^*$ in Lemma 5.14. We have seen that if $q \neq q^*$, then $\lim_{\tau\to 0} \mathcal{I}^*(\tau, \varpi) < \infty$ if and only if $q < q^*$.

Lemma 5.12. *If* $\lim_{\tau\to 0} \mathcal{I}^*(\tau,\varpi) < \infty$ *and* u *is a positive solution of* (1.3) *with*

$$\gamma^+ = \limsup_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^+(x)} \neq \infty, \tag{5.83}$$

then one of the following holds:

(a) $\gamma^+ = 0$ *and* $u(x)/\Phi_\lambda^-(x)$ *converges to some positive number as* $|x| \to 0$. *Moreover,* u *can be extended as a solution of* (1.3) *in* Ω.

(b) $\gamma^+ \in (0,\infty)$ *and* u *satisfies*

$$\lim_{|x|\to 0} \frac{u(x)}{\Phi_\lambda^+(x)} = \gamma^+, \quad \lim_{|x|\to 0} \frac{x\cdot \nabla u(x)}{\Phi_\lambda^+(x)} \to (2-N+p)\,\gamma^+. \tag{5.84}$$

Furthermore, if $0 < \lambda < (N-2)^2/4$, *then* u *can be extended as a solution of* (1.3) *in* Ω. *In turn, when* $\lambda = 0$, *we have* (2.20).

Proof. (a) When $\gamma^+ = 0$, the claim follows from (2.8) and Proposition 5.1.

(b) We now assume that $\gamma^+ \neq 0$ and prove the assertion of Lemma 5.12(b). Our argument here is similar to that for Lemma 5.4. We fix $r_0 > 0$ small such that $\overline{B_{2r_0}(0)} \subset \Omega$. From (5.83), we can find a positive constant C_1, which depends on r_0, such that

$$u(x) \le C_1 |x|^{2-N+p} \quad \text{for every } 0 < |x| \le 2r_0. \tag{5.85}$$

By (2.6), we have $\lim_{r\to 0} \Phi_\lambda^+(r)/\mathcal{K}(r) = 0$. Hence, by Lemma 4.9, there exist positive constants $C > 0$ and $\alpha \in (0,1)$ such that

$$|\nabla u(x)| \le C|x|^{1-N+p} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \le C|x|^{1-N+p-\alpha}|x-x'|^\alpha \tag{5.86}$$

for any x, x' in $\mathbb{R}^N$ with $0 < |x| \le |x'| < r_0$. For $r \in (0, r_0)$ fixed, we define

$$W_{(r)}(\xi) := \frac{u(r\xi)}{\Phi_\lambda^+(r)\Phi_\lambda^-(\xi)} = \frac{u(r\xi)|\xi|^p}{r^{2-N+p}} \quad \text{for } 0 < |\xi| < \frac{r_0}{r}. \tag{5.87}$$

Using the inequalities in (5.85) and (5.86), we find positive constants C_2 and C_3 such that for every fixed $r \in (0, r_0)$,

$$\begin{cases} 0 < W_{(r)}(\xi) \le C_1 |\xi|^{2-N+2p}, \qquad |\nabla W_{(r)}(\xi)| \le C_2 |\xi|^{1-N+2p}, \\[1ex] |\nabla W_{(r)}(\xi) - \nabla W_{(r)}(\xi')| \le |\xi-\xi'|^\alpha \dfrac{\left[C + C_3|\xi|^{\alpha-1}|\xi-\xi'|^{1-\alpha}\right]}{|\xi|^{N-2p-1+\alpha}}, \end{cases} \tag{5.88}$$

for every ξ, ξ' in $\mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$. The first two inequalities in (5.88) are easy to check. We only show the last inequality in (5.88). After a simple calculation, we arrive at

$$\nabla W_{(r)}(\xi) - \nabla W_{(r)}(\xi') = r^{N-1-p}\, T_1(r,\xi,\xi') + p\, r^{N-2-p}\, T_2(r,\xi,\xi'),$$

where we define $T_i(r,\xi,\xi')$ with $i = 1, 2$ as follows

$$\begin{cases} T_1(r,\xi,\xi') := |\xi|^p (\nabla u)(r\xi) - |\xi'|^p (\nabla u)(r\xi'); \\ T_2(r,\xi,\xi') := u(r\xi)|\xi|^{p-2}\xi - u(r\xi')|\xi'|^{p-2}\xi'. \end{cases}$$

The last inequality in (5.88) follows once we prove that there exist positive constants D_1 and D_2 such that

$$\begin{cases} |T_1(r,\xi,\xi')| \le C r^{1-N+p} \left[|\xi-\xi'|^\alpha\, |\xi|^{-N+2p+1-\alpha} + D_1\, |\xi|^{-N+2p}\, |\xi-\xi'| \right]; \\ |T_2(r,\xi,\xi')| \le D_2 r^{2-N+p} |\xi|^{-N+2p} |\xi-\xi'| \end{cases} \tag{5.89}$$

for every ξ, ξ' in $\mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$. From the definition of $T_i(r,\xi,\xi')$ with $i = 1, 2$, we obtain that

$$\begin{cases} |T_1(r,\xi,\xi')| \le |\xi|^p\, |(\nabla u)(r\xi) - (\nabla u)(r\xi')| + |(\nabla u)(r\xi')|\, \big||\xi|^p - |\xi'|^p\big|, \\ |T_2(r,\xi,\xi')| \le |\xi|^{p-1}\, |u(r\xi) - u(r\xi')| + \big||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'\big|\, u(r\xi'), \end{cases}$$

which, jointly with (5.85) and (5.86), imply that

$$\begin{cases} |T_1(r,\xi,\xi')| \le C r^{1-N+p} \left[\dfrac{|\xi-\xi'|^\alpha}{|\xi|^{N-1-2p+\alpha}} + \dfrac{\big||\xi|^p - |\xi'|^p\big|}{|\xi'|^{N-1-p}} \right], \\ |T_2(r,\xi,\xi')| \le r^{2-N+p} \left[C\, \dfrac{|\xi-\xi'|}{|\xi|^{N-2p}} + C_1 \dfrac{\big||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'\big|}{|\xi'|^{N-2-p}} \right]. \end{cases} \tag{5.90}$$

We conclude the inequalities in (5.89) from (5.90) by observing that for some positive constants D_0 and D_1, we have

$$\begin{cases} |\xi'|^{1-N+p}\, \big||\xi|^p - |\xi'|^p\big| \le D_1\, |\xi-\xi'|\, |\xi|^{-N+2p}, \\ |\xi'|^{2-N+p}\, \big||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'\big| \le D_0 |\xi|^{-N+2p}\, |\xi-\xi'| \end{cases}$$

for every ξ, ξ' in $\mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$. This proves (5.88).

We next want to show that

$$\lim_{r\to 0^+} W_{(r)}(\xi) = \gamma^+ |\xi|^{2p+2-N}, \quad \lim_{r\to 0^+} \nabla W_{(r)}(\xi) = \gamma^+ \nabla\left(|\xi|^{2p+2-N}\right) \tag{5.91}$$

for every $\xi \in \mathbb{R}^N \setminus \{0\}$. Since u is a solution of (1.3), we see that $W_{(r)}$ in (5.87) satisfies the following equation

$$\Delta W_{(r)}(\xi) - 2p \nabla W_{(r)}(\xi) \cdot \frac{\xi}{|\xi|^2} = |\xi|^p r^{N-p} b(r\xi)\, h(u(r\xi)) \quad \text{for } 0 < |\xi| < r_0/r. \tag{5.92}$$

For every fixed $\xi \in \mathbb{R}^N \setminus \{0\}$, we have $0 < |\xi| < r_0/r$ provided that $r > 0$ is small enough. The right-hand side of (5.92) converges to 0 as $r \to 0$, that is

$$\lim_{r\to 0} r^{N-p} b(r\xi)\, h(u(r\xi)) = 0 \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.93}$$

This follows from (1.5), Lemma A.10 and (5.85) by observing that

$$\lim_{r\to 0} r^{N-p+\theta} L_b(r) h_2(C_1 \Phi_\lambda^+(r)) = 0. \tag{5.94}$$

From (2.6) and $f \in RV_{q-1}(\infty)$, we have $r^{2+\theta} L_b(r) f(C_1 \Phi_\lambda^+(r)) \to 0$ as $r \to 0$. Hence, using that $h_2(t) \sim t f(t)$ as $t \to \infty$, we arrive at (5.94).

Next, we prove that $W_{(r)}$ converges along a sequence $r_n \to 0$. From (5.88), (5.92) and (5.93), we deduce that for any sequence $\overline{r}_n$ decreasing to zero, there exists a subsequence r_n such that

$$W_{(r_n)} \to W \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) \quad \text{as } n \to \infty \tag{5.95}$$

and W satisfies the equation

$$\Delta W(x) - 2p\nabla W(x) \cdot \frac{x}{|x|^2} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}). \tag{5.96}$$

We define

$$F^+(r) := \sup_{|x|=r} \frac{u(x)}{\Phi^+_\lambda(r)} \quad \text{for } r \in (0, r_0). \tag{5.97}$$

Let ξ_{r_n} be on the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ such that

$$F^+(r_n) = \frac{u(\xi_{r_n} r_n)}{\Phi^+_\lambda(r_n)}.$$

We may assume that $\xi_{r_n} \to \xi_0$ as $n \to \infty$. Using (5.87) and (5.97), we obtain that

$$\begin{cases} \dfrac{W_{(r_n)}(\xi)}{|\xi|^{2p+2-N}} = \dfrac{u(r_n\xi)}{|\xi r_n|^{2-N+p}} \leq F^+(r_n|\xi|) \quad \text{for } 0 < |\xi| < \dfrac{r_0}{r_n} \\ \dfrac{W_{(r_n)}(\xi_{r_n})}{|\xi_{r_n}|^{2p+2-N}} = F^+(r_n). \end{cases} \tag{5.98}$$

We find that $\lim_{r\to 0} F^+(r) = \gamma^+$ (similar to $\lim_{r\to 0} F^-(r) = \gamma^-$ in the proof of Lemma 5.4). Passing to the limit $n \to \infty$ in (5.98) and using (5.95), we obtain that

$$\frac{W(\xi)}{|\xi|^{2p+2-N}} \leq \gamma^+ \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\} \quad \text{and} \quad \frac{W(\xi_0)}{|\xi_0|^{2p+2-N}} = \gamma^+.$$

Since $W(\xi) - \gamma^+|\xi|^{2p+2-N}$ satisfies (5.96), by the strong maximum principle (see Theorem 8.19 in [**21**]), we conclude that

$$W(\xi) = \gamma^+|\xi|^{2p+2-N} \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}.$$

Hence, in light of (5.95), we find that

$$\begin{cases} \lim\limits_{n\to\infty} W_{(r_n)}(\xi) = \gamma^+|\xi|^{2p+2-N} \\ \lim\limits_{n\to\infty} \nabla W_{(r_n)}(\xi) = (2p+2-N)\gamma^+|\xi|^{2p-N}\xi \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \end{cases}$$

This proves (5.91). Letting $|\xi| = 1$ and $x = r\xi$ in (5.91), we conclude (5.84).

To complete the proof, we fix $\phi \in C^1_c(\Omega)$ and show that

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx - \int_\Omega \frac{\lambda}{|x|^2} u\phi \, dx + \int_\Omega b(x)\, h(u)\phi \, dx \tag{5.99}$$

is equal to 0 (respectively, $(N-2)N\omega_N\gamma^+\phi(0)$) if $0 < \lambda < (N-2)^2/4$ (respectively, $\lambda = 0$). This assertion for $\lambda = 0$ relies on (5.84) and thus can be proved as in Theorem 5.1 in [**15**]. We now assume that $0 < \lambda < (N-2)^2/4$. Since Φ^+_λ is a regular solution of (2.2) if and only if $0 < \lambda < (N-2)^2/4$, we infer from (5.84) that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $|x|^{-2}u(x)$ is in $L^1_{\mathrm{loc}}(\Omega)$. Hence, the first two integrals in (5.99) are well defined. Using (1.5) and $q \leq q^*$, we also find that $b(x)\, h(u) \in L^1_{\mathrm{loc}}(\Omega)$.

We proceed in the same manner as for proving (5.41) in Lemma 5.4. We need only justify (5.42), which follows since $\phi(x)|x|^{N-2-p}\nabla u\cdot x \to (2-N+p)\gamma^+\phi(0)$ as $|x|\to 0$ (from (5.84)) and

$$\int_{\{\epsilon<|x|<2\epsilon\}} |x|^{1-N+p} w_\epsilon'(|x|)\,dx = N\omega_N \int_\epsilon^{2\epsilon} r^p w_\epsilon'(r)\,dr \to 0 \quad \text{as } \epsilon\to 0. \tag{5.100}$$

Note that when $\lambda=0$ the integral in (5.100) equals $N\omega_N$ for every $\epsilon>0$, which implies that

$$\lim_{\epsilon\to 0}\int_\Omega \phi\nabla u\cdot\nabla w_\epsilon\,dx = (2-N)N\omega_N\gamma^+\phi(0).$$

Using $\phi w_\epsilon \in C^1_c(\Omega^*)$ as a test function in Definition 1.1, then letting $\epsilon\to 0$ we conclude (5.41) (respectively, (2.20)) for $0<\lambda<(N-2)^2/4$ (respectively, $\lambda=0$). This finishes the proof of Lemma 5.12. □

Lemma 5.13. *Let q in* (1.5) *satisfy $q<q^*$, where q^* is given by* (1.11). *Then all positive solutions u of* (1.3) *such that $\lim_{|x|\to 0} u(x)/\Phi^+_\lambda(x)=\infty$ are asymptotic at zero and they satisfy* (1.10), *that is*

$$u(x)\sim \ell^{1/(q-1)}\mathcal{K}(|x|) \quad \textit{as } |x|\to 0, \tag{5.101}$$

where $\mathcal{K}$ and ℓ are defined by (1.8) *and* (1.9), *respectively.*

Proof. Let u be a positive solution of (1.3) with $\lim_{|x|\to 0} u(x)/\Phi^+_\lambda(x)=\infty$. To show (5.101), we use a perturbation method for constructing sub-super-solutions that is inspired by the proof of Theorem 1.4 in [**15**]. However, our construction here is different and somehow simpler than that in [**15**]. Let $\nu_0\in(0,1)$ be fixed small enough. Let $\mathcal{K}$ be given by (1.8). For any $\nu\in[0,\nu_0]$ and $\epsilon\in(0,1)$ small enough, we define $W^\pm_{\epsilon,\nu}(r)$ on a small interval $(0,r_0)$ as follows

$$W^\pm_{\epsilon,\nu}(r) := \left(\ell^\pm_{\epsilon,\nu}\right)^{\frac{1}{q-1}}[\mathcal{K}(r)]^{1\pm\nu}, \quad \text{where } \ell^-_{\epsilon,\nu}<\ell<\ell^+_{\epsilon,\nu} \tag{5.102}$$

and $\lim_{\epsilon\to 0}\left(\lim_{\nu\to 0}\ell^\pm_{\epsilon,\nu}\right)=\ell$. We split the proof of (5.101) into three steps.

Step 1. *For any $\epsilon>0$ small, there exists $r_\epsilon>0$ small such that $W^+_{\epsilon,\nu}$ (respectively, $W^-_{\epsilon,\nu}$) is a super-solution (respectively, sub-solution) of* (1.3) *in $B_{r_\epsilon}(0)\setminus\{0\}$ for every $\nu\in[0,\nu_0]$.*

We fix $\epsilon>0$ sufficiently small. From (A.7) and (1.5), there exists $r_\epsilon>0$ such that for every $\nu\in[0,\nu_0]$ and every $0<|x|\le r_\epsilon$, we have

$$(1-\epsilon)|x|^\theta L_b(|x|)\,\tilde h(W^\pm_{\epsilon,\nu}) \le b(x)\,h(W^\pm_{\epsilon,\nu}) \le (1+\epsilon)|x|^\theta L_b(|x|)\,\tilde h(W^\pm_{\epsilon,\nu}). \tag{5.103}$$

It is enough to show that for every $\nu\in[0,\nu_0]$, the function $z=W^+_{\epsilon,\nu}$ satisfies

$$-\Delta z - \frac{\lambda}{|x|^2}z + (1-\epsilon)|x|^\theta L_b(|x|)\,\tilde h(z)\ge 0 \quad \text{for } 0<|x|<r_\epsilon. \tag{5.104}$$

Then by (5.103) it follows that $W^+_{\epsilon,\nu}$ is a super-solution of (1.3) in $B_{r_\epsilon}(0)\setminus\{0\}$. Similarly, $W^-_{\epsilon,\nu}$ is a sub-solution of (1.3) in $B_{r_\epsilon}(0)\setminus\{0\}$ provided that $z=W^-_{\epsilon,\nu}$ satisfies the following inequality for every $\nu\in[0,\nu_0]$

$$-\Delta z - \frac{\lambda}{|x|^2}z + (1+\epsilon)|x|^\theta L_b(|x|)\,\tilde h(z)\le 0 \quad \text{for } 0<|x|<r_\epsilon. \tag{5.105}$$

We observe that $z = W^{\pm}_{\epsilon,\nu}$ satisfies

$$
\begin{aligned}
&z''(r) + \frac{N-1}{r} z'(r) + \frac{\lambda}{r^2} z(r) \\
&= (\ell^{\pm}_{\epsilon,\nu})^{\frac{1}{q-1}} (1 \pm \nu) [\mathcal{K}(r)]^{\pm\nu} \left(\mathcal{K}''(r) + \frac{N-1}{r} \mathcal{K}'(r) + \frac{\lambda}{r^2} \mathcal{K}(r) \pm \nu T^{\pm}(r) \right)
\end{aligned} \tag{5.106}
$$

where we define

$$
T^{\pm}(r) := \frac{[\mathcal{K}'(r)]^2}{\mathcal{K}(r)} - \frac{\lambda}{1 \pm \nu} \frac{\mathcal{K}(r)}{r^2} \quad \text{for } r \in (0, r_\epsilon).
$$

From the proof of Lemma A.13 in Appendix A, we have (A.8) and

$$
T^{\pm}(r) \le \frac{[\mathcal{K}'(r)]^2}{\mathcal{K}(r)} + \frac{|\lambda|}{1 \pm \nu} \frac{\mathcal{K}(r)}{r^2} \sim \left[\left(\frac{\theta + 2}{q - 1} \right)^2 + \frac{|\lambda|}{1 \pm \nu} \right] r^{\theta} L_b(r) \tilde{h}(\mathcal{K}(r)), \tag{5.107}
$$

where the asymptotic equivalence in (5.107) holds as $r \to 0$. Since $f(t) = \tilde{h}(t)/t$ is increasing for large $t > 0$, we have

$$
\begin{cases}
\tilde{h}(W^{+}_{\epsilon,\nu}(r)) \ge [\mathcal{K}(r)]^{\nu}\, \tilde{h}((\ell^{+}_{\epsilon,\nu})^{1/(q-1)} \mathcal{K}(r)), \\
\tilde{h}(W^{-}_{\epsilon,\nu}(r)) \le [\mathcal{K}(r)]^{-\nu}\, \tilde{h}((\ell^{-}_{\epsilon,\nu})^{1/(q-1)} \mathcal{K}(r)).
\end{cases} \tag{5.108}
$$

From Proposition A.4 and the definition of $\ell^{\pm}_{\epsilon,\nu}$ in (5.102), it follows that

$$
\lim_{t \to \infty} \frac{L_h((\ell^{\pm}_{\epsilon,\nu})^{1/(q-1)} t)}{L_h(t)} = 1
$$

uniformly with respect to $\nu \in [0, \nu_0]$. Since $\tilde{h}(t) = t^q L_h(t)$, we obtain that

$$
\lim_{r \to 0} \frac{\tilde{h}((\ell^{\pm}_{\epsilon,\nu})^{1/(q-1)} \mathcal{K}(r))}{(\ell^{\pm}_{\epsilon,\nu})^{q/(q-1)} \tilde{h}(\mathcal{K}(r))} = 1
$$

uniformly with respect to $\nu \in [0, \nu_0]$. This, jointly with (5.106)–(5.108) and (A.8), proves that it is possible to choose $\ell^{\pm}_{\epsilon,\nu}$ as stated in (5.102) such that $z = W^{+}_{\epsilon,\nu}$ (respectively, $z = W^{-}_{\epsilon,\nu}$) satisfies (5.104) (respectively, (5.105)).

Step 2. *We have* $\lim_{|x| \to 0} u(x)/\mathcal{R}(|x|) = \infty$ *for every function* $\mathcal{R} \in RV_j(0+)$ *with* $j > -(\theta + 2)/(q - 1)$.

We use an argument similar to Lemma 7.1 in [**15**]. Let $\mathcal{R} \in RV_j(0+)$ with $j > -(\theta + 2)/(q - 1)$. We choose θ_1 and q_1 close enough to θ and q, respectively such that $-2 < \theta_1 < \theta$ and $q < q_1 < q^*(N, \lambda, \theta_1)$, where $q^*(N, \lambda, \theta_1)$ is given by (1.11) with θ_1 instead of θ. Moreover, our choice of θ_1 and q_1 is made so that

$$
j > -(\theta_1 + 2)/(q_1 - 1) > -(\theta + 2)/(q - 1). \tag{5.109}
$$

From (1.5) and Proposition A.5 in Appendix A, we have

$$
\lim_{t \to \infty} h(t)/t^{q_1} = 0 \quad \text{and} \quad \lim_{|x| \to 0} b(x)/|x|^{\theta_1} = 0.
$$

Thus there exists $r_0 > 0$ small such that $\overline{B_{r_0}(0)} \subset \Omega$ and

$$
b(x) h(u) \le |x|^{\theta_1} u^{q_1} \quad \text{for } 0 < |x| \le r_0.
$$

Hence, u is a super-solution for the equation

$$
-\Delta v - \lambda |x|^{-2} v + |x|^{\theta_1} v^{q_1} = 0 \quad \text{for } 0 < |x| < r_0. \tag{5.110}
$$

Without any loss of generality, we take $r_0 = 1$. For any positive integer n, we consider the problem

$$(5.111)\qquad \begin{cases} -v''(r) - \dfrac{N-1}{r}v'(r) - \dfrac{\lambda}{r^2}v(r) + r^{\theta_1}v^{q_1} = 0 \quad \text{for } 0 < r < 1, \\ \lim\limits_{r\to 0} v(r)/\Phi_\lambda^+(r) = n, \quad v(1) = \min\limits_{|x|=1} u(x). \end{cases}$$

By Proposition 3.1(c) and Lemma A.9, we have that (5.111) admits a unique positive solution v_n and $u \geq v_n$ on $B_1(0) \setminus \{0\}$ for every $n \geq 1$. Since $n \longmapsto v_n$ is increasing, using Lemma 4.1 and Lemma 4.9, we conclude that $v_n \to v_\infty$ in C^1 in every compact subset of $(0, 1]$ as $n \to \infty$ and v_∞ is a positive radial solution of (5.110) (with $r_0 = 1$) such that $\lim_{r\to 0} v_\infty(r)/\Phi_\lambda^+(r) = \infty$. Moreover, we also have $u \geq v_\infty$ in $B_1(0) \setminus \{0\}$. By Remark 3.2 in Chapter 3, we obtain that

$$\lim_{r\to 0} r^{\frac{\theta_1+2}{q_1-1}} v_\infty(r) \in (0, \infty).$$

This, jointly with the first inequality in (5.109), gives that $\lim_{r\to 0} v_\infty(r)/\mathcal{R}(r) = \infty$. Using $u(x) \geq v_\infty(|x|)$ for $0 < |x| < 1$, we conclude Step 2.

Step 3. *Proof of* (5.101) *completed.*

Since $W_{\epsilon,\nu}^{\pm}$ varies regularly at 0 with index $-(1 \pm \nu)(\theta + 2)/(q - 1)$, by Step 2 above and Corollary 4.3, we find that

$$(5.112)\qquad \lim_{|x|\to 0} \frac{u(x)}{W_{\epsilon,\nu}^-(|x|)} = \infty \quad \text{and} \quad \lim_{|x|\to 0} \frac{u(x)}{W_{\epsilon,\nu}^+(|x|)} = 0.$$

Let C_1 and C_2 be sufficiently large positive constants such that

$$\ell^{1/(q-1)}\,\mathcal{K}(r_\epsilon) \leq C_1\Phi_\lambda^-(r_\epsilon) \quad \text{and} \quad \max_{|x|=r_\epsilon} u(x) \leq C_2\Phi_\lambda^-(r_\epsilon).$$

Hence, $W_{\epsilon,\nu}^-(|x|) \leq u(x) + C_1\Phi_\lambda^-(x)$ on $|x| = r_\epsilon$ and $u(x) \leq W_{\epsilon,\nu}^+(|x|) + C_2\Phi_\lambda^-(x)$ on $|x| = r_\epsilon$, for every $\nu \in [0, \nu_0]$. Since $\lim_{|x|\to 0} u(x) = \infty$, we can assume that (5.103) holds with u instead of $W_{\epsilon,\nu}^{\pm}$. Hence, u satisfies

$$(5.113)\qquad \begin{cases} -\Delta u - \lambda|x|^{-2}u + (1-\epsilon)|x|^\theta L_b(|x|)\,\tilde{h}(u) \leq 0, \\ -\Delta u - \lambda|x|^{-2}u + (1+\epsilon)|x|^\theta L_b(|x|)\,\tilde{h}(u) \geq 0 \end{cases}$$

for every $0 < |x| < r_\epsilon$. We see that $z = W_{\epsilon,\nu}^+ + C_2\Phi_\lambda^-$ satisfies (5.104) for every $\nu \in [0, \nu_0]$ and the second inequality of (5.113) also holds with $u + C_1\Phi_\lambda^-$ instead of u, since $\tilde{h}(t)/t = f(t)$ is an increasing function at ∞ (see (A.7)). Therefore, using (5.112) and the comparison principle in Lemma A.9, we conclude that

$$u \leq W_{\epsilon,\nu}^+ + C_2\Phi_\lambda^- \quad \text{and} \quad u + C_1\Phi_\lambda^- \geq W_{\epsilon,\nu}^-$$

for $0 < |x| \leq r_\epsilon$ and every $\nu \in [0, \nu_0]$. By letting $\nu \to 0$ and using (5.102), we have

$$(5.114)\qquad \limsup_{|x|\to 0} \frac{u(x)}{\mathcal{K}(|x|)} \leq (\ell_\epsilon^+)^{1/(q-1)}, \quad \liminf_{|x|\to 0} \frac{u(x)}{\mathcal{K}(|x|)} \geq (\ell_\epsilon^-)^{1/(q-1)},$$

where $\ell_\epsilon^{\pm} := \lim_{\nu\to 0} \ell_{\epsilon,\nu}^{\pm}$. By letting ε go to zero in (5.114), we arrive at (5.101). This concludes the proof of Lemma 5.13. $\square$

LEMMA 5.14. *Let $q = q^*$ in* (1.5) *and* (1.12)*(a) hold. If* $\lim_{\tau\to 0} \mathcal{I}^*(\tau, \varpi) < \infty$, *then all positive solutions of* (1.3) *with* $\lim_{|x|\to 0} u(x)/\Phi_\lambda^+(x) = \infty$ *are asymptotic at zero and they satisfy* (2.21).

Proof. Let $R > 0$ be arbitrary such that $\overline{B_R(0)} \subset \Omega$. By Lemma 4.12, we need only show that

$$v(r) \sim \Phi_\lambda^+(r)\,[\mathcal{M}\mathcal{I}^*(r)]^{-1/(q-1)} \quad \text{as } r \to 0, \text{ where } \mathcal{I}^*(r) := \lim_{\tau\to 0} \mathcal{I}^*(\tau, r) \tag{5.115}$$

for every positive solution v of

$$\begin{cases} -\Delta v - \lambda |x|^{-2} v + |x|^\theta L_b(|x|)\,\tilde{h}(v) = 0 & \text{in } B_R(0) \setminus \{0\}, \\ \lim\limits_{|x|\to 0} v(x)/\Phi_\lambda^+(x) = \infty. \end{cases} \tag{5.116}$$

Using Lemma 4.4 and an argument similar to Lemma 5.2 or Lemma 5.8, we can construct positive radial solutions v_* and v^* of (5.116) with $v_* \le v \le v^*$ in $B_R(0) \setminus \{0\}$. So, it remains to prove (5.115) for any positive solution of

$$\begin{cases} -v''(r) - \dfrac{N-1}{r} v'(r) - \dfrac{\lambda}{r^2} v(r) + r^\theta L_b(r)\,\tilde{h}(v) = 0 & \text{in } (0, R), \\ \lim\limits_{r\to 0} v(r)/\Phi_\lambda^+(r) = \infty. \end{cases} \tag{5.117}$$

We take $R = 1$ for simplicity. Note that if we apply the change of variable (5.64) in the proof of Lemma 5.10, then we get (5.65). However, we cannot reason as in Lemma 5.10 (where $\lim_{s\to\infty} y(s)/s = \lim_{s\to\infty} y'(s) = 0$) since here we have

$$\lim_{r\to 0} \frac{v(r)}{\Phi_\lambda^+(r)} = \lim_{s\to\infty} \frac{y(s)}{s} = \lim_{s\to\infty} y'(s) = \infty.$$

Nor can we apply the perturbation technique employed in Lemma 5.13, where it was essential that $q < q^*$. The idea for $q = q^*$ is to somehow make use of Corollary 3.3. But it seems difficult, in general, to pass the conclusions from the special power case to the more general situations of regularly varying functions. We are able to achieve this because of our assumption (1.12)(a). Hence, we can write $L_h(e^t) = t^{\alpha_1} \tilde{L}(t)$, where $\tilde{L}$ is a slowly varying function at ∞.

Because $q = q^*$, we see that $\mathcal{K}$ in Lemma 4.1 is regularly varying at zero with the same index as Φ_λ^+ (cf., Remark 4.2). Consequently, we have $\log \mathcal{K}(r) \sim \log \Phi_\lambda^+(r)$ as $r \to 0$. Since $\lim_{r\to 0} v(r)/\Phi_\lambda^+(r) = \infty$, by Lemma 4.1 we deduce that

$$\frac{\log v(r)}{\log \Phi_\lambda^+(r)} \sim 1, \quad \frac{L_h(v(r))}{L_h(\Phi_\lambda^+(r))} = \left(\frac{\log v(r)}{\log \Phi_\lambda^+(r)}\right)^{\alpha_1} \frac{\tilde{L}(\log v(r))}{\tilde{L}(\log \Phi_\lambda^+(r))} \sim 1 \text{ as } r \to 0.$$

Hence, $\tilde{h}(v(r)) \sim L_h(\Phi_\lambda^+(r))[v(r)]^q$ as $r \to 0$. Since $\lim_{\tau\to 0} \mathcal{I}^*(\tau, \varpi) < \infty$, by applying Corollary 3.3 with

$$b_0(r) := r^\theta L_b(r)\, L_h(\Phi_\lambda^+(r)),$$

we conclude (5.115). This completes the proof of Lemma 5.14. □

CHAPTER 6

The analysis for the critical parameter

In this chapter, we assume that (1.5) holds. We investigate the asymptotic behaviour near zero for all positive solutions of (1.3) with $\lambda = (N-2)^2/4$, namely

$$-\Delta u - \left(\frac{N-2}{2}\right)^2 \frac{u}{|x|^2} + b(x)\, h(u) = 0 \quad \text{in } \Omega^*. \tag{6.1}$$

Our goal is to demonstrate the main results on (6.1), whose statements can be found in Section 2.2 of Chapter 2. We prove Theorem 2.5 in Section 6.1, whilst in Sections 6.2 and 6.3 we establish Theorems 2.6 and 2.7, respectively. For the reader's convenience, we recall that the fundamental solutions of (2.2) are given by $\Psi^{\pm}$ in (2.24) and the critical exponent for (6.1) is $q^* = (N+2+2\theta)/(N-2)$. We define $\mathcal{F}_*(\tau,\varpi)$ and $\mathcal{F}^*(\tau,\varpi)$ as in (2.25).

6.1. Proof of Theorem 2.5

The structure of this section is similar to that of Section 5.1 in Chapter 5. In Proposition 6.1, we prove the converse implication "$\Leftarrow$" of (2.29). The direct implications in (2.29) and (2.30) are demonstrated by Lemma 6.4. The remaining assertions of Theorem 2.5 are incorporated into Lemma 6.5.

6.1.1. The crux of Theorem 2.5.

Proposition 6.1. *Assume that* $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty$ *and u is a positive solution of* (6.1) *such that* $\lim_{|x|\to 0} u(x)/\Psi^+(x) = 0$. *Then, we have*

$$\lim_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} \in (0,\infty). \tag{6.2}$$

This result resembles Proposition 5.1, although we proceed here quite differently. We list the main ingredients (Lemmas 6.2 and 6.3), then we use them to complete the proof of Proposition 6.1, before we validate our auxiliary tools. The first ingredient, Lemma 6.2, is the analogue of Lemma 5.2, though here the proof is much simpler. The second one, Lemma 6.3, is an isotropy result comparable with Theorem 2.1 in [**22**], which extends Vázquez–Véron's isotropy theorems [**41, 42**] to the potential case.

Lemma 6.2. *Let* $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty$. *If u is a positive solution of* (6.1) *such that* $\lim_{|x|\to 0} u(x)/\Psi^+(x) = 0$, *then*

$$0 < \liminf_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} \le \limsup_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} < \infty. \tag{6.3}$$

Notation. Let $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ denote the standard $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$. Let $(r,\sigma) \in (0,\infty) \times \mathbb{S}^{N-1}$ stand for the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$. By $\bar{u}$ we denote the spherical average of u, that is

$$\bar{u}(r) = \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} u(r,\sigma)\, d\sigma.$$

LEMMA 6.3. *Let $b_0 \in RV_\theta(0+)$ with $\theta > -2$. Let $r_0 > 0$ be small such that $\overline{B_{r_0}(0)} \subset \Omega$ and b_0 is well defined on $(0, r_0)$. Assume that $q > 1$, as well as*

$$\frac{h(t)}{t} \text{ is increasing on } (0,\infty), \quad \lim_{t\to\infty} \frac{th'(t)}{h(t)} = q, \quad \lim_{t\to\infty} \frac{th''(t)}{h'(t)} = q-1 > 0. \tag{6.4}$$

Let u be a positive solution of (6.1) in $B_{r_0}(0) \setminus \{0\}$ with $b(x) = b_0(|x|)$, that is

$$-\Delta u - \left(\frac{N-2}{2}\right)^2 \frac{u}{|x|^2} + b_0(|x|)\, h(u) = 0 \quad \text{in } B_{r_0}(0) \setminus \{0\}. \tag{6.5}$$

If u satisfies the condition

$$\liminf_{r\to 0} r^{\frac{N-2}{2}+\sqrt{N-1}} \|u(r,\cdot) - \overline{u}(r)\|_{L^2(\mathbb{S}^{N-1})} = 0, \tag{6.6}$$

then $u(x)/\Psi^{\pm}(x)$ admits a limit in $[0,\infty]$ as $|x|$ tends to 0.

PROOF OF PROPOSITION 6.1. We first assume that h also satisfies (6.4) in Lemma 6.3 and there exists $r_0 > 0$ such that $b(x) = b_0(|x|)$ for $0 < |x| \le r_0$. The assertion of (6.2) follows from Lemmas 6.2 and 6.3. We next establish Proposition 6.1 without these extra hypotheses. Let $\epsilon \in (0,1)$ be fixed and v_* be given by Lemma 4.12 with $\lambda = (N-2)^2/4$. Since v_* satisfies (4.26) and (1.7) holds, we can apply Proposition 6.1 to v_*. Hence, $\lim_{|x|\to 0} v_*(x)/\Psi^-(x) \in (0,\infty)$ and

$$(1-\epsilon) \limsup_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} \le \lim_{|x|\to 0} \frac{v_*(x)}{\Psi^-(x)} \le (1+\epsilon) \liminf_{|x|\to 0} \frac{u(x)}{\Psi^-(x)}. \tag{6.7}$$

Thus we have $\lim_{|x|\to 0} u(x)/\Psi^-(x) \in (0,\infty)$, which finishes the proof. □

PROOF OF LEMMA 6.2. Assume that u is a positive solution of (6.1) satisfying $\lim_{|x|\to 0} u(x)/\Psi^+(x) = 0$. By Lemma A.9, we have $\limsup_{|x|\to 0} u(x)/\Psi^-(x) < \infty$. It remains to show that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} > 0. \tag{6.8}$$

We use ideas similar to those in Lemma 5.2 pertaining to $0 < \lambda < (N-2)^2/4$. But the proof is now much simplified because we already know that $\lim_{|x|\to 0} u(x) = \infty$ (see Remark 5.3). Let $C > 0$ be a large constant such that (5.17) holds. Hence, as in Step 4 of Lemma 5.2, we have $u \ge w_\infty$ in $B_1(0) \setminus \{0\}$ for some positive radial solution w_∞ of (5.18), where $\lambda = (N-2)^2/4$. Since $\lim_{r\to 0} w_\infty(r)/\Psi^-(r) \in [0,\infty)$, we need only show that $\lim_{r\to 0} w_\infty(r)/\Psi^-(r) \ne 0$. We see that (5.20) holds with $\lambda = (N-2)^2/4$. If we were to assume that $\lim_{r\to 0} w_\infty(r)/\Psi^-(r) = 0$, then by Proposition 3.4(b), we would have that

$$\lim_{\tau\to 0} \int_\tau^{\varpi} r^{\frac{N-m(N-2)}{2}+\theta} L_b(r) [w_\infty(r)]^{q-m} L_h(w_\infty(r)) \log(1/r)\, dr = \infty. \tag{6.9}$$

Since (5.19) holds with Ψ^- instead of Φ_λ^-, (6.9) leads to $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) = \infty$. This contradiction shows that $\lim_{r\to 0} w_\infty(r)/\Psi^-(r) \in (0,\infty)$, which proves (6.8). □

Proof of Lemma 6.3. Let u be a positive solution of (6.5) satisfying (6.6). Using the spherical average of u, we write

$$\frac{u(r,\sigma)}{\Psi^{\pm}(r)} = \frac{\bar{u}(r)}{\Psi^{\pm}(r)} + \frac{u(r,\sigma)-\bar{u}(r)}{\Psi^{\pm}(r)}. \tag{6.10}$$

The idea is to show that the first term in the right-hand side of (6.10) admits a limit in $[0,\infty]$ as $r \to 0$ (see Step 1 below), while the second term in the right-hand side of (6.10) converges to zero uniformly with respect to $\sigma \in \mathbb{S}^{N-1}$ (see Step 3). Our argument is divided into three steps.

Step 1. *The ratio $\bar{u}(r)/\Psi^{\pm}(r)$ admits a limit in $[0,\infty]$ as $r \to 0^+$.*

Let $(r,\sigma) \in (0,\infty) \times \mathbb{S}^{N-1}$ denote the spherical coordinates of $x \in \mathbb{R}^N \setminus \{0\}$. By averaging (6.5), we find

$$-\Delta \bar{u} - \left(\frac{N-2}{2}\right)^2 \frac{\bar{u}}{|x|^2} + b_0(r)\, \overline{h(u(r,\sigma))} = 0 \quad \text{in } B_{r_0}(0) \setminus \{0\}, \tag{6.11}$$

where $\overline{h(u(r,\sigma))}$ denotes the spherical average of $h(u(r,\sigma))$. From Remark 5.3, we have $\lim_{|x|\to 0} u(x) = \infty$. By (6.4), we see that h is convex on $[t_1,\infty)$ with $t_1 > 0$ sufficiently large. Thus we infer that $\overline{h(u)} \geq h(\bar{u})$ for $r \in (0,r_1)$, where $r_1 > 0$ is chosen suitably small such that $u(x) \geq t_1$ for every $x \in \mathbb{R}^N$ with $0 < |x| \leq r_1$. Using (6.11), we have

$$-\Delta \bar{u} - \left(\frac{N-2}{2}\right)^2 \frac{\bar{u}}{|x|^2} + b_0(|x|)\, h(\bar{u}) \leq 0 \quad \text{in } B_{r_1}(0) \setminus \{0\}. \tag{6.12}$$

From (6.12) we conclude Step 1 by contradiction proceeding as in Step 2 in the proof of Lemma 5.2 with regard to $v_\infty(r)/\Phi_\lambda^-(r)$ admitting a limit as $r \to 0$.

Step 2. *We prove that* $\limsup_{r\to 0} r^{\frac{N-2}{2}-\sqrt{N-1}} \|u(r,\cdot) - \bar{u}(r)\|_{L^2(\mathbb{S}^{N-1})} < \infty$.

We proceed similarly to Lemma 2.1 in Guerch and Véron [**22**]. We write

$$y(s,\sigma) = r^{(N-2)/2} u(r,\sigma) \quad \text{with } s = \log(1/r) \quad \text{for } r \in (0,r_*].$$

Here $r_* > 0$ is small such that $r_* < \min\{1, r_0\}$. Hence, y satisfies the equation

$$\frac{\partial^2 y}{\partial s^2} + \Delta_{\mathbb{S}^{N-1}} y = e^{-(2+N)s/2} b_0(e^{-s})\, h(e^{(N-2)s/2} y) \tag{6.13}$$

for $(s,\sigma) \in [\log R_*, \infty) \times \mathbb{S}^{N-1}$, where $R_* = 1/r_*$ and $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$. Let $\bar{y}$ and $\overline{h(e^{(N-2)s/2}y)}$, respectively denote the spherical average of y and $h(e^{(N-2)s/2}y)$ on $\mathbb{S}^{N-1}$, respectively. For every $s \in [\log R_*, \infty)$, we define

$$E(s) := \|y(s,\sigma) - \bar{y}(s)\|_{L^2(\mathbb{S}^{N-1})} = \left(\int_{\mathbb{S}^{N-1}} |y(s,\sigma) - \bar{y}(s)|^2\, d\sigma\right)^{1/2}. \tag{6.14}$$

The assertion of Step 2 means that there exists a positive constant $C > 0$ such that

$$E(s) \leq C e^{-s\sqrt{N-1}} \quad \text{on } [\log R_*, \infty). \tag{6.15}$$

To achieve (6.15), we want to prove that

$$E''(s) - (N-1)E(s) \geq 0 \quad \text{for any } s \in [\log R_*, \infty). \tag{6.16}$$

By averaging (6.13), we obtain that

$$\frac{\partial^2 \bar{y}}{\partial s^2} = e^{-(2+N)s/2} b_0(e^{-s})\, \overline{h(e^{(N-2)s/2} y)} \quad \text{for } s \in [\log R_*, \infty). \tag{6.17}$$

Since h is non-decreasing (from (6.4)) and $\int_{\mathbb{S}^{N-1}} (y - \bar{y})\, d\sigma = 0$, we infer that

$$\begin{aligned} &\int_{\mathbb{S}^{N-1}} (y-\bar{y}) \left[h(e^{(N-2)s/2} y) - \overline{h(e^{(N-2)s/2} y)} \right] d\sigma \\ &= \int_{\mathbb{S}^{N-1}} (y-\bar{y}) \left[h(e^{(N-2)s/2} y) - h(e^{(N-2)s/2} \bar{y}) \right] d\sigma \geq 0. \end{aligned} \tag{6.18}$$

We multiply (6.13) and (6.17) by $(y - \bar{y})$, then integrate over $\mathbb{S}^{N-1}$ with respect to σ. By subtracting the equations obtained in this way and using (6.18), we have

$$\int_{\mathbb{S}^{N-1}} (y-\bar{y}) \frac{\partial^2 (y-\bar{y})}{\partial s^2}\, d\sigma + \int_{\mathbb{S}^{N-1}} (y-\bar{y}) \Delta_{\mathbb{S}^{N-1}} y \, d\sigma \geq 0. \tag{6.19}$$

We now recall that $(N-1)$ is the first eigenvalue of the Laplace–Beltrami operator $\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}^{N-1}$. So, we have

$$- \int_{\mathbb{S}^{N-1}} (y-\bar{y}) \Delta_{S^{N-1}} y \, d\sigma \geq (N-1) \int_{\mathbb{S}^{N-1}} (y-\bar{y})^2 \, d\sigma. \tag{6.20}$$

From (6.19) and (6.20), it follows that

$$\int_{\mathbb{S}^{N-1}} (y-\bar{y}) \frac{\partial^2 (y-\bar{y})}{\partial s^2}\, d\sigma - (N-1) \int_{\mathbb{S}^{N-1}} (y-\bar{y})^2 \, d\sigma \geq 0. \tag{6.21}$$

Using (6.14), we observe that $[E'(s)]^2$ is bounded above by $\int_{\mathbb{S}^{N-1}} \left[\frac{\partial}{\partial s} (y-\bar{y}) \right]^2 d\sigma$ and the following identity holds

$$[E'(s)]^2 + E(s) E''(s) = \int_{\mathbb{S}^{N-1}} \left[\frac{\partial}{\partial s} (y-\bar{y}) \right]^2 d\sigma + \int_{\mathbb{S}^{N-1}} (y-\bar{y}) \frac{\partial^2}{\partial s^2} (y-\bar{y})\, d\sigma. \tag{6.22}$$

Hence, the second integral in the right-hand side of (6.22) is bounded above by $E(s)E''(s)$. So, using (6.21), we conclude the proof of (6.16). Let $C > 0$ be large enough such that $E(\log R_*) \leq C (R_*)^{-\sqrt{N-1}}$. For any $\epsilon > 0$, we define

$$Q_\epsilon(s) := \epsilon e^{s\sqrt{N-1}} + C e^{-s\sqrt{N-1}} \quad \text{for } s \in [\log R_*, \infty). \tag{6.23}$$

Clearly, Q_ϵ verifies the following equation

$$Q_\epsilon''(s) - (N-1) Q_\epsilon(s) = 0 \quad \text{for } s \in [\log R_*, \infty). \tag{6.24}$$

If u satisfies (6.6), then there exists a sequence $(s_n)_{n \geq 1}$ such that $s_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} e^{-s_n \sqrt{N-1}} E(s_n) = 0$. Hence, there exists a large positive integer n_ϵ such that $E(s_n) \leq \epsilon e^{s_n \sqrt{N-1}}$ for every $n \geq n_\epsilon$. Consequently, $E(s) \leq Q_\epsilon(s)$ for all $s = s_n$ with $n \geq n_\epsilon$ and also for $s = \log R_*$ (from the choice of C and (6.23)). In view of (6.16) and (6.24), we can apply the comparison principle on each interval $[\log R_*, s_n]$ with $n \geq n_\epsilon$. Hence, $E(s) \leq Q_\epsilon(s)$ for every $s \in [\log R_*, \infty)$. Since $\epsilon > 0$ is arbitrary, by letting $\epsilon \to 0$, we conclude the proof of (6.15).

Step 3. *We show that* $\limsup_{r\to 0^+} r^{\frac{N-2}{2} - \sqrt{N-1}} \|u(r,\cdot) - \bar{u}(r)\|_{L^\infty(\mathbb{S}^{N-1})} < \infty$.

By defining $Y(s,\sigma) := y(s,\sigma) - \bar{y}(s)$, the claim of Step 3 can be restated as

$$\limsup_{s\to\infty} e^{s\sqrt{N-1}} \|Y(s,\cdot)\|_{L^\infty(\mathbb{S}^{N-1})} < \infty. \tag{6.25}$$

From (6.13) and (6.17), we see that Y satisfies the equation

$$\frac{\partial^2 Y}{\partial s^2} + \Delta_{\mathbb{S}^{N-1}} Y = F \text{ on } [\log R_*, \infty) \times \mathbb{S}^{N-1}, \tag{6.26}$$

where F is defined by

$$F := e^{-(2+N)s/2} b_0(e^{-s}) \left[h(e^{(N-2)s/2} y) - \overline{h(e^{(N-2)s/2} y)} \right]. \tag{6.27}$$

We apply Remark 6.6 in [**43**] to the functions Y and F defined above. We have

$$\int_{\mathbb{S}^{N-1}} Y(s,\sigma)\, d\sigma = \int_{\mathbb{S}^{N-1}} F(s,\sigma)\, d\sigma = 0 \quad \text{for every } s \geq \log R_*.$$

In the preceding Step 2, we proved that

$$\limsup_{s\to\infty} e^{s\sqrt{N-1}} \|Y(s,\cdot)\|_{L^2(\mathbb{S}^{N-1})} < \infty.$$

Thus to complete Step 3, we need to show for some constant $c > 0$, we have

$$\|F(s,\cdot)\|_{L^p(\mathbb{S}^{N-1})} \leq c \|Y(s,\cdot)\|_{L^p(\mathbb{S}^{N-1})} \quad \text{for every } s \geq \log R_* \text{ and all } p \geq 2. \tag{6.28}$$

Revisiting Step 1, we see that $h'(t)$ is increasing for $t \geq t_1$ and $e^{(N-2)s/2} y(s,\sigma) \geq t_1$ for every $(s,\sigma) \in [\log R_*, \infty) \times \mathbb{S}^{N-1}$. Since u is a solution of (6.5) and $\bar{u}$ is a sub-solution of (6.5), we can apply Lemma 4.1 with $b(x) = b_0(|x|)$. Hence, there exists a positive constant C_* such that $|x|^2 b_0(|x|)\, h'(u)$ and $|x|^2 b_0(|x|)\, h'(\bar{u})$ are bounded above by C_* for every $0 < |x| \leq r_*$. Using that $u(r,\sigma) = e^{(N-2)s/2} y(s,\sigma)$ with $r = e^{-s}$ and the mean value theorem, we obtain that

$$e^{-\frac{(2+N)s}{2}} b_0(e^{-s}) \left| h(e^{\frac{(N-2)s}{2}} y) - h(e^{\frac{(N-2)s}{2}} \bar{y}) \right| \leq C_* |y - \bar{y}| = C_* |Y(s,\sigma)|, \tag{6.29}$$

for every $(s,\sigma) \in [\log R_*, \infty) \times \mathbb{S}^{N-1}$. On the other hand, we have

$$\begin{aligned} \left| h(e^{(N-2)s/2} y) - \overline{h(e^{(N-2)s/2} y)} \right| &\leq \left| h(e^{(N-2)s/2} y) - h(e^{(N-2)s/2} \bar{y}) \right| \\ &\quad + \left| h(e^{(N-2)s/2} \bar{y}) - \overline{h(e^{(N-2)s/2} y)} \right|, \end{aligned} \tag{6.30}$$

where the second term in the right-hand side of (6.30) is bounded above by

$$\frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} \left| h(e^{(N-2)s/2} y) - h(e^{(N-2)s/2} \bar{y}) \right| d\sigma.$$

Multiplying (6.30) by $e^{-(2+N)s/2} b_0(e^{-s})$ and using (6.29), we infer that

$$|F(s,\sigma)| \leq C_* \left[|Y(s,\sigma)| + \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} |Y(s,\sigma)|\, d\sigma \right] \tag{6.31}$$

for every $(s,\sigma) \in [\log R_*, \infty) \times \mathbb{S}^{N-1}$, where F is defined by (6.27). From (6.31), we conclude (6.28). This completes the proof of Step 3. $\square$

6.1.2. Proof of Theorem 2.5 completed.

Lemma 6.4. *If* (6.1) *admits positive solutions such that*

$$\lim_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} \in (0,\infty) \text{ and } \lim_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} \in (0,\infty), \text{ respectively} \tag{6.32}$$

then we have

$$\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) < \infty \text{ and } \lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty, \text{ respectively.} \tag{6.33}$$

Proof. We assume that (6.1) possesses a positive solution u such that the first limit in (6.32) holds. We prove that (6.33) holds, that is $\lim_{\tau\to 0}\mathcal{F}^*(\tau,\varpi)<\infty$. Since $\lim_{|x|\to 0}\Psi^-(x)=\infty$, from (1.5) and the assumption in (6.32), we have

$$b(x)\,h(u)\sim |x|^\theta L_b(|x|)\,L_h(\Psi^-(x))\,u^q \quad \text{as } |x|\to 0.$$

This fact and an argument similar to Lemma 4.12 in Chapter 4 show that for some $\delta>0$, we can construct a positive solution V of

$$-\Delta V-\lambda\frac{V}{|x|^2}+|x|^\theta\,L_b(|x|)\,L_h(\Psi^-(x))\,V^q=0 \ \text{ in } B_\delta(0)\setminus\{0\} \tag{6.34}$$

such that $u/2\le V\le 2u$ in $B_\delta(0)\setminus\{0\}$. In (6.34) we have $\lambda=(N-2)^2/4$. Proceeding as in Step 2 in the proof of Lemma 5.2 (replacing (5.4) by (6.34)), we can find a positive radial solution V_∞ of (6.34) such that $V_\infty(|x|)\le V(x)$ for $0<|x|<\delta$. Moreover, using the Harnack-type inequality in Lemma 4.4, we also obtain that $V_\infty(|x|)\ge \kappa V(x)$ for some positive constant κ with $\kappa<1$. Therefore, V_∞ satisfies

$$\frac{\kappa}{2}u(x)\le V_\infty(|x|)\le 2u(x) \ \text{ for } 0<|x|<\delta.$$

Since V_∞ is a positive radial solution of (6.34) and $\lim_{|x|\to 0}u(x)/\Psi^-(x)\in(0,\infty)$, we obtain that $\lim_{r\to 0}V_\infty(r)/\Psi^-(r)\in(0,\infty)$ (see a similar argument for $v_\infty(r)/\Phi^-_\lambda(r)$ in Step 2 of Lemma 5.2). We conclude that $\lim_{\tau\to 0}\mathcal{F}^*(\tau,\varpi)<\infty$ by applying Proposition 3.4(b) for V_∞ with $b_0(r)=r^\theta\,L_b(r)\,L_h(\Psi^-(r))$.

To show that $\lim_{|x|\to 0}u(x)/\Psi^+(x)\in(0,\infty)$ implies that $\lim_{\tau\to 0}\mathcal{F}_*(\tau,\varpi)<\infty$, we can proceed in exactly the same manner as above working with Ψ^+ instead of Ψ^- and using Proposition 3.4(a) with $b_0(r)=r^\theta\,L_b(r)\,L_h(\Psi^+(r))$. □

Lemma 6.5. *Assume that $h(t)/t$ is increasing on $(0,\infty)$ and $\vartheta\in C^1(\partial B_1(0))$ is a non-negative function.*

(a) If $\lim_{\tau\to 0}\mathcal{F}_*(\tau,\varpi)<\infty$ and γ is any positive number, then

$$\begin{cases} -\Delta u-\dfrac{(N-2)^2}{4}\dfrac{u}{|x|^2}+b(x)h(u)=0 & \text{in } B^*:=B_1(0)\setminus\{0\},\\ \lim\limits_{|x|\to 0}u(x)/\Psi^+(x)=\gamma, \quad u=\vartheta & \text{on } \partial B_1(0),\\ u>0 \ \text{ in } B^*, \end{cases} \tag{6.35}$$

has a unique solution u_γ, which is in $C^{1,\alpha}_{\rm loc}(B^*)$ for some $\alpha\in(0,1)$. The same assertion holds for $\gamma=\infty$ if we are in either of the three cases of Theorem 2.7(C).

(b) If ϑ is a non-trivial function, then (6.35) with $\gamma=0$ admits $C^{1,\alpha}_{\rm loc}(B^*)$–solutions for some $\alpha\in(0,1)$.

Proof. (a) In the proof of Lemma 5.6, we replace λ and $\Phi^\pm_\lambda$, respectively by $(N-2)^2/4$ and $\Psi^\pm$, respectively. The existence and uniqueness of the solution u_* for the new problem (5.45) follows from Proposition 3.4(c) and Lemma A.9. The rest of the argument is the same and thus is left to the reader.

(b) We prove the existence of $C^{1,\alpha}_{\rm loc}(\Omega^*)$–solutions of (6.35) with $\gamma=0$ as in Lemma 5.7. We need only replace λ and $\Phi^\pm_\lambda$ by $(N-2)^2/4$ and $\Psi^\pm$, respectively. □

6.2. Proof of Theorems 2.6

Theorem 2.6(a) is proved by Proposition 6.1 and Lemma 6.6(i).

LEMMA 6.6. *Assume that* (2.31) *is satisfied when* $q = q^*$ *in* (1.5).

(i) *If* $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) = \infty$, *then every positive solution* u *of* (6.1) *satisfies*

$$\lim_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} = 0. \tag{6.36}$$

(ii) *If* $\lim_{\tau\to 0} \mathcal{F}^*(\tau, \varpi) = \infty$, *then every positive solution* u *of* (6.1) *satisfies*

$$\lim_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} = 0. \tag{6.37}$$

Moreover, all positive solutions of (6.1) *are asymptotic as* $|x| \to 0$.

REMARK 6.7. As observed in Chapter 2, when $q \neq q^*$, then

$$\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) = \infty \ (\lim_{\tau\to 0} \mathcal{F}^*(\tau, \varpi) = \infty) \quad \text{if and only if } q > q^*.$$

In the case $q > q^*$, we conclude that $\lim_{|x|\to 0} u(x)/\Psi^\pm(x) = 0$ from Corollary 4.3 since $\Psi^\pm \in RV_{(2-N)/2}(0+)$ and $(2-N)/2 < -(\theta+2)/(q-1)$. However, Corollary 4.3 is not useful for the critical case $q = q^*$, as shown, for example, by $h(t) = t^q$ and $L_b(r) \sim [\log(1/r)]^\alpha$ as $r \to 0$:

- If $-1 - q \le \alpha < 1 - q$, then $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) = \infty$ and (2.31) holds, but
$$\lim_{r\to 0} \mathcal{K}(r)/\Psi^+(r) = \lim_{r\to 0} [\log(1/r)]^{-1-\alpha/(q-1)} = \infty.$$
- If $-2 \le \alpha < 0$, then $\lim_{\tau\to 0} \mathcal{F}^*(\tau, \varpi) = \infty$ and (2.31) holds, but
$$\lim_{r\to 0} \mathcal{K}(r)/\Psi^-(r) = \lim_{r\to 0} [\log(1/r)]^{-\alpha/(q-1)} = \infty.$$

PROOF. Let u be an arbitrary positive solution of (6.1). To prove (6.36) or (6.37), we could consider only $q = q^*$, but our proof works for $q \ge q^*$.

Proof of (i). We assume that $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) = \infty$. By Corollary 4.5 (see also Remark 4.6), the proof of (6.36) reduces to showing that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} = 0.$$

We proceed similarly to Lemma 5.8, where we replace λ, p and Φ_λ^+ by $(N-2)^2/4$, $(N-2)/2$ and Ψ^+, respectively. Suppose by contradiction that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} \neq 0. \tag{6.38}$$

We choose m such that

$$q - 2(\theta+2)/(N-2) < m < q. \tag{6.39}$$

Note that $m > 1$ (since $q \ge q^*$). As before, we define $\chi(t) = t^{q-m} L_h(t)$, which satisfies (5.54). We regain (5.52) and (5.55) with Ψ^+ instead of Φ_λ^+. We now denote

$$b_0(r) := c_1 r^\theta L_b(r)\, \chi(\Psi^+(r)),$$

which is regularly varying at 0 with index $\theta - (N-2)(q-m)/2$ greater than -2 from (6.39). As in Lemma 5.8, we obtain a positive solution U_∞ of

$$-U''(r) - \frac{N-1}{r} U'(r) - \frac{(N-2)^2}{4} \frac{U(r)}{r^2} + b_0(r)[U(r)]^m = 0 \quad \text{for } 0 < r < 1, \tag{6.40}$$

such that $u \le U_\infty$ in $B_1(0) \setminus \{0\}$. Using $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) = \infty$, we find that

$$\lim_{\tau\to 0} \int_\tau^\varpi r^{\frac{N-m(N-2)}{2}} [\log(1/r)]^m \, b_0(r)\, dr = \infty. \tag{6.41}$$

Replacing q by m in both (3.17) and $\lim_{\tau\to 0} \mathcal{F}_1(\tau,\varpi) = \infty$ (with $\mathcal{F}_1(\tau,\varpi)$ given by (3.12)), we arrive at (2.31) and (6.41). Thus, by Proposition 3.4(e2) applied to U_∞, we find that $\lim_{r\to 0} U_\infty(r)/\Psi^+(r) = 0$. Hence, $\lim_{|x|\to 0} u(x)/\Psi^+(x) = 0$, which is a contradiction with our assumption (6.38). This proves the assertion of (i).

Proof of (ii). We now assume that $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) = \infty$. We first prove (6.37). From (i), we have (6.36). By the comparison principle (Lemma A.9), it follows that $\limsup_{|x|\to 0} u(x)/\Psi^-(x) < \infty$. We assume by contradiction that

$$\liminf_{|x|\to 0} \frac{u(x)}{\Psi^-(x)} > 0. \tag{6.42}$$

Using the previous argument with Ψ^- instead of Ψ^+, we conclude that $u \le U_\infty$ in $B_1(0) \setminus \{0\}$, where U_∞ is a positive solution of (6.40) and b_0 is given by

$$b_0(r) := c_1 r^\theta L_b(r)\, \chi(\Psi^-(r)) \quad \text{for } r \in (0,1].$$

From $\lim_{\tau\to 0} \mathcal{F}^*(\tau,\varpi) = \infty$, we check easily that

$$\lim_{\tau\to 0} \int_\tau^\varpi r^{\frac{N-m(N-2)}{2}} \, b_0(r) \log(1/r)\, dr = \infty, \tag{6.43}$$

which corresponds to $\lim_{\tau\to 0} \mathcal{F}_2(\tau,\varpi) = \infty$ in which q is replaced by m (see (3.12) for the definition of $\mathcal{F}_2(\tau,\varpi)$). We want to prove that

$$\lim_{r\to 0} \frac{U_\infty(r)}{\Psi^+(r)} = 0. \tag{6.44}$$

Note that we cannot apply Proposition 3.4(e2) to the solution U_∞ of (6.40) without the monotonicity requirement corresponding to (3.17) in which q is replaced by m. However, we do not need such an hypothesis. Since (6.36) holds, it is enough to prove that $U_\infty \le Ku$ in $B^* := B_1(0) \setminus \{0\}$ for some large constant $K > 1$. Let $n \ge 2$ and $\mathcal{A}_n$ be defined as in (5.5). We recall (from the proof of Lemma 5.8) that $U_n \to U_\infty$ in $C^1_{\rm loc}(B^*)$ as $n \to \infty$, where we denote by U_n the unique positive solution of the boundary value problem

$$\begin{cases} -\Delta U - \dfrac{(N-2)^2}{4} \dfrac{U}{|x|^2} + b_0(|x|) U^m = 0 & \text{in } \mathcal{A}_n, \\ U(x) = \max\limits_{|y|=|x|} u(y) \quad \text{for } |x| = 1/n \quad \text{and} \quad |x| = 1. \end{cases} \tag{6.45}$$

By the Harnack inequality (Lemma 4.4), there exists a constant $K > 1$ such that

$$\max_{|x|=r} u(x) \le K \min_{|x|=r} u(x) \quad \text{for every } r \in (0,1].$$

Using (5.54), $\lim_{|x|\to 0} u(x) = \infty$ and $\limsup_{|x|\to 0} u(x)/\Psi^-(x) < \infty$, we find that

$$\frac{h(u(x))}{[u(x)]^m} \le C \chi(\Psi^-(x)) \quad \text{for } 0 < |x| \le 1,$$

where $C > 0$ is a constant. This, jointly with $b(x) \sim |x|^\theta L_b(|x|)$ as $|x| \to 0$, ensures that we can increase K such that

$$b_0(|x|) K^{m-1} u^m \ge b(x) h(u) \quad \text{for } 0 < |x| \le 1.$$

This yields that Ku is a super-solution of (6.45). Hence, using Lemma A.9 we find that $U_n \le Ku$ in $\mathcal{A}_n$ for every $n \ge 2$. Consequently, $U_\infty \le Ku$ in B^*, proving (6.44). We now apply Proposition 3.4(b) to the solution U_∞ of (6.40) to conclude that $\lim_{r\to 0} U_\infty(r)/\Psi^-(r) = 0$. This leads to a contradiction with our assumption (6.42). This proves (6.37) for every positive solution u of (6.1).

Finally, we show that all positive solutions of (6.1) are asymptotic at zero. We follow an argument comparable to that in Step 2 of Lemma 5.10, where we replace λ by $(N-2)^2/4$. So, we need to prove that all positive solutions of

$$-v''(r) - \frac{N-1}{r}v'(r) - \frac{(N-2)^2}{4}\,\frac{v(r)}{r^2} + r^\theta L_b(r)\,\tilde{h}(v) = 0 \quad \text{for } 0 < r < 1 \tag{6.46}$$

are asymptotic as $r \to 0$. Now instead of (5.64), we apply the change of variable

$$y(s) = v(r)/\Psi^-(r) = r^{\frac{N-2}{2}} v(r) \quad \text{with } s = \log(1/r). \tag{6.47}$$

The philosophy of the proof remains the same, though the various equations in the proof of Lemma 5.10 change due to (6.47). Instead of (5.65), we have that $y(s)$ satisfies the differential equation

$$y''(s) = \varphi(s)\,\tilde{h}(e^{\frac{(N-2)s}{2}} y(s)) \ \text{ for } s > 0, \ \text{ where } \varphi(s) := e^{-\left(\theta + \frac{N+2}{2}\right)s} L_b(e^{-s}). \tag{6.48}$$

Let y_1 and y_2 be two positive solutions of (6.48). If these solutions coincide at some point $s_0 > 0$, then $y_1(s) = y_2(s)$ for any $s \in [s_0, \infty)$. This follows as before because $y''(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} y(s) = 0$ (from the first part that we proved in Lemma 6.6(ii)). Hence, without loss of generality, we need only show that $y_1(s) \sim y_2(s)$ as $s \to \infty$ in the case $y_1(s) < y_2(s)$ on some interval $[s_0, \infty)$ with $s_0 > 0$ large. Defining $z(t) = y_1(s)/y_2(s)$ for t as in (5.68), we find that

$$\frac{d^2 z}{dt^2} = \varphi(s)[y_2(s)]^3 \left[\tilde{h}(e^{\frac{(N-2)s}{2}} y_2(s) z(t)) - z(t)\,\tilde{h}(e^{\frac{(N-2)s}{2}} y_2(s))\right]. \tag{6.49}$$

From (6.49), we conclude that $\lim_{t\to\infty} z(t) = \beta$ for some $\beta \in (0,1]$. Since

$$v_i(r) = e^{\frac{(N-2)s}{2}} y_i(s)$$

are solutions of (6.46), by Remark 5.3 we have $\lim_{r\to 0} v_i(r) = \infty$ for $i = 1, 2$. In view of $\lim_{r\to 0} v_1(r)/v_2(r) = \beta$, from (6.48), we recover (5.71). Hence, by L'Hôpital's rule, we find that $\beta = 1$, completing the proof of Lemma 6.6. ☐

To end the proof of Theorem 2.6(b), we use Lemma 6.6(ii) and Lemma A.13, together with the next result.

Lemma 6.8. *Let $q = q^*$ in* (1.5). *Assume that $\lim_{\tau\to 0} \mathcal{F}^*(\tau, \varpi) = \infty$ and* (2.31) *is verified. Let u be an arbitrary positive solution of* (6.1).

(i) *If* (1.12) *holds, then $\alpha_1 + \alpha_2 \ge -2$ and u satisfies* (2.32).
(ii) *If* (1.12)*(a) and* (1.16)*(a) are verified, then u satisfies* (2.33).
(iii) *If* (1.12)*(b) holds, jointly with* (1.16)*(c) such that S is regularly varying at ∞ with index η, then we have* (2.34).

Proof. From the proof of Lemma 6.6(ii), we know that every positive solution u of (6.1) is asymptotically equivalent to any positive solution v of (6.46). We apply (6.47) and arrive at (6.48). Since $\tilde{h}(t) = t^q L_h(t)$ and $q = q^*$, we have

$$y''(s) = L_b(e^{-s})\,L_h(e^{(N-2)s/2} y(s))\,[y(s)]^q \quad \text{for } s > 0. \tag{6.50}$$

In fact, all positive solutions of (6.50) are asymptotically equivalent at ∞ to any positive C^2–function $\mathcal{Y}(s)$ satisfying

$$\mathcal{Y}''(s) \sim L_b(e^{-s})\, L_h(e^{(N-2)s/2}\mathcal{Y}(s))\, [\mathcal{Y}(s)]^q \quad \text{as } s \to \infty. \tag{6.51}$$

Hence, for every positive solution u of (6.1), we have

$$\frac{u(x)}{\Psi^-(x)} \sim \mathcal{Y}(\log(1/|x|)) \ \text{ as } |x| \to 0 \ \text{ for any } \mathcal{Y} \ \text{ as in (6.51)}. \tag{6.52}$$

We use this fact to obtain each of the asymptotic behaviour specified by Lemma 6.8.

Proof of (i). The assumption (1.12) implies that

$$\phi(t) := L_b(e^{-t})\, L_h(e^{(N-2)t/2}) \tag{6.53}$$

is regularly varying at ∞ with index $\alpha_1 + \alpha_2$. By Remark A.3, there exists a C^1–function $\widehat{\phi}$ such that $\widehat{\phi}(t) \sim \phi(t)$ and $t\widehat{\phi}'(t) \sim (\alpha_1+\alpha_2)\widehat{\phi}(t)$ as $t \to \infty$. By the definition of $\mathcal{F}^*(\tau, \varpi)$ in (2.25) with $q = q^*$, we have

$$\mathcal{F}^*(e^{-t}, \varpi) = \int_{\log(1/\varpi)}^t \xi\, \phi(\xi)\, d\xi \ \text{ for large } t > 0. \tag{6.54}$$

Since $t \longmapsto t\,\phi(t)$ belongs to $RV_{\alpha_1+\alpha_2+1}(\infty)$ and $\lim_{t\to\infty} \mathcal{F}^*(e^{-t}, \varpi) = \infty$, we must have $\alpha_1 + \alpha_2 \geq -2$. We now define $\mathcal{Y}(t)$ for large $t > 0$ as follows

$$\mathcal{Y}(t) := \left[\frac{(q-1)^2}{q + \alpha_1 + \alpha_2 + 1} \int_{\log(1/\varpi)}^t \xi\, \widehat{\phi}(\xi)\, d\xi \right]^{-1/(q-1)}. \tag{6.55}$$

In view of (6.52) and (6.54), we conclude (2.32) by showing that $\mathcal{Y}$ satisfies (6.51). Clearly, $\log \mathcal{Y}(t)$ is slowly varying at ∞ so that $\lim_{t\to\infty}(1/t) \log \mathcal{Y}(t) = 0$. Hence, from (1.12)(a), we deduce that

$$L_h(e^{(N-2)t/2}\mathcal{Y}(t)) \sim L_h(e^{(N-2)t/2}) \ \text{ as } t \to \infty. \tag{6.56}$$

By Karamata's Theorem in Appendix A, we have

$$t^2 \widehat{\phi}(t) \sim (\alpha_1 + \alpha_2 + 2) \int_{\log(1/\varpi)}^t \xi\, \widehat{\phi}(\xi)\, d\xi \ \text{ as } t \to \infty.$$

Using the properties of $\widehat{\phi}$ and a simple calculation, we find that

$$\mathcal{Y}''(t) \sim \phi(t) [\mathcal{Y}(t)]^q \ \text{ as } t \to \infty. \tag{6.57}$$

This, jointly with (6.56) and (6.53), proves (6.51). Hence, u satisfies (2.32).

Proof of (ii). By the assumption (1.12)(a), $t \longmapsto L_h(e^{(N-2)t/2})$ is regularly varying at ∞ with real index α_1. By Remark A.3, there exists a C^1-function ζ belonging to $RV_{\alpha_1}(\infty)$ such that

$$t\zeta'(t)/\zeta(t) \to \alpha_1 \ \text{ as } t \to \infty \quad \text{and} \quad \zeta(t) \sim L_h(e^{(N-2)t/2}) \ \text{ as } t \to \infty. \tag{6.58}$$

We fix $c > 0$ sufficiently large and for any $t > c$, we define

$$\mathcal{Y}(t) := \left[(q-1)^2 \int_c^t S(\xi)\, \Lambda(\xi)\, \zeta(\xi)\, d\xi \right]^{-1/(q-1)}. \tag{6.59}$$

Here, Λ is the function which appears in the hypothesis (1.16)(a). Let ϕ be given by (6.53). From the properties of Λ and ζ, we find

$$\int_c^t S(\xi)\, \Lambda(\xi)\, \zeta(\xi)\, d\xi \sim [S(t)]^2 \Lambda(t)\, \zeta(t) \sim [S(t)]^2 \phi(t) \ \text{ as } t \to \infty. \tag{6.60}$$

Hence, to conclude the assertion of (ii), it is enough to show that $\mathcal{Y}$ in (6.59) satisfies (6.51). Clearly, we have

$$\lim_{t\to\infty}(1/t)\log S(t)=\lim_{t\to\infty}(1/t)\log\Lambda(t)=\lim_{t\to\infty}(1/t)\log\zeta(t)=0.$$

Thus, using (6.59) and (6.60), we find that $\lim_{t\to\infty}(1/t)\log\mathcal{Y}(t)=0$ so that we recover (6.56). A simple calculation shows that

$$\mathcal{Y}''(t)\sim\Lambda(t)\,\zeta(t)[\mathcal{Y}(t)]^q\quad\text{as } t\to\infty.$$

Using (6.58) and (1.16)(a), we regain (6.57). This concludes the proof of (6.51).

Proof of (iii). We shall use an idea similar to Lemma 5.11(iii). Since (1.16)(c) holds and $S\in RV_\eta(\infty)$, there exists a C^1 function $\widehat{S}$ such that $\widehat{S}(t)\sim S(t)$ as $t\to\infty$ and $\lim_{t\to\infty}t\widehat{S}'(t)/\widehat{S}(t)=\eta$. We can assume that $\lim_{t\to\infty}tS'(t)/S(t)=\eta$, since we could use $\widehat{S}$ instead of S in our argument. We define

$$V(r):=f^{-1}\left(\frac{[(N-2)/2]^{2-2\eta}(q-1)^{-2}}{\mathcal{J}(r)\,[S(\log(1/r))]^2}\right)\quad\text{for } r>0\text{ small.}\tag{6.61}$$

To prove (2.34), it suffices to show that $u(x)\sim V(|x|)$ as $|x|\to 0$. Since $f(t)=t^{q-1}L_h(t)$, we have $V(r):=\Psi^-(r)Z(r)$ with $Z(r)$ defined by

$$Z(r):=\left\{(q-1)^2[(N-2)/2]^{2\eta-2}\,L_b(r)\,L_h(V(r))\,[S(\log(1/r))]^2\right\}^{-\frac{1}{q-1}}.\tag{6.62}$$

In light of (6.52), we conclude the proof of (2.34) by constructing a C^2 function $\mathcal{Y}$ which satisfies (6.51) and

$$\mathcal{Y}(t)\sim Z(e^{-t})\quad\text{as } t\to\infty.\tag{6.63}$$

Since $q=q^*$, it follows that $r\longmapsto V(r)$ is regularly varying at 0 with index $-(N-2)/2$. By Remark A.3, there exists a C^1–function $\widehat{V}$ such that

$$\widehat{V}(r)\sim V(r)\quad\text{and}\quad r\widehat{V}'(r)/\widehat{V}(r)\sim-(N-2)/2\quad\text{as } r\to 0.\tag{6.64}$$

We fix $c>0$ large enough. Using $\widehat{V}$, we now introduce $\mathcal{Y}(t)$ for any $t>c$ as follows

$$\mathcal{Y}(t):=\left\{(q-1)^2\left(\frac{N-2}{2}\right)^{\eta-1}\int_c^t L_b(e^{-\xi})\Lambda(\log\widehat{V}(e^{-\xi}))S(\xi)\,d\xi\right\}^{-\frac{1}{q-1}}.\tag{6.65}$$

Using that L_h is slowly varying at ∞ and (1.16)(c) holds, we obtain that

$$L_h(V(e^{-t}))\sim L_h(\widehat{V}(e^{-t}))\sim\Lambda(\log\widehat{V}(e^{-t}))\quad\text{as } t\to\infty.\tag{6.66}$$

From (6.64), we have $\log\widehat{V}(e^{-t})\sim[(N-2)/2]\,t$ as $t\to\infty$. Recall that $t\longmapsto S(t)$ and $t\longmapsto L_b(e^{-t})$ are regularly varying at ∞ with index η and α_2, respectively. Since Λ is Γ-varying at ∞ with auxiliary function S, we have

$$\lim_{t\to\infty}\frac{\int_c^t L_b(e^{-\xi})\Lambda(\log\widehat{V}(e^{-\xi}))S(\xi)\,d\xi}{L_b(e^{-t})\Lambda(\log\widehat{V}(e^{-t}))[S(t)]^2}=\left(\frac{N-2}{2}\right)^{\eta-1}.\tag{6.67}$$

By (6.66) and (6.67), we conclude (6.63) (see (6.62) and (6.65)). Thus, we have $V(e^{-t})\sim e^{(N-2)t/2}\mathcal{Y}(t)$ as $t\to\infty$, since $V(r)=\Psi^-(r)Z(r)$. Hence, (6.66) yields

$$\Lambda(\log\widehat{V}(e^{-t}))\sim L_h(e^{(N-2)t/2}\mathcal{Y}(t))\quad\text{as}\quad t\to\infty.\tag{6.68}$$

From (6.67) and the definition of $\mathcal{Y}$ in (6.65), we find that

$$\mathcal{Y}''(t)\sim L_b(e^{-t})\,\Lambda(\log\widehat{V}(e^{-t}))\,[\mathcal{Y}(t)]^q\quad\text{as } t\to\infty.\tag{6.69}$$

Using (6.68) in (6.69), we conclude (6.51). This ends the proof of Lemma 6.8. $\square$

6.3. Proof of Theorem 2.7

We assume that (1.5) holds, $\lambda = (N-2)^2/4$ and $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty$. Let u be an arbitrary positive solution of (6.1). By Corollary 4.5, the behaviour of u near zero falls into one of the cases:

$$\text{(A)} \lim_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} = 0; \text{ (B)} \limsup_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} \in (0,\infty); \text{ (C)} \lim_{|x|\to 0} \frac{u(x)}{\Psi^+(x)} = \infty. \tag{6.70}$$

Case (A) of (6.70): By (2.27) and Proposition 6.1, we obtain that $u(x)/\Psi^-(x)$ converges to some positive number as $|x| \to 0$.

Case (B) of (6.70): We show that $u(x)/\Psi^+(x)$ has a limit as $|x| \to 0$. This claim follows immediately from Lemma 6.3 provided that h also satisfies (6.4) and there exists $r_0 > 0$ such that $b(x) = b_0(|x|)$ for every $0 < |x| \le r_0$. Without these restriction, we fix $\epsilon \in (0,1)$ and let v_* be prescribed by Lemma 4.12 with $\lambda = (N-2)^2/4$. We now can apply Proposition 6.1 to v_*. Therefore, we infer that $\lim_{|x|\to 0} v_*(x)/\Psi^+(x) \in (0,\infty)$ and (6.7) holds with Ψ^+ instead of Ψ^-. This proves that $\lim_{|x|\to 0} u(x)/\Psi^+(x) \in (0,\infty)$.

Case (C) of (6.70): The hypothesis $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty$ implies that $q \le q^*$. We finish the proof by separating $q < q^*$ in (C1) from $q = q^*$ in (C2) and (C3).

(C1) If $q < q^*$, then $\lim_{\tau\to 0} \mathcal{F}_*(\tau,\varpi) < \infty$ is automatically satisfied. In this case, we can conclude that u satisfies (1.10) by following the same ideas as in the proof of Lemma 5.13, where λ and Φ_λ^+ are replaced by $(N-2)^2/4$ and Ψ^+, respectively. Therefore, we leave this task to the reader.

(C2) Let $q = q^*$. Assuming that (1.12) holds, as well as (2.36), we prove below that u satisfies (2.37).

(C3) Let $q = q^*$. We shall prove that if (1.12)(a) holds, jointly with (1.16)(b), then u satisfies (2.33).

In both (C2) and (C3), we assume that $q = q^*$ and (1.12)(a) holds. By Lemma 4.12 and an argument similar to Lemma 5.14, it is enough to show the assertions of (C2) and (C3) only for the positive solutions u of

$$\begin{cases} u''(r) + \dfrac{N-1}{r}\, u'(r) + \dfrac{(N-2)^2}{4}\, \dfrac{u(r)}{r^2} = r^\theta L_b(r)\, \tilde{h}(u(r)) \quad \text{in } (0,1), \\ \lim\limits_{r\to 0} \dfrac{u(r)}{\Psi^+(r)} = \infty. \end{cases} \tag{6.71}$$

By applying Lemma 4.1 to (6.71), we find that $\log u(r) \sim \log \Psi^+(r)$ as $r \to 0$. Since $t \longmapsto L_h(e^t)$ in (1.12)(a) is regularly varying at ∞ with index α_1, we infer that

$$L_h(u(r)) \sim L_h(\Psi^+(r)) \sim [(N-2)/2]^{\alpha_1} L_h(1/r) \quad \text{as } r \to 0. \tag{6.72}$$

Thus, for (C2) and (C3), it follows that

$$\tilde{h}(u(r)) \sim [(N-2)/2]^{\alpha_1} L_h(1/r)\, [u(r)]^q \quad \text{as } r \to 0.$$

Taking $b_0(r) := [(N-2)/2]^{\alpha_1} r^\theta L_b(r)\, L_h(1/r)$ in Remark 3.5, then using the change of variable $y(s) = r^{(N-2)/2} u(r)$ with $s = \log(1/r)$, we conclude that

$$u(r)/\Psi^-(r) \sim \mathcal{Y}(\log(1/r)) \quad \text{as } r \to 0 \tag{6.73}$$

for any positive C^2–function $\mathcal{Y}$ satisfying $\mathcal{Y}(s)/s \to \infty$ as $s \to \infty$ and

$$\mathcal{Y}''(s) \sim [(N-2)/2]^{\alpha_1} L_h(e^s)\, L_b(e^{-s})[\mathcal{Y}(s)]^q \quad \text{as } s \to \infty. \tag{6.74}$$

We point out that the monotonicity assumption needed to apply Proposition 3.4(d) or (e) is satisfied because (2.36) holds for both (C2) and (C3).

Case (C2). The definition of $\mathcal{F}_*(\tau, \varpi)$ in (2.25) with $q = q^*$ yields that

$$\mathcal{F}_*(\tau, \varpi) = \int_{\log(1/\varpi)}^{\log(1/\tau)} \xi^q L_h(\xi e^{(N-2)\xi/2})\, L_b(e^{-\xi})\, d\xi. \tag{6.75}$$

Since (1.12) holds, the integrand in (6.75) is a regularly varying function at ∞ (in ξ) with index $q + \alpha_1 + \alpha_2$. Hence, $\lim_{\tau\to 0} \mathcal{F}_*(\tau, \varpi) < \infty$ yields that $\alpha_1 + \alpha_2 + q \leq -1$. From (1.12) and Remark A.3, there exists a C^1–function ϕ such that

$$\phi(t) \sim L_h(e^t)\, L_b(e^{-t}) \quad \text{and} \quad t\phi'(t) \sim (\alpha_1 + \alpha_2)\, \phi(t) \quad \text{as } t \to \infty. \tag{6.76}$$

We define $\mathcal{H}(t)$ for large $t > 0$ as follows

$$\mathcal{H}(t) := \frac{(q-1)^2}{-2-\alpha_1-\alpha_2} \left(\frac{N-2}{2}\right)^{\alpha_1} \int_t^\infty \xi^q \phi(\xi)\, d\xi. \tag{6.77}$$

Using (6.76), we see that $\mathcal{H}$ satisfies

$$\frac{t\mathcal{H}'(t)}{\mathcal{H}(t)} \sim \alpha_1 + \alpha_2 + q + 1, \quad \frac{t\mathcal{H}''(t)}{\mathcal{H}'(t)} \sim \alpha_1 + \alpha_2 + q \quad \text{as } t \to \infty.$$

Thus after a simple calculation, we find that $\mathcal{Y}(t) := t[\mathcal{H}(t)]^{-1/(q-1)}$ satisfies (6.74). Since $\mathcal{F}_*(e^{-t}) = \lim_{\tau \to 0} \mathcal{F}_*(\tau, e^{-t})$, using (1.12) and (6.75)–(6.77), we arrive at

$$\mathcal{Y}(t) \sim t \left(\frac{(q-1)^2}{-2-\alpha_1-\alpha_2} \mathcal{F}_*(e^{-t}) \right)^{-1/(q-1)} \quad \text{as } t \to \infty. \tag{6.78}$$

Hence, $\mathcal{Y}(t)/t \to \infty$ as $t \to \infty$ since $\mathcal{F}_*(r) \to 0$ as $r \to 0$. We now conclude (2.37) based on (6.73) and (6.78).

Case (C3). Since (1.12)(a) holds, there exists a C^1 function ζ as in the proof of Lemma 6.8(ii) (see (6.58)). For $t > 0$ large, we define

$$\mathcal{Y}(t) = \left[(q-1)^2 \int_t^\infty \frac{S(\xi)}{\Lambda(\xi)}\, \zeta(\xi)\, d\xi \right]^{-1/(q-1)}. \tag{6.79}$$

Using (6.58) and the properties of Λ appearing in (1.16)(b), we easily find that

$$\int_t^\infty \frac{S(\xi)}{\Lambda(\xi)}\, \zeta(\xi)\, d\xi \sim \frac{[S(t)]^2}{\Lambda(t)} \zeta(t) \sim \left(\frac{N-2}{2}\right)^{\alpha_1} [S(t)]^2 L_b(e^{-t})\, L_h(e^t) \tag{6.80}$$

as $t \to \infty$. From (6.79) and (6.80), we obtain that

$$\mathcal{Y}(t) \sim \left\{ (q-1)^2 \left(\frac{N-2}{2}\right)^{\alpha_1} [S(t)]^2 L_b(e^{-t})\, L_h(e^t) \right\}^{-\frac{1}{q-1}} \quad \text{as } t \to \infty. \tag{6.81}$$

From (1.12)(a), we have $\log L_h(e^t) \sim \alpha_1 \log t$ as $t \to \infty$ so that $\log L_h(e^t)$ and $\log S(t)$ are dominated by $\log \Lambda(t)$ as $t \to \infty$. Hence, $\log \mathcal{Y}(t) - \log t \to \infty$ as $t \to \infty$, which proves that $\mathcal{Y}(t)/t \to \infty$ as $t \to \infty$. By a simple calculation, we find

$$\mathcal{Y}''(t) \sim \frac{\zeta(t)}{\Lambda(t)} [\mathcal{Y}(t)]^q \quad \text{as } t \to \infty.$$

Using now (6.80), (6.58) and (1.16)(b), we obtain (6.74). From (6.73) and (6.81), we conclude the proof of (2.33). This completes the proof of Theorem 2.7. □

CHAPTER 7

Illustration of our results

7.1. On a prototype model

We give below a complete classification of all the positive solutions of (1.3) on the example of (1.15). For $\lambda < (N-2)^2/4$, we define p and ℓ as in (1.9), whereas q^* and q^{**} are given by (1.11).

COROLLARY 7.1. *Let $0 < \lambda < (N-2)^2/4$. Assume that*

$$\begin{cases} h(t) \sim t^q (\log t)^{\alpha_1} & \text{as } t \to \infty \quad \text{for some } q > 1, \quad \alpha_1 \in \mathbb{R}, \\ b(x) \sim |x|^\theta \left(\log \dfrac{1}{|x|}\right)^{\alpha_2} & \text{as } |x| \to 0 \quad \text{for some } \theta > -2, \quad \alpha_2 \in \mathbb{R}. \end{cases} \tag{7.1}$$

Let u be any positive solution of (1.3)*.*

(i) *If $1 < q < q^*$, then exactly one of the following occurs as $|x| \to 0$:*
  (A) *$|x|^p u(x)$ converges to a positive number;*
  (B) *$|x|^{N-2-p} u(x)$ converges to a positive number;*
  (C) *$|x|^{N-2-p} u(x) \to \infty$ and, moreover, u satisfies* (1.13)*, that is*

$$u(x) \sim \left[\frac{1}{\ell}\left(\frac{\theta+2}{q-1}\right)^{\alpha_1} |x|^{\theta+2} \left(\log \frac{1}{|x|}\right)^{\alpha_1+\alpha_2}\right]^{-\frac{1}{q-1}} \quad \text{as } |x| \to 0. \tag{7.2}$$

(ii) *If $q = q^*$ and $\alpha_1 + \alpha_2 < -1$, then the conclusion of (i) above holds except for* (7.2)*, which is replaced by*

$$u(x) \sim \left[\frac{(q-1)(N-2-p)^{\alpha_1}}{-(\alpha_1+\alpha_2+1)(N-2-2p)} \left(\log \frac{1}{|x|}\right)^{\alpha_1+\alpha_2+1}\right]^{\frac{-1}{q-1}} |x|^{2-N+p} \quad \text{as } |x| \to 0.$$

(iii) *We have $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$ in any of the following three cases:*
  (a) *$q = q^*$ and $\alpha_1 + \alpha_2 \ge -1$;*
  (b) *$q^* < q < q^{**}$;*
  (c) *$q = q^{**}$ and $\alpha_1 + \alpha_2 < -1$.*

(iv) *If $q = q^{**}$ and $\alpha_1 + \alpha_2 = -1$, then*

$$u(x) \sim \left[\frac{(q-1)p^{\alpha_1}}{N-2-2p} \log\left(\log \frac{1}{|x|}\right)\right]^{-\frac{1}{q-1}} |x|^{-p} \quad \text{as } |x| \to 0.$$

(v) *If $q = q^{**}$ and $\alpha_1 + \alpha_2 > -1$, then*

$$u(x) \sim \left[\frac{(q-1)p^{\alpha_1}}{(\alpha_1+\alpha_2+1)(N-2-2p)} \left(\log \frac{1}{|x|}\right)^{\alpha_1+\alpha_2+1}\right]^{\frac{-1}{q-1}} |x|^{-p} \quad \text{as } |x| \to 0.$$

(vi) *If $q > q^{**}$, then* (7.2) *applies for every $\alpha_1, \alpha_2 \in \mathbb{R}$.*

Proof. The assertions of (i) and (ii) follow from Theorem 2.4. To obtain (7.2), we also use Remark 1.5. We invoke Theorem 2.2 to conclude the claim of (iii). By using Theorem 2.3(b) (respectively, Theorem 2.3(a)), we establish the asymptotics stated in (iv) and (v) (respectively, (vi)). □

Remark 7.2. If in Corollary 7.1 we assume that $-\infty < \lambda \le 0$, then the statements (i) and (ii) remain valid, while the conclusion of (iii) applies when

$$\text{(a) } q = q^* \ \text{ and } \ \alpha_1 + \alpha_2 \ge -1; \qquad \text{(b) for any } q > q^*.$$

This change is justified by (2.10).

We next completely classify the positive solutions of (1.3) on the example in (1.15) when $\lambda = (N-2)^2/4$. In this case, q^* and ℓ are given by (2.26). Our conclusions here are very different from those pertaining to $\lambda < (N-2)^2/4$.

Corollary 7.3. *Let $\lambda = (N-2)^2/4$. Assume that (7.1) holds. Let u be any positive solution of (1.3).*

(i) *If $1 < q < q^*$, then exactly one of the following occurs as $|x| \to 0$:*
  (A) $\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} u(x) \in (0,\infty)$;
  (B) $\lim_{|x|\to 0} u(x)|x|^{\frac{N-2}{2}} / \log(1/|x|) \in (0,\infty)$;
  (C) (7.2) *holds for all $\alpha_1, \alpha_2 \in \mathbb{R}$.*

(ii) *If $q = q^*$ and $\alpha_1 + \alpha_2 < -q - 1$, then the conclusion of* (i) *remains valid with the exception* (7.2) *in* (i)(C) *above, which must be replaced here by*

$$u(x) \sim |x|^{-\frac{N-2}{2}} \left[\frac{(q-1)^2 \left(\frac{N-2}{2}\right)^{\alpha_1} \left(\log \frac{1}{|x|}\right)^{\alpha_1+\alpha_2+2}}{(\alpha_1+\alpha_2+2)(\alpha_1+\alpha_2+q+1)} \right]^{-\frac{1}{q-1}} \quad \textit{as } |x| \to 0. \tag{7.3}$$

(iii) *If $q = q^*$ and $-q-1 \le \alpha_1 + \alpha_2 < -2$, then* $\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} u(x) \in (0,\infty)$.

(iv) *If $q = q^*$ and $\alpha_1 + \alpha_2 = -2$, then*

$$u(x) \sim |x|^{-\frac{N-2}{2}} \left[(q-1) \left(\frac{N-2}{2} \right)^{\alpha_1} \log \left(\log \frac{1}{|x|} \right) \right]^{-\frac{1}{q-1}} \quad \textit{as } |x| \to 0.$$

(v) *If $q = q^*$ and $\alpha_1 + \alpha_2 > -2$, then u satisfies* (7.3).

(vi) *If $q > q^*$, then for every $\alpha_1, \alpha_2 \in \mathbb{R}$, we have* (7.2).

Proof. For (i) and (ii), we use Theorem 2.7. The assertions of (iii) and (vi) follow from Theorem 2.6(a) and Theorem 2.6(b1), respectively. We conclude (iv) and (v) by applying Theorem 2.6(b2). □

Corollary 7.1, jointly with Remark 7.2, and Corollary 7.3 extend the classification results for the power model

$$\begin{cases} h(t) \sim t^q & \text{as } t \to \infty \quad \text{for } q > 1, \\ b(x) \sim |x|^\theta & \text{as } |x| \to 0 \quad \text{for } \theta > -2. \end{cases} \tag{7.4}$$

Corollary 7.4. *Let $-\infty < \lambda < (N-2)^2/4$ and (7.4) hold. Let u be any positive solution of* (1.3).

(1) *If $1 < q < q^*$, then as $|x| \to 0$, exactly one of the following holds:*
  (A) *$|x|^p u(x)$ converges to a positive number;*
  (B) *$|x|^{N-2-p} u(x)$ converges to a positive number;*

(C) $|x|^{\frac{\theta+2}{q-1}}u(x)$ *converges to* $\ell^{\frac{1}{q-1}}$, *where* ℓ *is defined by* (1.9).

(2) *If* $q \geq q^*$, *then* $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$ *assuming any of the cases*

$$\text{(a)}\ -\infty < \lambda \leq 0; \qquad \text{(b)}\ \lambda > 0 \ \text{ and } \ q < q^{**}.$$

(3) *If* $\lambda > 0$ *and* $q = q^{**}$, *then we have*

$$\lim_{|x|\to 0} [\log(1/|x|)]^{1/(q-1)}\, |x|^p u(x) = \left(\frac{N-2-2p}{q-1}\right)^{1/(q-1)}.$$

(4) *If* $\lambda > 0$ *and* $q > q^{**}$, *then* $|x|^{\frac{\theta+2}{q-1}}u(x)$ *converges to* $\ell^{\frac{1}{q-1}}$ *as* $|x| \to 0$.

Corollary 7.5. *Let* $\lambda = (N-2)^2/4$ *and* (7.4) *hold. Let* u *be any positive solution of* (1.3).

(I) *If* $1 < q < q^*$, *then exactly one of the following holds as* $|x| \to 0$:
  (a) $|x|^{\frac{N-2}{2}}u(x)$ *converges to a positive number;*
  (b) $|x|^{\frac{N-2}{2}}u(x)/\log(1/|x|)$ *converges to a positive number;*
  (c) $|x|^{\frac{\theta+2}{q-1}}u(x)$ *converges to* $\ell^{\frac{1}{q-1}}$.

(II) *If* $q = q^*$, *then* $|x|^{\frac{N-2}{2}}[\log(1/|x|)]^{\frac{2}{q-1}}u(x) \to \left[\frac{2(q+1)}{(q-1)^2}\right]^{\frac{1}{q-1}}$ *as* $|x| \to 0$;

(III) *If* $q > q^*$, *then* $|x|^{\frac{\theta+2}{q-1}}u(x)$ *converges to* $\ell^{\frac{1}{q-1}}$ *as* $|x| \to 0$.

7.2. In other settings

In this section, we apply our results in various situations when hypothesis (1.16) comes into play.

Corollary 7.6. *Let* $-\infty < \lambda \leq (N-2)^2/4$ *and* (1.5) *hold. Assume that* (1.12)*(a) and* (1.16)*(a) are satisfied. Let* u *be any positive solution of* (1.3).

(1) *If* $\lambda \neq (N-2)^2/4$ *and* $1 < q < q^*$, *then one of the following occurs:*
  (A) $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$;
  (B) $\lim_{|x|\to 0} |x|^{N-2-p} u(x) \in (0,\infty)$;
  (C) (1.10) *holds.*

(2) *If* $\lambda \leq 0$ *and* $q \geq q^*$, *then* $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$.

(3) *If* $0 < \lambda < (N-2)^2/4$, *then we have*
  (i) $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$ *if* $q^* \leq q < q^{**}$;
  (ii) (2.19) *holds if* $q = q^{**}$;
  (iii) (1.10) *holds if* $q > q^{**}$.

(4) *If* $\lambda = (N-2)^2/4$ *and* $1 < q < q^*$, *then one of the three cases occurs:*
  (A) $\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} u(x) \in (0,\infty)$;
  (B) $\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} u(x)/\log(1/|x|) \in (0,\infty)$;
  (C) (1.10) *holds.*

(5) *If* $\lambda = (N-2)^2/4$ *and* $q \geq q^*$, *then*
  (i) (2.33) *holds for* $q = q^*$;
  (ii) (1.10) *holds for* $q > q^*$.

Proof. The first assertion follows from Theorem 2.4. Applying Theorem 2.2, we deduce the claim of (2) and (3)(i). The statements of (3)(ii) and (3)(iii) are proved by Theorem 2.3(b) and Theorem 2.3(a), respectively. We conclude the claim of (4) based on Theorem 2.7. Finally, Theorem 2.6(b3) (respectively, Theorem 2.6(b1)) proves the validity of the statement (5)(i) (respectively, (5)(ii)). $\square$

If in the settings of Corollary 7.6, we replace (1.16)(a) by (1.16)(b), we obtain.

COROLLARY 7.7. *Let* $-\infty < \lambda \le (N-2)^2/4$ *and* (1.5) *hold. Assume that* (1.12)*(a) and* (1.16)*(b) are satisfied. Let* u *be any positive solution of* (1.3).

(I) *If* $0 < \lambda < (N-2)^2/4$ *and* $q = q^{**}$, *then* $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$.

(II) *If* $-\infty < \lambda < (N-2)^2/4$ *and* $q = q^*$, *then* u *satisfies one of the following*
  - (A) $\lim_{|x|\to 0} |x|^p u(x) \in (0,\infty)$;
  - (B) $\lim_{|x|\to 0} |x|^{N-2-p} u(x) \in (0,\infty)$;
  - (C) (2.23) *holds.*

(III) *If* $\lambda = (N-2)^2/4$ *and* $q = q^*$, *then exactly one of the following occurs*
  - (A) $\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} u(x) \in (0,\infty)$;
  - (B) $\lim_{|x|\to 0} |x|^{\frac{N-2}{2}} u(x)/\log(1/|x|) \in (0,\infty)$;
  - (C) (2.33) *holds.*

(IV) *In the remaining non-critical situations, the conclusions of Corollary 7.6 remain valid.*

PROOF. For (I), we apply Theorem 2.2. The assertions of (II) and (III) follow by applying Theorem 2.4 and Theorem 2.7, respectively. □

COROLLARY 7.8. *Let* $-\infty < \lambda \le (N-2)^2/4$ *and* (1.5) *hold. Assume that* (1.12)*(b) and* (1.16)*(c) are verified such that* S *is regularly varying at* ∞ *with index* η. *Then for every positive solution* u *of* (1.3), *we have:*

(I) *If* $0 < \lambda < (N-2)^2/4$ *and* $q = q^{**}$, *then* u *satisfies* (2.17).

(II) *If* $\lambda = (N-2)^2/4$ *and* $q = q^*$, *then* (2.34) *holds.*

(III) *In the remaining situations, the conclusions of Corollary 7.6 remain valid.*

PROOF. Using Theorem 2.3(d) and Theorem 2.6(b4), we obtain the assertion of (I) and (II), respectively. For the remaining situations, we can proceed as in Corollary 7.6. □

REMARK 7.9. In Corollaries 7.6 and 7.7 (but not in Corollary 7.8) we can, in fact, use (1.13) instead of (1.10) (whenever it appears) since (1.12)(a) is assumed (see Remark 1.5 in Chapter 1).

APPENDIX A

Regular variation theory and related results

A.1. Properties of regularly varying functions

If $\mathcal{R}$ is a positive measurable function defined in a neighbourhood of infinity and the limit $\lim_{t\to\infty} \mathcal{R}(\xi t)/\mathcal{R}(t)$ exists in $(0,\infty)$ for every $\xi > 0$, then there exists $m \in \mathbb{R}$ such that $\lim_{t\to\infty} \mathcal{R}(\xi t)/\mathcal{R}(t) = \xi^m$ for every $\xi > 0$ (see [**33**]). Functions with this property were first introduced by Karamata [**24**] and are called *regularly varying functions* at ∞ with index m. The space of such functions will be denoted by $RV_m(\infty)$, where the subscript stands for the index of regular variation. A function is called slowly varying at ∞ if it is regularly varying at ∞ with index zero. Note that a function $\mathcal{R}$ is regularly varying at ∞ with index m if and only if $\mathcal{R}(t)/t^m$ is slowly varying at ∞. This means that for almost all intents and purposes, it is enough to study the properties of slowly varying functions.

The theory of regular variation, which was later extended and developed by many others, plays an important role in certain areas of probability theory such as in the theory of domains of attraction and max-stable distributions.

For the reader's convenience, we include here some basic properties of regularly varying functions. For detailed accounts of the theory of regular variation, its extensions and many of its applications, we refer to [**33**], [**5**] and [**31**].

Proposition A.1 (Representation Theorem). *A function L is slowly varying at ∞ if and only if*

$$L(t) = T(t)\exp\left\{\int_{t_0}^{t} \frac{\phi(\xi)}{\xi}\,d\xi\right\} \quad (t \geq t_0 > 0) \tag{A.1}$$

where $\phi \in C[t_0,\infty)$ satisfies $\lim_{t\to\infty}\phi(t) = 0$ and T is measurable on $[t_0,\infty)$ such that $\lim_{t\to\infty} T(t) := \widehat{T} \in (0,\infty)$.

Proposition A.2 (see Theorem 1.3.3 in [**5**]). *Let L be slowly varying at ∞. Then $L(t) \sim L_1(t)$ as $t\to\infty$, where $L_1 \in C^\infty[t_0,\infty)$ and $h_1(t) := \log L_1(e^t)$ has the property*

$$h_1^{(n)}(t) \to 0 \quad \text{as } t\to\infty \quad \text{for } n = 1,2,\ldots.$$

(Hence, L_1 is slowly varying at ∞ with the representation

$$L_1(t) = \exp\left(c_1 + \int_{t_0}^{t} \frac{\phi(\xi)}{\xi}\,d\xi\right)$$

in which $c_1 = h_1(\log t_0)$ and $\phi(\xi) = h_1'(\log\xi)$.)

Remark A.3. For any $\mathcal{R} \in RV_m(\infty)$, there exists a C^1-function $\widehat{\mathcal{R}} \in RV_m(\infty)$ such that

$$\lim_{t\to\infty} \frac{\widehat{\mathcal{R}}(t)}{\mathcal{R}(t)} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{t\widehat{\mathcal{R}}'(t)}{\widehat{\mathcal{R}}(t)} = m. \tag{A.2}$$

Proposition A.4 (Uniform Convergence Theorem). *If L is slowly varying at ∞, then $L(\xi t)/L(t) \to 1$ as $t \to \infty$, uniformly on each compact ξ-set in $(0, \infty)$.*

Proposition A.5 (Properties of slowly varying functions). *Assume that L is slowly varying at ∞. The following hold:*

(1) *$\log L(t)/\log t$ converges to 0 as $t \to \infty$;*
(2) *For any $j > 0$, we have $t^j L(t) \to \infty$ and $t^{-j} L(t) \to 0$ as $t \to \infty$;*
(3) *$(L(t))^j$ varies slowly at ∞ for every $j \in \mathbb{R}$;*
(4) *If L_1 varies slowly at ∞, so does the product (respectively the sum) of L and L_1.*

Proposition A.6 (Karamata's Theorem). *If $\mathcal{R} \in RV_m(\infty)$ is locally bounded in $[A, \infty)$, then*

(1) $\displaystyle\lim_{t\to\infty} \frac{t^{j+1}\mathcal{R}(t)}{\int_A^t \xi^j \mathcal{R}(\xi)\, d\xi} = j + m + 1$ *for any $j \geq -(m+1)$;*
(2) *for any $j < -(m+1)$ (and for $j = -(m+1)$ if $\int^\infty \xi^{-(m+1)}\mathcal{R}(\xi)\, d\xi < \infty$) we have*

$$\lim_{t\to\infty} \frac{t^{j+1}\mathcal{R}(t)}{\int_t^\infty \xi^j \mathcal{R}(\xi)\, d\xi} = -(j + m + 1).$$

Notation. As in [**31**], let $\mathcal{R}^{\leftarrow}$ denote the (left continuous) inverse of a non-decreasing function $\mathcal{R}$ on $\mathbb{R}$, namely

$$\mathcal{R}^{\leftarrow}(t) = \inf\{s : \quad \mathcal{R}(s) \geq t\}.$$

Proposition A.7 (see Proposition 0.8 in [**31**]). *We have*

(1) *If $\mathcal{R} \in RV_m(\infty)$, then $\lim_{t\to\infty} \log \mathcal{R}(t)/\log t = m$.*
(2) *If $\mathcal{R}_1 \in RV_{m_1}(\infty)$ and $\mathcal{R}_2 \in RV_{m_2}(\infty)$ with $\lim_{t\to\infty} \mathcal{R}_2(t) = \infty$, then*

$$\mathcal{R}_1 \circ \mathcal{R}_2 \in RV_{m_1 m_2}(\infty).$$

(3) *Suppose $\mathcal{R}$ is non-decreasing, $\mathcal{R}(\infty) = \infty$, and $\mathcal{R} \in RV_m(\infty)$ with $0 < m < \infty$. Then*

$$\mathcal{R}^{\leftarrow} \in RV_{1/m}(\infty).$$

We see next that any function $\mathcal{R}$ varying regularly at ∞ with *positive* index is asymptotic to a monotone function.

Proposition A.8 (see Theorem 1.5.3 in [**5**]). *Let $\mathcal{R} \in RV_m(\infty)$ and choose $t_0 \geq 0$ so that $\mathcal{R}$ is locally bounded on $[t_0, \infty)$. If $m > 0$, then we have*

(a) *$\overline{\mathcal{R}}(t) := \sup\{\mathcal{R}(s) : \ t_0 \leq s \leq t\} \sim \mathcal{R}(t)$ as $t \to \infty$;*
(b) *$\underline{\mathcal{R}}(t) := \inf\{\mathcal{R}(s) : \ s \geq t\} \sim \mathcal{R}(t)$ as $t \to \infty$.*

A.2. Other results

We shall frequently use the following comparison principle, which follows from Lemma 2.1 in [**16**].

Lemma A.9 (Comparison principle). *Let $N \geq 3$ and Ω be a smooth bounded domain in $\mathbb{R}^N$ with $\overline{\Omega} \subset \mathbb{R}^N \setminus \{0\}$. Assume that g is continuous on $(0, \infty)$ such that*

$g(t)/t$ is increasing for $t>0$. Let λ be a real parameter. If u_1 and u_2 are positive $C^1(\Omega)$-functions such that

$$\begin{cases} -\Delta u_1 - \dfrac{\lambda}{|x|^2}u_1 + g(u_1) \le 0 \le -\Delta u_2 - \dfrac{\lambda}{|x|^2}u_2 + g(u_2) \quad \text{in } \mathcal{D}'(\Omega), \\ \limsup\limits_{x\to\partial\Omega}[u_1(x)-u_2(x)] \le 0, \end{cases} \tag{A.3}$$

then $u_1 \le u_2$ in Ω.

Since we do not require $h(t)/t$ be increasing for $t>0$, we give the following.

Lemma A.10. *Let h be as in Assumption A in Chapter 1. We can construct two functions h_1 and h_2 that are continuous on $[0,\infty)$, positive on $(0,\infty)$ with $h_1(0)=h_2(0)=0$ such that*

$$\begin{cases} h_1(t) \le h(t) \le h_2(t) \text{ for any } t\ge 0, \\ \dfrac{h_1(t)}{t} \text{ and } \dfrac{h_2(t)}{t} \text{ are both increasing for } t>0. \end{cases} \tag{A.4}$$

Proof. Let $q>1$. To define h_1, we set

$$g(t) = t^{(-q-1)/2}h(t) \quad \text{for } t>0 \;\text{ and }\; g_*(t) = \inf_{s\ge t} g(s) \;\text{ for } t>0.$$

Hence, $g_* \le g$ on $(0,\infty)$ and g_* is non-decreasing on $(0,\infty)$. Define h_1 on $[0,\infty)$ by

$$h_1(t) = g_*(t)t^{(q+1)/2} \quad \text{for any } t>0 \;\text{ and } h_1(0)=0. \tag{A.5}$$

Using the monotonicity of g_* and $q>1$, we infer that $h_1(t)/t = g_*(t)t^{(q-1)/2}$ is increasing on $(0,\infty)$. Moreover, $h_1(t)\le h(t)$ for any $t\ge 0$. We construct h_2 on $[0,\infty)$ as follows

$$h_2(t) = t\left(\sup_{0<s\le t}\frac{h(s)}{s} + t^{(q-1)/2}\right) \quad \text{for any } t>0 \quad \text{and} \quad h_2(0)=0. \tag{A.6}$$

Since $h(0)=0$ and $h(t)/t$ is bounded for small $t>0$, we see that h_2 is well defined and satisfies (A.4). □

Remark A.11. If, in addition, $h\in RV_q$ with $q>1$, then by Proposition A.8, h_1 and h_2 given by (A.5) and (A.6) satisfy $\lim_{s\to\infty} h_i(s)/h(s)=1$ for $i=1,2$.

Remark A.12. The functions f and $\mathcal{J}$ introduced in (1.8) satisfy:

$$\begin{cases} \lim\limits_{t\to\infty}\dfrac{h(t)}{tf(t)}=1, \quad \lim\limits_{t\to\infty}\dfrac{tf'(t)}{f(t)} = \lim\limits_{t\to\infty}\dfrac{f'(t)h(t)}{f^2(t)} = q-1, \\ \lim\limits_{t\to\infty}\dfrac{tf''(t)}{f'(t)} = q-2, \quad \lim\limits_{t\to\infty}\dfrac{f(t)f''(t)}{[f'(t)]^2} = \dfrac{q-2}{q-1}; \\ \lim\limits_{|x|\to 0}\dfrac{\mathcal{J}(|x|)}{|x|^2b(x)} = 1, \quad \lim\limits_{r\to 0}\dfrac{r\mathcal{J}'(r)}{\mathcal{J}(r)} = \theta+2, \quad \lim\limits_{r\to 0}\dfrac{r\mathcal{J}''(r)}{\mathcal{J}'(r)} = \theta+1. \end{cases} \tag{A.7}$$

Lemma A.13. *Let $-\infty<\lambda\le (N-2)^2/4$. Assume that $q>1$ and $\theta>-2$. Let $\mathcal{K}(r)$ and ℓ be given by (1.8) and (1.9), respectively. If $\ell>0$, then (5.59) is satisfied by $\mathcal{U}(r) := \ell^{1/(q-1)}\mathcal{K}(r)$, which is defined for any small $r>0$.*

Proof. Since $\lim_{r\to 0}\mathcal{K}(r)=\infty$ and $\tilde{h}\in RV_q(\infty)$, it remains to show that

$$\mathcal{K}''(r) + \frac{N-1}{r}\mathcal{K}'(r) + \frac{\lambda}{r^2}\mathcal{K}(r) \sim \ell r^\theta L_b(r)\,\tilde{h}(\mathcal{K}(r)) \quad \text{as } r\to 0. \tag{A.8}$$

From the definition of $\mathcal{K}$ in (1.8), we obtain that

$$\begin{cases} \mathcal{K}'(r) = -\dfrac{\mathcal{J}'(r)}{\mathcal{J}(r)} \dfrac{f(\mathcal{K}(r))}{f'(\mathcal{K}(r))}, \\ \mathcal{K}''(r) = \mathcal{K}'(r) \left(\dfrac{\mathcal{J}''(r)}{\mathcal{J}'(r)} - \dfrac{f''(\mathcal{K}(r))\, \mathcal{K}'(r)}{f'(\mathcal{K}(r))} - 2\dfrac{\mathcal{J}'(r)}{\mathcal{J}(r)} \right) \end{cases}$$

for any small $r > 0$. As in (1.9), we set $\Theta := (\theta + 2)/(q - 1)$. Using the properties of f and $\mathcal{J}$ in (A.7), we find

$$\begin{cases} \mathcal{K}'(r) \sim -\Theta \dfrac{\mathcal{K}(r)}{r} & \text{as } r \to 0, \\ \mathcal{K}''(r) \sim \Theta\,(1 + \Theta)\, \dfrac{\mathcal{K}(r)}{r^2} & \text{as } r \to 0. \end{cases} \tag{A.9}$$

Using (1.8) and (A.7), we deduce that

$$\frac{\mathcal{K}(r)}{\tilde{h}(\mathcal{K}(r))} \sim \frac{1}{f(\mathcal{K}(r))} = \mathcal{J}(r) = r^{\theta+2} L_b(r) \quad \text{as } r \to 0.$$

Hence, we have $\mathcal{K}(r)/r^2 \sim r^\theta L_b(r)\, \tilde{h}(\mathcal{K}(r))$ as $r \to 0$, which jointly with (A.9), proves the assertion of (A.8). This concludes the proof of Lemma A.13. $\square$

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