ALEXANDER POLYNOMIALS OF RIBBON LINKS

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Abstract. We give a simple argument to show that every polynomial $f(t) \in \mathbb{Z}[t]$ such that $f(1) = 1$ is the Alexander polynomial of some ribbon 2-knot whose group is a 1-relator group, and we extend this result to links.

It is well known that every Laurent polynomial $f(t) \in \Lambda = \mathbb{Z}[t, t^{-1}]$ with $f(1) = 1$ is the Alexander polynomial of some ribbon 2-knot [7]. (See also [1, 2], for the fibred case, and §7H of [11], for a construction of knot polynomials by surgery.) We shall give another argument, which seems particularly simple, and which gives a slightly stronger result. We shall then extend this result to higher-dimensional links.

In higher dimensions the term "Alexander polynomial" is potentially ambiguous. Let $X = S^{n+2} - intK \times D^2$ be the knot exterior, $\pi = \pi K = \pi_1(X)$ the knot group and X' the maximal abelian covering space of X. The homology groups $H_q(X'; \mathbb{Z})$ are finitely generated torsion Λ-modules under the action of $Aut(X'/X) \cong Z$. They each have a sequence of "Alexander polynomial" invariants $\Delta_i^q(K)$ such that $\Delta_i^q(K)$ divides $\Delta_{i+1}^q(K)$ in Λ [8]. Poincaré duality implies that $\Delta_i^{n+1-q}(K) = \overline{\Delta_i^q(K)}$ for $q \leq [\frac{n+1}{2}]$, where the overbar is the involution defined by inverting the generators t_i. More generally, if L is a μ-component n-link there are similar invariants in $\Lambda_\mu = \mathbb{Z}[t_1^{\pm}, \ldots, t_\mu^{\pm}]$.

In this paper "Alexander polynomial" shall mean the greatest common divisor $\Delta(\pi)$ of the first nonzero elementary ideal of the "Alexander module" $A(\pi)$ of π. A presentation for this module may be derived from a presentation for π by the free differential calculus. If $n > 1$ the module has rank μ and $\Delta(L) = \Delta_\mu^1(L)$, but when $n = 1$ it has rank $\leq \mu$, with equality if L is concordant to a boundary link. (See [6] for more on Alexander modules.)

Let $\varepsilon : \Lambda_\mu \to \mathbb{Z}$ be the augmentation homomorphism defined by $\varepsilon(t_i) = 1$ for all i. Then $\varepsilon(\Delta(\pi)) = 1$, since $\pi/\pi' \cong Z^\mu$. The burden of this note is that this is the only constraint on such link polynomials, if $n > 1$. The case $n = 2$ is of particular interest, for then $H_1(X'; \mathbb{Z})$ and duality determine the other homology modules. (When $n = 1$ and L is a boundary link we must also have $\overline{\Delta} = \Delta$; there is as yet no such characterization for other classical links.)

1. knots

An n-knot is a ribbon knot of 1-fusion if it bounds the sum of two disjoint copies of D^{n+1} along a band $D^n \times [0, 1]$ which meets the discs only at its ends.

Theorem 1. *Let $f = f(t) \in \Lambda$ be such that $f(1) = 1$, and let $n > 1$. Then there is an n-knot K which is a ribbon knot of 1-fusion such that $\pi'/\pi'' \cong \Lambda/(f)$, where $\pi = \pi K$.*

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Proof. We may assume that $f \in \mathbb{Z}[t]$ and $f(0) \neq 0$. Let d be the degree of f and let $g = g(t) = (f(t) - 1)/(t - 1)$. Then $g(t) = \Sigma_{i=0}^{i=d-1} g_i t^i \in \mathbb{Z}[t]$. Let

$$\pi = \langle a, t \mid a = wtw^{-1}t^{-1} \rangle,$$

where $w = w(a,t) = \Pi_{i=0}^{i=d-1} t^i a^{g_i} t^{-i} = a^{g_0} \dots t^{d-1} a^{g_{d-1}} t^{1-d} t^{-g(1)}$. (The final term ensures that $w(1,t) = 1$ in $\langle t \rangle$.) Clearly $\pi/\pi' \cong Z$, and it is easily seen that $\pi'/\pi'' \cong \Lambda/(f)$, since $f = (t-1)g + 1$.

Let $x = at$ and let $w_k \dots w_1$ be a word of length k in the alphabet $\{t, t^{-1}, x, x^{-1}\}$ representing $w(xt^{-1}, t)$ Then π also has the deficiency-1 Wirtinger presentation

$$\langle t, x_0, \dots, x_k, x \mid t = x_0,\ x = x_k,\ x_i = w_i x_{i-1} w_i^{-1}\ \forall\ 1 \leq i \leq k \rangle.$$

We may then use the elementary construction of §1.8 of [6] to obtain an n-ribbon $R : D^{n+1} \to S^{n+2}$ with π as its ribbon group, for any $n \geq 1$. This has k parallel throughcuts, and the corresponding slits are in the two extreme components of the complement of the throughcuts. Hence $K = R|_{\partial D^{n+1}}$ is the fusion of a 2-component trivial link along a single band, and so is a ribbon knot of 1-fusion. If $n \geq 2$ then $\pi K \cong \pi$. □

The group π is a 1-relator group. (This is so for the group of any ribbon knot of 1-fusion.) Since $\pi/\pi' \cong Z$ the relator $atwt^{-1}w^{-1}$ is not a proper power. Therefore $c.d.\pi \leq 2$ [9]. The conditions $c.d.\pi = 1$, $\pi \cong Z$ and $f(t) = 1$ are clearly equivalent for groups with such presentations.

When $n = 1$ the knot $K = R|_{\partial D^2}$ provided by the construction of §1.8 of [6] bounds a disc knot $D^2 \subset D^4$ with group π, obtained by desingularizing the ribbon immersion R, and so K has Alexander polynomial $f\overline{f}$. Ribbon knots realizing such polynomials were first constructed in [12]. In fact, the Alexander polynomial of any classical slice knot has this form [4]. However the ribbon immersion R realizing π is not uniquely determined, and we do not know whether we can arrange that πK be a 1-relator group.

Addendum. *The knot constructed in Theorem* 1 *is fibred if and only if the extreme coefficients of* f *are* ± 1.

Proof. If f is a monic polynomial with $f(0) = \pm 1$ then π' is free with basis represented by $\{t^i a t^{-i} \mid 0 \leq i < d\}$. Since $n > 1$ and K is a ribbon knot of 1-fusion it is fibred, by a theorem of Yoshikawa [1, 13].

The converse is clear. □

By taking connected sums of knots we may realize arbitrary finite sequences δ_i with δ_{i+1} dividing δ_i in Λ as the higher polynomial invariants associated to $A(\pi)$. If the summands are all fibred so is their sum.

2. LINKS

Let $A = \{a_1, \dots, a_\mu\}$ and $T = \{t_1 \dots, t_\mu\}$, and let $\partial_i : \mathbb{Z}F(A \cup T) \to \Lambda_\mu$ be the composite of the free derivation $\frac{\partial}{\partial a_i}$ of $\mathbb{Z}F(A \cup T)$ with respect to the generator a_i with the retraction onto $\mathbb{Z}[F(T)]$ which sends a_j to 1 and t_j to t_j for $j \leq \mu$.

Lemma. *Given* $f_i \in \mathbb{Z}[F(T)]$, *there is a word* $W \in F(A \cup T)$ *with trivial image in* $F(T)$ *and such that* $\partial_i(W) = f_i$ *for all* $i \leq \mu$.

Proof. Suppose that $f_i = \Sigma_{m\in F(T)} f_{im} m$. Let $v_i = \Pi_{m\in F(T)} m a_i^{f_{im}} m^{-1}$, where the factors are taken in some fixed order, and let $W = \Pi v_i = v_1 \dots v_\mu$. Then the v_i and W have trivial image in $F(T)$. Moreover, $\partial_i(v_i) = \Sigma_{m\in F(T)} f_{im} m = f_i$, and $\partial_j(v_i) = 0$, if $j \neq i$, so $\partial_i(W) = \partial_i(v_i) = f_i$, for all $i \leq \mu$. □

The order on $F(T)$ used in this lemma is not important.

Theorem 2. *Let $f \in \Lambda_\mu$ be such that $\varepsilon(f) = 1$, and let $n > 1$. Then there is a μ-component ribbon boundary n-link L with $\Delta(\pi L) = f$.*

Proof. We may write $f = 1 - \Sigma(t_i - 1) f_i$. Choose $F_i \in \mathbb{Z}[F(T)]$ with image $f_i \in \Lambda_\mu$. There are words $W_i \in F(A \cup T)$ with trivial image in $F(T)$ and such that $\partial_j W_i = F_i$ for all $j \leq \mu$, by the lemma. Let π be the group with presentation

$$\langle a_i, t_i, 1 \leq i \leq \mu \mid a_i = t_i W_i t_i^{-1} W_i^{-1},\ \forall\ i \rangle.$$

The free differential calculus gives a presentation matrix $[I_\mu - D, 0_\mu]$ for $A(\pi)$, where D is a $\mu \times \mu$ matrix with $D_{ij} = (t_i - 1) f_i$ for all $i, j \leq \mu$ and 0_μ is a null $\mu \times \mu$ matrix. As the columns of D are all equal, it is easy to see that $\Delta(\pi) = \det(I_\mu - D) = 1 - \Sigma(t_i - 1) f_i = f$.

As in Theorem 1, the group π has an equivalent Wirtinger presentation of deficiency μ, and the elementary construction of §1.8 of [6] gives a μ-component ribbon n-link L with group $\pi L \cong \pi$ and meridians corresponding to the generators t_i. Since the projection of π onto $\pi/\langle\langle a_1, \dots, a_\mu \rangle\rangle \cong F(T)$ carries the meridians to a free basis, L is a boundary link. □

Let Y be the finite 2-complex corresponding to the above presentation, and let Z be the complex obtained by adjoining 2-cells along maps corresponding to the generators t_i. Then Z is 1-connected and $\chi(Z) = 1$, and so it is a finite contractible 2-complex. Thus if the Whitehead Conjecture is true Y is aspherical, and so $c.d.\pi L \leq 2$.

When $n = 1$ this construction gives a ribbon boundary link L with $\Delta(\pi) = f\overline{f}$. This condition is satisfied by the first nonzero Alexander polynomial of every classical slice link. Every $f \in \Lambda_\mu$ such that $\varepsilon f = 1$ and $\overline{f} = f$ is $\Delta(\pi)$ for some μ-component boundary 1-link [5].

3. A NON-COMMUTATIVE ANALOGUE?

A μ-component link L is an homology boundary link if there is an epimorphism from $\pi = \pi L$ to $F(\mu)$. (It is a boundary link if and only if there is such an epimorphism which takes the images of a set of meridians to a basis for the free group.) The kernel of any such epimorphism is $\pi_\omega = \cap \pi_{[n]}$, the intersection of the lower central series. Let k be a field and $k\Gamma_\mu = k[F(\mu)]$. The homology groups $H_q = H_q(X^\omega; k)$ of the covering space X^ω with group π_ω are finitely generated left $k\Gamma_\mu$-modules. If $1 \leq q < n$ then H_q satisfies the Sato property: $k \otimes_{k\Gamma_\mu} H_q = Tor_1^{k\Gamma_\mu}(k, H_q) = 0$. (See Chapter 9 of [6].)

Farber constructed invariants of such modules with values “noncommutative rational functions”. When $\mu = 1$ these are equivalent to the usual Alexander polynomials Δ_0^q (although closer in form to the logarithmic derivative) [3]. His work has been reformulated in terms of Gelfand-Retakh quasideterminants [10]. Is there a realization result analogous to Theorem 2 for the invariants of $H_1(X^\omega; k) = k \otimes_{\mathbb{Z}} (\pi_\omega/\pi_\omega{}')$?

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