Isolated Singularities for Weighted Quasilinear Elliptic Equations☆

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Abstract

We classify all the possible asymptotic behavior at the origin for positive solutions of quasilinear elliptic equations of the form $\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = b(x)h(u)$ in $\Omega \setminus \{0\}$, where $1 < p \leq N$ and Ω is an open subset of $\mathbb{R}^N$ with $0 \in \Omega$. Our main result provides a sharp extension of a well-known theorem of Friedman and Véron for $h(u) = u^q$ and $b(x) \equiv 1$, and a recent result of the authors for $p = 2$ and $b(x) \equiv 1$. We assume that the function h is regularly varying at ∞ with index q (that is, $\lim_{t\to\infty} h(\lambda t)/h(t) = \lambda^q$ for every $\lambda > 0$) and the weight function $b(x)$ behaves near the origin as a function $b_0(|x|)$ varying regularly at zero with index θ greater than $-p$. This condition includes $b(x) = |x|^\theta$ and some of its perturbations, for instance, $b(x) = |x|^\theta(-\log|x|)^m$ for any $m \in \mathbb{R}$. Our approach makes use of the theory of regular variation and a new perturbation method for constructing sub- and super-solutions.

Key words: Quasilinear elliptic equations, isolated singularities, regular variation theory
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1. Introduction

Let $1 < p \leq N$ and Ω be an open subset of $\mathbb{R}^N$ such that the origin is contained in Ω. Motivated by [7], [3], [17] and our recent work [4], we classify here all the possible asymptotic behavior at the origin for positive solutions of quasilinear elliptic equations of the form

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + b(x)h(u) = 0 \quad \text{in } \Omega^* := \Omega \setminus \{0\}, \tag{1.1}$$

under suitable assumptions on $b(x)$ and $h(u)$. Unless stated otherwise, the functions h and b always satisfy the following conditions.

Assumption A. *The function h is continuous on $\mathbb{R}$ and positive on $(0, \infty)$ with $h(0) = 0$, and $h(t)/t^{p-1}$ is bounded for small $t > 0$, while b is a positive continuous function on $\overline{\Omega} \setminus \{0\}$.*

By a solution of (1.1), we mean the following.

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Definition 1.1. A function u is said to be a *solution* (*sub-solution, super-solution*) of (1.1) if $u(x) \in C^1(\Omega^*)$ and for all functions (non-negative functions) $\varphi(x)$ in $C^1_c(\Omega^*)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx + \int_\Omega b(x)h(u)\varphi\, dx = 0 \qquad (\leq 0,\ \geq 0). \tag{1.2}$$

By $C^1_c(\Omega^*)$ we denote the space of functions in $C^1(\Omega^*)$ having compact support in Ω^*.

Friedman and Véron considered in [7] the following special case of (1.1):

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + |u|^{q-1}u = 0 \quad \text{in } \Omega^*. \tag{1.3}$$

They obtained a complete classification of the behavior near zero for all positive solutions when $p-1 < q < \frac{(p-1)N}{N-p}$ (any $q > p-1$ if $p = N$). The homogeneity of the power non-linearity and various scaling arguments were key ingredients in the approach of [7] and other related papers such as [18, 19] and [3]. These arguments can be easily modified to treat a more general case where $h(u)$ behaves like u^q near infinity, but it is crucial that in the limit it behaves like a pure power, that is, $\lim_{t\to\infty} h(t)/t^q = c > 0$; see Remark 2.3 in [7].

Our main goal is to extend the classification result of Friedman and Véron [7] to weighted equations of the type (1.1) when the nonlinearity h *needs not behave like a pure power* at infinity. For such h the scaling arguments used before fail to work in several key steps. The condition near infinity we impose on h is the following:

$$\lim_{t\to\infty} \frac{h(\lambda t)}{h(t)} = \lambda^q \qquad \text{for every } \lambda > 0 \text{ and some } q > p-1. \tag{1.4}$$

Functions satisfying condition (1.4) are known as regularly varying functions at ∞ with index q. More precisely, a positive measurable function h defined on an interval (A, ∞) with $A > 0$ is called regularly varying at ∞ with index q, written $h \in RV_q$, provided that the equation in (1.4) holds for some $q \in \mathbb{R}$. A regularly varying function of index zero is called a *slowly varying* function. Any positive constant function is trivially a slowly varying function. Other non-trivial examples of slowly varying functions include:

(a) The logarithm $\log t$, its m-iterates $\log_m t$ (defined as $\log \log_{m-1} t$) and powers of $\log_m t$ for any integer $m \geq 1$.
(b) $\exp\left(\frac{\log t}{\log\log t}\right)$.
(c) $\exp((\log t)^\alpha)$ with $\alpha \in (0,1)$.

We have $h \in RV_q$ if and only if $h(t) = t^q L(t)$ for a function L that is slowly varying at ∞.

The concept of regular variation can be applied *at zero* as follows.

Definition 1.2 (see [12]). We say that b_0 is *regularly varying at (the right of) zero* with index $\theta \in \mathbb{R}$ (written as $b_0 \in RV_\theta(0+)$) if $t \to b_0(1/t)$ is regularly varying at ∞ with index $-\theta$.

Thus $b_0 \in RV_\theta(0+)$ if and only $b_0(r) = r^\theta L(1/r)$ for $r > 0$ small, where L is a slowly varying function at ∞. Note that $\lim_{r\to 0} b_0(r) = 0$ if $\theta > 0$, whereas $\lim_{r\to 0} b_0(r) = \infty$ if $\theta < 0$. However, if b_0 is slowly varying at zero (that is, $\theta = 0$), then the above examples show that the limit of b_0 at zero in general cannot be determined, and it may not even exist. For instance, if

$$b_0(r) = \exp\{(-\log r)^{1/3} \cos((-\log r)^{1/3})\} \quad \text{for } r \in (0,1),$$

then b_0 is slowly varying at zero, but $\liminf_{r\to 0} b_0(r) = 0$ and $\limsup_{r\to 0} b_0(r) = +\infty$.

Our hypothesis on b involves regular variation at zero, namely

$$\lim_{|x|\to 0} \frac{b(x)}{b_0(|x|)} = 1 \quad \text{for some } b_0 \in RV_\theta(0+) \text{ and } \theta > -p. \tag{1.5}$$

Let $\mu(x)$ denote (as in [7]) the fundamental solution of the p-harmonic equation

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = \delta_0 \ \text{ in } \mathcal{D}'(\mathbb{R}^N) \text{ (in the sense of distributions in } \mathbb{R}^N),$$

where δ_0 denotes the Dirac mass at 0. If ω_N denotes the volume of the unit ball in $\mathbb{R}^N$, then

$$\mu(x) = \mu(|x|) = \begin{cases} \dfrac{p-1}{N-p}(N\omega_N)^{-1/(p-1)}|x|^{(p-N)/(p-1)} & \text{for } 1 < p < N, \\ (N\omega_N)^{-1/(N-1)} \log(1/|x|) & \text{for } p = N. \end{cases}$$

From (1.5), the function b is locally in $L^{\frac{N}{p-\varepsilon}}(\Omega)$ for some $\varepsilon > 0$ small. Hence Theorem 1 of Serrin [14] is applicable to (1.1) whenever $h(t)/t^{p-1}$ is *bounded* in a neighbourhood of $+\infty$. In this case, if u is any given positive solution of (1.1), then one of the following holds

(a) u can be defined at 0 so that the resulting function is a continuous solution of (1.1) in all of Ω (that is, $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ such that (1.2) holds for all $\varphi \in C^1_c(\Omega)$);
(b) there exists a constant $C > 0$ such that $C^{-1}\mu(x) \le u(x) \le C\mu(x)$ near $x = 0$.

To ensure that $h(t)/t^{p-1}$ is *unbounded* at ∞, we require $q > p-1$ in (1.4). We define

$$C_{N,p,\theta} := \frac{(p-1)(N+\theta)}{N-p} \text{ if } 1 < p \le N \qquad (C_{N,p,\theta} = \infty \text{ if } p = N), \tag{1.6}$$

where θ appears in (1.5). In Theorem 1.2 we show that if $p-1 < q < C_{N,p,\theta}$, then a new type of behavior near zero arises (in the sense of solutions u satisfying $\lim_{|x|\to 0} u(x)/\mu(x) = \infty$).

Our central result (Theorem 1.1) establishes a complete classification of the positive solutions of (1.1), assuming that $p-1 < q < C_{N,p,\theta}$. We also show that the restriction $q < C_{N,p,\theta}$ is *sharp* (cf., Theorem 1.3) and that there exist solutions in each of the categories of Theorem 1.1 under suitable regularity and monotonicity assumptions (see Theorem 1.2).

We now state precisely our main results.

Theorem 1.1. *Let* (1.4) *and* (1.5) *hold with* $1 < p \le N$ *and* $p-1 < q < C_{N,p,\theta}$. *If* u *is a positive solution of* (1.1)*, then as* $|x| \to 0$ *exactly one of the following applies:*

(i_1) $|x|^p b(x)\dfrac{h(u(x))}{u^{p-1}(x)}$ *converges to the following positive number*

$$\xi_{N,p,q,\theta} := \left(\frac{p+\theta}{q+1-p}\right)^{p-1}\left(\frac{pq}{q+1-p} - N + \frac{(p-1)\theta}{q+1-p}\right).$$

(i_2) $u(x)/\mu(x)$ *converges to a positive constant* γ *and*

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + b(x)h(u) = \gamma^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.7}$$

(i_3) $u(x)$ *has a finite limit and* u *can be extended as a continuous solution of* (1.1) *in all* Ω.

Theorem 1.2. *Let* (1.4) *and* (1.5) *hold with* $1 < p \le N$ *and* $p - 1 < q < C_{N,p,\theta}$. *Assume that* Ω *is a bounded domain with* C^1*-boundary and* $\vartheta \in C^1(\partial\Omega)$ *is a non-negative function. If* $h(t)/t^{p-1}$ *is non-decreasing for* $t > 0$, *then for every* $\gamma \in [0, \infty) \cup \{+\infty\}$, *the following problem*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(x)h(u) = 0 \quad \text{in } \Omega^*, \\ \lim_{|x|\to 0} \dfrac{u(x)}{\mu(x)} = \gamma, \qquad u = \vartheta \ \text{ on } \partial\Omega, \end{cases} \tag{1.8}$$

admits a unique non-negative solution u_γ, *which is in* $C^{1,\alpha}_{\mathrm{loc}}(\Omega^*)$ *for some* $\alpha \in (0, 1)$. *Moreover, if* $\gamma \in [0, \infty)$, *then* (1.7) *holds with* $u = u_\gamma$.

Theorem 1.3. *Let* (1.4) *and* (1.5) *hold with* $1 < p < N$ *and* $q \ge C_{N,p,\theta}$. *If* $q = C_{N,p,\theta}$, *then we assume in addition that*

$$\liminf_{t\to\infty} \frac{h(t)}{t^{C_{N,p,\theta}}} > 0 \quad \text{and} \quad \liminf_{|x|\to 0} \frac{b(x)}{|x|^\theta} > 0. \tag{1.9}$$

Then any positive solution of (1.1) *can be extended as a continuous solution of* (1.1) *in all* Ω.

Remark 1.1. We extend several results in papers such as [7], [3], [17] and [4].

(a) Theorem 1.1 with $b \equiv 1$ and $h(t) = t^q$ reduces to Theorem 2.1 of Friedman and Véron [7] on Eq. (1.3), which for $p = 2$ was proved earlier by Véron [18, 19] and also by Brezis and Oswald [2] (with a different approach to [18, 19]).
(b) Theorem 1.2 with $h(t) = t^q$ and $b(x) \equiv 1$ is due to Friedman and Véron [7].
(c) Theorem 1.3 extends results given for $b(x) \equiv 1$ by Brezis–Véron [3] ($p = 2$) and Vázquez–Véron [17] ($1 < p < N$). Our proof is somehow different than in [3] and [17].

In Theorem 5.1 we prove that if u is a positive solution of (1.1) and $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} \neq \infty$, then either (i_2) or (i_3) holds in the settings of Theorem 1.1. However, the most difficult part in the proof of Theorem 1.1 is the next result dealing with the case $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$.

Theorem 1.4. *Let* (1.4) *and* (1.5) *hold with* $1 < p \le N$ *and* $p - 1 < q < C_{N,p,\theta}$. *If* u *is a positive solution of* (1.1) *such that* $\limsup_{|x|\to 0} u(x)/\mu(x) = \infty$, *then*

$$\lim_{|x|\to 0} \frac{u(x)}{\Upsilon(|x|)} = \eta, \quad \text{where } \eta := \left(\frac{pq}{q+1-p} - N + \frac{(p-1)\theta}{q+1-p} \right)^{\frac{1}{q+1-p}}, \tag{1.10}$$

and the function Υ *is defined by*

$$\int_{\Upsilon(r)}^\infty \frac{dt}{[h(t)]^{\frac{1}{p-1}}} = \int_0^r [s b_0(s)]^{\frac{1}{p-1}}\, ds \quad \text{for small } r > 0. \tag{1.11}$$

The statement of (i_1) in Theorem 1.1 is equivalent to (1.10). This can be easily checked using (A.6) and (A.7) in Appendix A. Theorem 1.4 determines the precise asymptotic limit of solutions with strong singularities at zero (that is, solutions u satisfying $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$). Understanding the blow-up behavior at zero for such solutions is more intricate than in [7] due to the lack of homogeneity of h in (1.4) and the richness of the admissible class for the weight function b in (1.5). We recently made progress in [4] by treating such a nonlinearity h in the special case $b \equiv 1$ and $p = 2$. More exactly, we extended Véron's classification result in [18, 19] to positive solutions of $\Delta u = h(u)$ in Ω^* when $h \in RV_q$ with $q > 1$. To overcome the difficulty caused by the lack of homogeneity of h, we introduced in [4] a perturbation method that enabled us to construct crucial sub-super-solutions to the equation. These were used to obtain the precise limiting behavior of the solutions u with a strong singularity at zero. But the perturbation method in [4] seems difficult to apply if $p \neq 2$.

In this paper, we introduce a different perturbation method, which not only applies to the general case $1 < p \leq N$, but can also tackle a weight function $b(x)$ in the equation. Moreover, even in the special case $p = 2$ and $b \equiv 1$, this new method is much simpler to use than the earlier perturbation method of [4]. In Section 2 by assuming two facts (to be validated later in Section 3 and Section 7), we prove Theorem 1.4. Our key ingredient is given by the construction of sub- and super-solutions via the new perturbation method. The super-solutions will be used to obtain a key sharp upper bound (see (2.9)), while the sub-solutions are instrumental in proving a sharp lower bound for positive solutions with strong singularities at zero.

The rest of the paper is organized as follows. In Section 3 we show that for $r_0 > 0$ small, every positive sub-solution $u(x)$ of (1.1) is bounded above by $C_1\Upsilon(|x|)$ for $0 < |x| < r_0$, where $C_1 = C_1(r_0) > 0$ is a constant independent of u (see (3.1)). This validates our first assumed fact and enables us to prove that every positive solution u satisfies a Harnack-type inequality (see Lemma 3.1). Section 4 proves a regularity result that is to be frequently used in compactness arguments in later sections. One such application is in Section 5, where we prove Theorem 5.1 that treats the case of positive solutions (1.1) satisfying $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} \neq \infty$. Section 6 gives several results for the power case $b(x) = |x|^\theta$ and $h(t) = t^q$ for $t > 0$, which will be useful for the general case later. The arguments here are based on ideas in [7]. In Section 7 we complete the proof of Theorem 1.4 by validating the second fact assumed true in Section 2. The proof of Theorem 1.1 rests on Theorem 5.1 if $\limsup_{|x|\to 0} u(x)/\mu(x) \neq \infty$, otherwise we use Theorem 1.4. The above ingredients will also serve to prove Theorem 1.2 in Section 8 and Theorem 1.3 in Section 9. In Appendix A, we include properties of regularly varying functions used in the paper, along with some known comparison results (Lemma A.8 and Lemma A.9).

2. Solutions with strong singularities at zero

We first assume that (1.4) and (1.5) hold with $1 < p \leq N$ and $q > p - 1$.

Remark 2.1. The function Υ in (1.11) is regularly varying at zero of index $-\frac{\theta+p}{q-p+1}$. Hence, $\lim_{r\to 0} \frac{\Upsilon(r)}{f(r)} = 0$ for every $f \in RV_\sigma(0+)$ with $\sigma < -\frac{\theta+p}{q-p+1}$.

Consequently, Lemma 3.1 (a) implies the following.

Fact 1: *Any positive sub-solution u of* (1.1) *satisfies*

$$\lim_{|x|\to 0} \frac{u(x)}{f(|x|)} = 0 \quad \text{for every } f \in RV_\sigma(0+) \text{ with } \sigma < -\frac{\theta + p}{q - p + 1}. \tag{2.1}$$

For the remainder of Section 2, we assume in addition that $q < C_{N,p,\theta}$. We shall later prove

Fact 2: *If u is a positive solution of* (1.1) *such that* $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$, *then*

$$\lim_{|x|\to 0} \frac{u(x)}{f(|x|)} = \infty \quad \text{for every } f \in RV_\sigma(0+) \text{ with } \sigma > -\frac{\theta + p}{q - p + 1}. \tag{2.2}$$

We postpone the validation of Fact 2 to Lemma 7.1 in Section 7. We can now proceed with the proof of Theorem 1.4, which relies on the construction of sub-super-solutions in Section 2.2.

2.1. Proof of Theorem 1.4 *(assuming Facts 1 and 2)*

Let u denote a positive solution of (1.1) such that $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$. Without loss of generality, we can assume that $h(t)$ is increasing for large $t > 0$, the function $\Upsilon \in C^2(0, r_0)$ for small $r_0 > 0$ and (A.8) holds (see Remark A.2 and Remark A.4 of Appendix A).

Fix $\epsilon > 0$ sufficiently small. We can find $\eta_\epsilon^- < \eta < \eta_\epsilon^+$ such that $\eta_\epsilon^\pm \to \eta$ as $\epsilon \to 0$ and $\eta_\epsilon^- \Upsilon$ (respectively, $\eta_\epsilon^+ \Upsilon$) is a sub-solution (respectively, super-solution) of (1.1) in $B_{r_\varepsilon}(0) \setminus \{0\}$ for some small $r_\varepsilon > 0$. This assertion follows from Lemma 2.1 (with $\nu = 0$). If we could show that

$$\begin{cases} u(x) \text{ \textit{is dominated} by } \eta_\epsilon^+ \Upsilon(|x|) \text{ near zero,} \\ u(x) \text{ \textit{dominates} } \eta_\epsilon^- \Upsilon(|x|) \text{ near zero,} \end{cases} \tag{2.3}$$

then we could use the comparison principle (Lemma A.8) to conclude that

$$\eta_\epsilon^+ \Upsilon(|x|) + C_\epsilon \geq u(x) \quad \text{and} \quad u(x) + C'_\epsilon \geq \eta_\epsilon^- \Upsilon(|x|) \qquad \text{for every } 0 < |x| \leq r_\varepsilon, \tag{2.4}$$

where $C_\varepsilon = \max_{|x|=r_\varepsilon} u(x)$ and $C'_\varepsilon := \eta \Upsilon(r_\varepsilon)$. From (2.4), we would immediately get

$$\eta_\varepsilon^+ \geq \limsup_{|x|\to 0} \frac{u(x)}{\Upsilon(|x|)} \quad \text{and} \quad \liminf_{|x|\to 0} \frac{u(x)}{\Upsilon(|x|)} \geq \eta_\epsilon^-. \tag{2.5}$$

By letting $\epsilon \to 0$ in (2.5), we would get (1.10). However, it is difficult to obtain (2.3) since we do not have enough control of $u(x)$ near $x = 0$ to compare it with $\eta_\epsilon^\pm \Upsilon(|x|)$. Thus we introduce a perturbation method that uses the weaker information from Facts 1 and 2 above. In Section 2.2 we construct a one-parameter family of functions $\left(\eta_{\epsilon,\nu}^+ \Upsilon_\nu(r)\right)_{\nu \in (0,\nu_0]}$ (respectively, $\left(\eta_{\epsilon,\nu}^- \Upsilon_{-\nu}(r)\right)_{\nu \in (0,\nu_0]}$) such that $\lim_{\nu\to 0} \eta_{\epsilon,\nu}^\pm = \eta_\epsilon^\pm$, and $\lim_{\nu\to 0} \Upsilon_{\pm\nu}(r) = \Upsilon(r)$ for every r in a small interval $(0, r_0)$ (see (2.10) and (2.14)). Moreover, for each $\nu \in (0, \nu_0]$, we have:

(P1) $\Upsilon_\nu(r) \geq \Upsilon(r) \geq \Upsilon_{-\nu}(r)$ for all $r \in (0, r_0)$ (see (2.11)).
(P2) $r \longmapsto \Upsilon_\nu(r)$ is regularly varying at zero of index *less* than $-\frac{\theta+p}{q-p+1}$ (using (2.10)).
(P3) $r \longmapsto \Upsilon_{-\nu}(r)$ is regularly varying at zero of index *greater* than $-\frac{\theta+p}{q-p+1}$.
(P4) $\eta_{\epsilon,\nu}^+ \Upsilon_\nu(r)$ (respectively, $\eta_{\epsilon,\nu}^- \Upsilon_{-\nu}(r)$) is a super-solution (respectively, sub-solution) of (1.1) in $B_{r_\varepsilon}(0) \setminus \{0\}$ for some small $r_\varepsilon > 0$ that is independent of ν (see Lemma 2.1).

The facts assumed early in the section can now be used to compare u and $\eta_{\varepsilon,\nu}^\pm \Upsilon_{\pm\nu}$ near zero. Let $\nu \in (0, \nu_0]$ be arbitrary. Using (P2) and (P3), jointly with (2.1) and (2.2), we obtain

$$\lim_{|x|\to 0} \frac{u(x)}{\Upsilon_\nu(|x|)} = 0, \qquad \lim_{|x|\to 0} \frac{u(x)}{\Upsilon_{-\nu}(|x|)} = \infty. \tag{2.6}$$

We prove below that (2.4) holds when $\eta_\varepsilon^+ \Upsilon(|x|)$ (respectively, $\eta_\varepsilon^- \Upsilon(|x|)$) is replaced by $\eta_{\varepsilon,\nu}^+ \Upsilon_\nu(|x|)$ (respectively, $\eta_{\varepsilon,\nu}^- \Upsilon_{-\nu}(|x|)$). Notice that $\eta_{\epsilon,\nu}^+ \Upsilon_\nu(r) + C_\varepsilon$ is a super-solution of (1.1) in $B_{r_\varepsilon}(0) \setminus \{0\}$. By (2.6), we see that $u(x)$ is *dominated* by $\eta_{\epsilon,\nu}^+ \Upsilon_\nu(|x|)$ near $x = 0$. By applying the comparison principle (Lemma A.8), we find

$$u(x) \leq \eta_{\epsilon,\nu}^+ \Upsilon_\nu(|x|) + C_\varepsilon \quad \text{for every } 0 < |x| \leq r_\varepsilon. \tag{2.7}$$

Using $\eta > \eta_\varepsilon^-$ and $\lim_{\nu\to 0} \eta_{\varepsilon,\nu}^- = \eta_\varepsilon^-$, by (P1) we find $C'_\varepsilon \geq \eta_{\varepsilon,\nu}^- \Upsilon_{-\nu}(r_\varepsilon)$ for every $\nu \in (0, \nu_0]$ (if needed, we reduce $\nu_0 > 0$). Since $u(x)$ *dominates* $\eta_{\epsilon,\nu}^- \Upsilon_{-\nu}(|x|)$ near $x = 0$ and $u(x) + C'_\varepsilon$ is a super-solution of (1.1) in $B_{r_\varepsilon}(0) \setminus \{0\}$, by applying Lemma A.8 again, we obtain

$$u(x) + C'_\varepsilon \geq \eta_{\epsilon,\nu}^- \Upsilon_{-\nu}(|x|) \quad \text{for every } 0 < |x| \leq r_\varepsilon. \tag{2.8}$$

Letting $\nu \to 0$ in (2.7) and (2.8), we arrive at (2.4). This completes the proof of (1.10).

Remark 2.2. Using Fact 1, we proved that any positive *sub-solution* u of (1.1) satisfies

$$\limsup_{|x|\to 0} \frac{u(x)}{\Upsilon(|x|)} \le \eta, \quad \text{where } \eta \text{ is given by (1.10).} \tag{2.9}$$

2.2. Sub- and super-solutions via a new perturbation method

Our construction of sub-super-solutions uses a suitable perturbation of the function Υ defined by (1.11). Fix $\nu_0 \in (0,1)$ suitably small. For every $\nu \in [0,\nu_0]$, we define $\Upsilon_\nu(r)$ by

$$\int_{\Upsilon_\nu(r)}^{\infty} \frac{dt}{[h(t)]^{\frac{1}{p-1}}} = \left(\int_0^r [s b_0(s)]^{\frac{1}{p-1}}\, ds \right)^{1+\nu} \quad \text{for } r \in (0,r_0). \tag{2.10}$$

We assume that $r_0 > 0$ is small such that $b_0(r) > 0$ and $\int_0^r [s b_0(s)]^{\frac{1}{p-1}}\, ds < 1$ for all $r \in (0,r_0)$. Clearly $\Upsilon = \Upsilon_0$. Let $\Upsilon_{-\nu}$ be obtained from the definition of Υ_ν with ν replaced by $-\nu$. Hence,

$$\Upsilon_\nu \ge \Upsilon \ge \Upsilon_{-\nu} \ge \Upsilon_{-\nu_0} \quad \text{for every } 0 \le \nu \le \nu_0. \tag{2.11}$$

From (2.10), we see that Υ_ν (respectively, $\Upsilon_{-\nu}$) is regularly varying at zero of index $-\frac{(1+\nu)(\theta+p)}{q-p+1}$ (respectively, $-\frac{(1-\nu)(\theta+p)}{q-p+1}$). Since $p-1 < q < C_{N,p,\theta}$, the constant η in (1.10) is positive. In what follows, ε and r_ε will denote small positive constants, and $B_{r_\varepsilon} := B_{r_\varepsilon}(0)$. We will define

$$\Phi_\varepsilon^+(r) := \eta_\varepsilon^+ \Upsilon(r), \quad \Phi_\varepsilon^-(r) := \eta_\varepsilon^- \Upsilon(r) \quad \text{for } r \in (0,r_0) \tag{2.12}$$

with suitable $\eta_\varepsilon^\pm > 0$ satisfying $\lim_{\varepsilon\to 0} \eta_\varepsilon^\pm = \eta$, and then show that Φ_ε^+ (respectively, Φ_ε^-) is a radial super-solution (respectively, sub-solution) of (1.1) in $B_{r_\varepsilon} \setminus \{0\}$ for $r_\varepsilon > 0$ small. This is achieved by a perturbation method involving $\Upsilon_{\pm\nu}$ given above.

Construction of $\Phi_{\varepsilon,\nu}^\pm$. For any $\nu \in (0,\nu_0]$, we define $\Phi_{\varepsilon,\nu}^\pm(r)$ for $r \in (0,r_0)$ by

$$\Phi_{\varepsilon,\nu}^+(r) := \eta_{\varepsilon,\nu}^+ \Upsilon_\nu(r) \quad \text{and} \quad \Phi_{\varepsilon,\nu}^-(r) := \eta_{\varepsilon,\nu}^- \Upsilon_{-\nu}(r), \tag{2.13}$$

where $\eta_{\varepsilon,\nu}^\pm > 0$ is suitably chosen such that $\lim_{\nu\to 0} \eta_{\varepsilon,\nu}^\pm = \eta_\varepsilon^\pm$ and $\lim_{\varepsilon\to 0} \eta_\varepsilon^\pm = \eta$. We will take

$$\eta_{\varepsilon,\nu}^+ = \left[\frac{(1+\nu)^{p-1}}{(1-\varepsilon)^2} \left(\eta^{q+1-p} + o_{\varepsilon,\nu}^+ \right) \right]^{\frac{1}{q+1-p}}, \quad \eta_{\varepsilon,\nu}^- = \left[\frac{(1-\nu)^{p-1}}{(1+\varepsilon)^2} \left(\eta^{q+1-p} + o_{\varepsilon,\nu}^- \right) \right]^{\frac{1}{q+1-p}} \tag{2.14}$$

for some $o_{\varepsilon,\nu}^+ > 0$ and $o_{\varepsilon,\nu}^- < 0$ satisfying $\lim_{\nu\to 0} o_{\varepsilon,\nu}^\pm = o_\varepsilon^\pm$ and $o_\varepsilon^\pm \to 0$ as $\varepsilon \to 0$.

For $\nu = 0$ we identify $\Phi_{\varepsilon,0}^\pm$ with $\Phi_\varepsilon^\pm$ given by (2.12). Hence the one-parameter family $(\Phi_{\varepsilon,\nu}^\pm)_\nu$ can be regarded as a "perturbation" of $\Phi_\varepsilon^\pm$, which converges to $\Phi_\varepsilon^\pm$ as ν goes to 0.

Lemma 2.1. *For any small $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that $\Phi_{\varepsilon,\nu}^+$ (respectively, $\Phi_{\varepsilon,\nu}^-$) is a radial super-solution (respectively, sub-solution) of (1.1) in $B_{r_\varepsilon} \setminus \{0\}$ for every $\nu \in [0,\nu_0]$.*

Proof. We fix $\varepsilon > 0$ sufficiently small. By (1.5), there exists $r_\varepsilon > 0$ small such that

$$(1-\varepsilon) b_0(|x|) \le b(x) \le (1+\varepsilon) b_0(|x|) \quad \text{for every } 0 < |x| \le r_\varepsilon.$$

By reducing $r_\varepsilon > 0$ if needed, we will show that for any $\nu \in [0,\nu_0]$, the function $v = \Phi_{\varepsilon,\nu}^+$ satisfies

$$-(r^{N-1}|v_r|^{p-2} v_r)_r + (1-\varepsilon) r^{N-1} b_0(r) h(v(r)) \ge 0 \quad \text{for } r \in (0,r_\varepsilon). \tag{2.15}$$

This clearly implies that $\Phi^+_{\varepsilon,\nu}$ is a super-solution of (1.1) in $B_{r_\varepsilon} \setminus \{0\}$. Since $h \in RV_q$, there exists a function L which varies slowly at ∞ such that $h(t) = t^q L(t)$ for $t > 0$ large enough. From (2.14) and Proposition A.2, it follows that

$$\lim_{t\to\infty} \frac{L(\eta^\pm_{\varepsilon,\nu} t)}{L(t)} = 1 \quad \text{uniformly with respect to } \nu \in [0, \nu_0],$$

provided that $\nu_0 > 0$ is small enough. So, by taking $t_\varepsilon > 0$ large enough, the ratio $L(\eta^\pm_{\varepsilon,\nu} t)/L(t)$ is bounded below by $1 - \varepsilon$ for all $t \geq t_\varepsilon$ and every $\nu \in [0, \nu_0]$. Since $\Upsilon(r) \to \infty$ as $r \to 0$, we can reduce $r_\varepsilon > 0$ such that $\Upsilon(r) \geq t_\varepsilon$ for all $r \in (0, r_\varepsilon)$. By (2.11) and (2.13), we get

$$h(\Phi^+_{\varepsilon,\nu}(r)) \geq (1-\varepsilon)(\eta^+_{\varepsilon,\nu})^q h(\Upsilon_\nu(r)) \quad \text{for every } r \in (0, r_\varepsilon) \text{ and any } \nu \in [0, \nu_0].$$

Hence to prove (2.15) for $v = \Phi^+_{\varepsilon,\nu}$, it suffices to show that for every $\nu \in [0, \nu_0]$, we have

$$(r^{N-1}|v_r|^{p-2}v_r)_r \leq (1-\varepsilon)^2 (\eta^+_{\varepsilon,\nu})^q r^{N-1} b_0(r) h(\Upsilon_\nu(r)) \quad \text{for every } r \in (0, r_\varepsilon). \tag{2.16}$$

Let $\mathcal{J}$, $\mathcal{B}$ and $\mathcal{F}$ be given by (A.6) and (A.8). For small $r > 0$, we set

$$\mathcal{P}_\nu(r) := -N - \frac{r b_0'(r)}{b_0(r)} + (1+\nu)\mathcal{B}(r)\mathcal{J}(\Upsilon_\nu(r))\mathcal{F}(\Upsilon_\nu(r)) - \nu(p-1)\mathcal{B}(r). \tag{2.17}$$

Using (2.10) and (2.13), after some calculations, we find that for $v = \Phi^+_{\varepsilon,\nu}$ the left-hand side of (2.16) is given by

$$[(1+\nu)\eta^+_{\varepsilon,\nu}]^{p-1}\mathcal{P}_\nu(r)\left[\int_0^r (s b_0(s))^{\frac{1}{p-1}}\, ds\right]^{\nu(p-1)} r^{N-1} b_0(r) h(\Upsilon_\nu(r)). \tag{2.18}$$

In view of (2.17), we write $\mathcal{P}_\nu(r) = T_{1,\nu}(r) + \nu T_{2,\nu}(r)$, where $T_{1,\nu}(r)$ is given by

$$T_{1,\nu}(r) := -N - \frac{r b_0'(r)}{b_0(r)} + \mathcal{B}(r)\mathcal{J}(\Upsilon_\nu(r))\mathcal{F}(\Upsilon_\nu(r)).$$

From (2.11) and the convergence properties in (A.7) and (A.8), we deduce that as $r \to 0$ the function $T_{1,\nu}(r)$ (respectively, $T_{2,\nu}(r)$) converges to η^{q-p+1} (respectively, $\frac{(\theta+p)(p-1)}{q-p+1}$), uniformly with respect to $\nu \in [0, \nu_0]$. Hence, there exists $r_\varepsilon > 0$ such that for every $\nu \in [0, \nu_0]$

$$0 < \mathcal{P}_\nu(r) \leq \eta^{q-p+1} + o^+_{\varepsilon,\nu} \quad \text{for every } r \in (0, r_\varepsilon), \tag{2.19}$$

where $o^+_{\varepsilon,\nu} > 0$ satisfies $\lim_{\nu\to 0} o^+_{\varepsilon,\nu} = o^+_\varepsilon > 0$ and $o^+_\varepsilon \to 0$ as $\varepsilon \to 0$. From (2.19) and (2.14), we find that the quantity in (2.18) is bounded above by the right-hand side of (2.16). This ends the proof of (2.15) for $v = \Phi^+_{\varepsilon,\nu}$. One can similarly check that $v = \Phi^-_{\varepsilon,\nu}$ satisfies the reverse inequality in (2.15) (i.e., "$\leq$" instead of "$\geq$") with $-\varepsilon$ replaced by $+\varepsilon$. Since the argument follows the same ideas as for (2.15), we omit the details. This completes the proof of Lemma 2.1.

3. A priori estimates and Harnack inequality

In this section, we assume that (1.4) and (1.5) hold with $1 < p \leq N$ and $q > p - 1$. Note that here we do not impose any upper bound restriction on q. We first extend Lemma 2.1 and Lemma 2.2 in [7], where the special case $b \equiv 1$ and $h(t) = |t|^{q-1}t$ is treated. In Lemma 3.1 we prove that every positive sub-solution of (1.1) satisfies a priori estimates of the type (3.1), which will be used to derive a Harnack inequality for positive solutions u of (1.1). If $\lim_{|x|\to 0} \frac{u(x)}{\mu(x)} = 0$, then we show that u can be extended as a continuous solution of (1.1) in all Ω (cf. Lemma 3.2).

Lemma 3.1. *Fix $r_0 > 0$ such that $B_{2r_0}(0) \subset\subset \Omega$. Then there exist positive constants C_1 and C_2 (which depend on r_0) such that*

(a) *(A priori estimates) For every positive sub-solution u of* (1.1)*, we have*

$$u(x) \le C_1 \Upsilon(|x|) \quad \textit{for every } 0 < |x| \le r_0. \tag{3.1}$$

(b) *(Harnack-type inequality) For every positive solution u of* (1.1)*, it holds*

$$\max_{|x|=r} u(x) \le C_2 \min_{|x|=r} u(x) \quad \textit{for all } 0 < r \le r_0/2. \tag{3.2}$$

Proof. Without any loss of generality, we can take h to be increasing on $(0,\infty)$ (see Remark A.2). Using the convention in Remark A.4, we may assume that (A.8) holds.

To conclude (3.1), it is enough to prove that there exists a constant $C > 0$ such that

$$\int_{u(x)}^{\infty} \frac{dt}{[h(t)]^{\frac{1}{p-1}}} \ge [C|x|^p b(x)]^{\frac{1}{p-1}} \quad \text{for every } 0 < |x| \le r_0. \tag{3.3}$$

Then we can find a large constant $C_1 > 0$ such that

$$\frac{[C|x|^p b(x)]^{\frac{1}{p-1}}}{\int_{C_1\Upsilon(|x|)}^{\infty} [h(t)]^{-\frac{1}{p-1}}\, dt} \ge 1 \quad \text{for every } 0 < |x| \le r_0. \tag{3.4}$$

Indeed, by (A.7) and (1.11), it follows that as $|x| \to 0$, the left-hand side of (3.4) converges to $C^{\frac{1}{p-1}} C_1^{\frac{q-p+1}{p-1}} \left(\frac{\theta+p}{p-1}\right)$. Hence, by choosing a suitable large constant $C_1 > 0$, the inequality in (3.4) holds for $|x| \in (0,\varepsilon)$ and some $\varepsilon > 0$. Then for $|x| \in [\varepsilon, r_0]$, the inequality in (3.4) holds by possibly enlarging $C_1 > 0$ (since $\int_{C_1\Upsilon(r_0)}^{\infty} [h(t)]^{-\frac{1}{p-1}}\, dt \to 0$ as $C_1 \to \infty$). This proves (3.4) for some constant $C_1 > 0$ sufficiently large. By combining (3.3) and (3.4), we reach (3.1).

We now prove (3.3). Fix $x_0 \in \mathbb{R}^N$ with $0 < |x_0| \le r_0$. We set $p' = p/(p-1)$ and define

$$\zeta(x) := 1 - \left(\frac{2|x - x_0|}{|x_0|}\right)^{p'} \quad \text{for } x \in B_{\frac{|x_0|}{2}}(x_0).$$

We have $\zeta(x_0) = 1$ and $0 < \zeta \le 1$ in $B_{|x_0|/2}(x_0)$. For some $C > 0$, we define S as follows

$$\int_{S(x)}^{\infty} \frac{dt}{[h(t)]^{\frac{1}{p-1}}} = [C|x_0|^p b(x_0)]^{\frac{1}{p-1}}\, [\zeta(x)]^{p'} \quad \text{for } x \in B_{\frac{|x_0|}{2}}(x_0). \tag{3.5}$$

The right-hand side of (3.5) equals zero for $x \in \partial B_{|x_0|/2}(x_0)$. Hence $S = \infty$ on $\partial B_{|x_0|/2}(x_0)$. We shall choose in (3.5) a constant $C > 0$, which is independent of x_0, such that S satisfies

$$-\mathrm{div}(|\nabla S|^{p-2}\nabla S) + b(x)h(S) \ge 0 \quad \text{in } B_{|x_0|/2}(x_0). \tag{3.6}$$

Then we can apply the comparison principle (see Lemma A.8 in Appendix A) to deduce that

$$u(x) \le S(x) \quad \text{for every } x \in B_{|x_0|/2}(x_0). \tag{3.7}$$

Using $x = x_0$ in (3.7) and (3.5), we get the inequality in (3.3) with $x = x_0$. This proves (3.3) since x_0 is arbitrarily fixed with $0 < |x_0| \le r_0$. To end our proof, we need to show (3.6).

Proof of (3.6). Using (1.5) and Proposition A.2, we can find a constant $c > 0$ such that

$$b(x_0) \leq c\, b(x) \quad \text{for every } x, x_0 \text{ such that } 0 < |x_0| \leq r_0 \text{ and } |x_0|/2 \leq |x| \leq 3|x_0|/2. \tag{3.8}$$

We next show that S defined by (3.5) satisfies

$$\operatorname{div}(|\nabla S|^{p-2}\nabla S) \leq Cc(p')^{2(p-1)}2^p \left[N + (p')^2 \mathcal{J}(S)\mathcal{F}(S)\right] b(x)h(S), \quad \forall x \in B_{\frac{|x_0|}{2}}(x_0), \tag{3.9}$$

where $\mathcal{J}$ and $\mathcal{F}$ are given by (A.6) and (A.8), respectively. Using (3.5), we obtain

$$\nabla S = (p')^2 \left\{2^p Cb(x_0)h(S)|x-x_0|^{2-p}\zeta(x)\right\}^{\frac{1}{p-1}} (x - x_0) \quad \text{in } B_{|x_0|/2}(x_0). \tag{3.10}$$

Hence, using (3.5), (A.6) and (A.8), it follows that

$$\frac{h'(S)}{h(S)}\zeta(x)\nabla S \cdot (x - x_0) = (p')^2(1 - \zeta(x))\mathcal{J}(S)\mathcal{F}(S) < (p')^2\mathcal{J}(S)\mathcal{F}(S) \quad \text{in } B_{|x_0|/2}(x_0). \tag{3.11}$$

By (3.10), we find that the left-hand side of (3.9) equals

$$C(p')^{2(p-1)}2^p \left[N\zeta(x) + \frac{h'(S)}{h(S)}\zeta(x)\nabla S \cdot (x - x_0) - p'(1 - \zeta(x))\right] b(x_0)h(S). \tag{3.12}$$

Using (3.11), (3.8) and $0 < \zeta \leq 1$ in $B_{|x_0|/2}(x_0)$, we obtain that the quantity in (3.12) is bounded above by the right-hand side of (3.9). This concludes the proof of (3.9).

From (1.5), we have $\lim_{|x|\to 0} |x|^p b(x) = 0$ so that $\sup_{0<|x|\leq r_0} |x|^p b(x) < \infty$. From the definition of S in (3.5), the minimum of S on the ball $B_{|x_0|/2}(x_0)$ can be made as large as desired by choosing a sufficiently small constant $C > 0$, which is independent of x_0. From (A.7) and (A.8), we have $\lim_{t\to\infty} \mathcal{J}(t)\mathcal{F}(t) = \frac{q(p-1)}{q-p+1}$. Using (3.9), we see that (3.6) holds for a small positive constant C that is independent of x_0. This proves the claim of (a).

(b) We rewrite the equation (1.1) in the form

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + [b_2(x)]^p u^{p-1} = 0 \quad \text{for } 0 < |x| < r_0, \tag{3.13}$$

where $b_2(x)$ is a positive function defined by

$$[b_2(x)]^p := \frac{b(x)h(u(x))}{[u(x)]^{p-1}} \quad \text{for every } x \in \mathbb{R}^N \text{ with } 0 < |x| < r_0. \tag{3.14}$$

Using (3.1), (3.14) and (A.3) in Lemma A.7, we find

$$|x|^p[b_2(x)]^p \leq (C_1)^{1-p}|x|^p b(x)\frac{h_2(C_1\Upsilon(|x|))}{[\Upsilon(|x|)]^{p-1}} \quad \text{for every } 0 < |x| \leq r_0. \tag{3.15}$$

By (1.11), (A.7) and Remark A.2, we find that as $|x| \to 0$, the right-hand side of (3.15) converges to $(C_1)^{q-p+1}\left(\frac{\theta+p}{q-p+1}\right)^{p-1}$. Hence, for some constant $A > 0$, we have

$$|x|^p[b_2(x)]^p \leq A \quad \text{for all } 0 < |x| \leq r_0. \tag{3.16}$$

Fix $x_0 \in \mathbb{R}^N$ such that $0 < |x_0| \leq r_0/2$. By applying the Harnack inequality (Theorem 1.1) of Trudinger [16] for (3.13) on $B_{|x_0|/2}(x_0)$, there exists a constant $c_0 > 0$ depending only on p, N and $|x_0|\|b_2(x)\|_{L^\infty(B_{|x_0|/2}(x_0))}$ such that

$$\sup_{x\in B_{|x_0|/6}(x_0)} u(x) \leq c_0 \inf_{x\in B_{|x_0|/6}(x_0)} u(x). \tag{3.17}$$

Using (3.16), we derive that $|x_0|\|b_2(x)\|_{L^\infty(B_{|x_0|/2}(x_0))}$ is bounded above by $2A^{1/p}$, which is independent of x_0. Hence $c_0 = c_0(p, N, A) > 0$ is independent of x_0 with $0 < |x_0| \le r_0/2$. To deduce (3.2), we use a standard covering argument as in [7]. If x_1 and x_2 are any points in $\mathbb{R}^N$ such that $0 < |x_1| = |x_2| \le r_0/2$, then x_1 and x_2 can be joined by 10 overlapping balls of radius $|x_1|/6$ with centers on $\partial B_{|x_1|}(0)$. By (3.17), we obtain (3.2) with $C_2 = c_0^{10}$.

Lemma 3.2. *Let u be a positive solution of* (1.1) *and* $\gamma := \limsup_{|x|\to 0} u(x)/\mu(x)$.

(i) *If $\gamma \neq 0$, then $\lim_{|x|\to 0} u(x) = \infty$;*
(ii) *If $\gamma = 0$, then $\lim_{|x|\to 0} u(x)$ is finite, and u can be extended as a continuous solution of* (1.1) *in all Ω.*

Proof. (i) Clearly, $\gamma \neq 0$ implies that $\limsup_{|x|\to 0} u(x) = \infty$. Suppose by contradiction that $d_0 := \liminf_{|x|\to 0} u(x) < \infty$. Then there exists a sequence $\{x_n\}_{n\ge 1}$ in $\mathbb{R}^N$ which converges to zero such that $\lim_{n\to\infty} u(x_n) = d_0$. Without loss of generality, we can take $|x_n|$ to be decreasing to zero as $n \to \infty$ and $0 < |x_n| \le r_0/2$ for some $r_0 > 0$ small such that $B_{2r_0}(0) \subset\subset \Omega$. Let $n_0 \ge 1$ be large enough such that $u(x_n) \le d_0 + 1$ for every $n \ge n_0$. By Lemma 3.1 (b), there exists a constant $C_2 > 0$ such that (3.2) holds. Thus we obtain that

$$\max_{|x|=|x_n|} u(x) \le C_2 \min_{|x|=|x_n|} u(x) \le C_2\, u(x_n) \le C_2(d_0 + 1), \quad \forall n \ge n_0.$$

Since $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \le 0$ for $0 < |x| < |x_{n_0}|$, by the weak maximum principle for the p-Laplace operator ([8]) applied on $\{x \in \mathbb{R}^N : \ |x_n| < |x| < |x_{n_0}|\}$ with $n > n_0$, we find $u(x) \le C_2(d_0 + 1)$ for all $0 < |x| \le |x_{n_0}|$. This is a contradiction with $\limsup_{|x|\to 0} u(x) = \infty$.

(ii) Let u satisfy $\lim_{|x|\to 0} u(x)/\mu(x) = 0$. We rewrite the equation (1.1) in the form

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + d(x)u^{p-1} = 0 \quad \text{in } \Omega, \tag{3.18}$$

where $d(x) := b(x)h(u)/u^{p-1}$ for $x \in \Omega$. Let $r_0 > 0$ be small such that $B_{r_0}(0) \subset\subset \Omega$. We first prove that $\limsup_{|x|\to 0} u(x) < \infty$. We set $C := \max_{|x|=r_0} u(x)$. For any integer $n \ge 1$, we define

$$v_n(x) := (1/n)\mu(x) + C \quad \text{for every } 0 < |x| \le r_0.$$

Since $\gamma = 0$, we see that for any integer $n \ge 1$, there exists $r_n > 0$ such that $u(x) \le v_n(x)$ for every $x \in \mathbb{R}^N$ with $0 < |x| \le r_n$. We may assume that $\{r_n\}_{n\ge 1}$ decreases to zero and $r_n < r_0$ for every $n \ge 1$. Set $Q_n := \{x \in \mathbb{R}^N : r_n < |x| < r_0\}$. Then we have

$$u \le v_n \text{ on } \partial Q_n \quad \text{and} \quad -\mathrm{div}(|\nabla u|^{p-2}\nabla u) \le 0 = -\mathrm{div}(|\nabla v_n|^{p-2}\nabla v_n) \text{ in } Q_n.$$

By the maximum principle, we find $u \le v_n$ in Q_n for any $n \ge 1$. For $x \in \mathbb{R}^N$ with $0 < |x| < r_0$, we have $u(x) \le v_n(x)$ for all $n \ge 1$ sufficiently large. Since $\lim_{n\to\infty} v_n(x) = C$, we conclude that $u(x) \le C$ for $0 < |x| \le r_0$. By (1.5), we find that $b(x) \in L^{\frac{N}{p-\varepsilon}}(B_{r_0}(0))$ for some small $\varepsilon > 0$. Using Assumption A, it follows that $d(x) \in L^{\frac{N}{p-\varepsilon}}(B_{r_0}(0))$. We can then apply Theorem 1 of Serrin [14] to the solution u of (3.18) and conclude the assertion of (ii).

4. A regularity result

Our aim is to extend the regularity result of Lemma 1.1 in [7] on (1.3) to equations of the form (1.1). We let $r_0 > 0$ be small such that $B_{4r_0}(0) \subset\subset \Omega$ and let g be a positive continuous function defined on $(0, 4r_0]$. We prove here the following result.

Lemma 4.1. *Let* (1.4) *and* (1.5) *hold for* $q > p-1$ *and* $p > 1$. *Assume that* $N > 1$ *and* $0 \le \delta \le \frac{\theta+p}{q+1-p}$. *Let* $g \in RV_{-\delta}(0+)$ *satisfy* $\limsup_{r\to 0} \frac{g(r)}{\Upsilon(r)} < \infty$, *where* Υ *is defined by* (1.11).

If u *is a positive solution of* (1.1) *such that, for some constant* $C_1 > 0$,

$$0 < u(x) \le C_1 g(|x|) \quad \textit{for } 0 < |x| < 2r_0, \tag{4.1}$$

then there exist constants $C > 0$ *and* $\alpha \in (0,1)$ *such that*

$$|\nabla u(x)| \le C\frac{g(|x|)}{|x|} \quad \textit{and} \quad |\nabla u(x) - \nabla u(x')| \le C\frac{g(|x|)}{|x|^{1+\alpha}}|x-x'|^{\alpha}, \tag{4.2}$$

for any x, x' *in* $\mathbb{R}^N$ *satisfying* $0 < |x| \le |x'| < r_0$.

Remark 4.1. (i) If $1 < p \le N$ in Lemma 4.1, then there exists a constant $C_1 > 0$ such that (4.1) holds with $g \equiv \Upsilon$ for *every* positive solution u of (1.1) (cf., Lemma 3.1).

(ii) If $g \in RV_{-\delta}(0+)$ with $0 \le \delta < \frac{\theta+p}{q+1-p}$, then $\lim_{r\to 0} \frac{g(r)}{\Upsilon(r)} = 0$ since $\Upsilon \in RV_{-\frac{\theta+p}{q+1-p}}(0+)$.

Proof. We use a line of thought similar to Lemma 1.1 of [7] based upon a $C^{1,\alpha}$-regularity result of Tolksdorf [15] applied to nonlinear degenerate elliptic equations of the form

$$-\mathrm{div}(|\nabla\Psi|^{p-2}\nabla\Psi) + B = 0 \quad \text{in } \Gamma, \quad \text{where } \Gamma := \{y \in \mathbb{R}^N : \ 1 < |y| < 7\} \tag{4.3}$$

and $B \in L^\infty(\Gamma)$. If $\Psi \in L^\infty(\Gamma) \cap W^{1,p}(\Gamma)$ is a weak solution of (4.3), then there exist constants $\alpha = \alpha(N,p) \in (0,1)$ and $\widetilde{C} = \widetilde{C}(N, p, \|\Psi\|_{L^\infty(\Gamma)}, \|B\|_{L^\infty(\Gamma)}) \ge 0$ such that

$$\|\nabla\Psi\|_{C^{0,\alpha}(\Gamma^*)} \le \widetilde{C}, \quad \text{where } \Gamma^* := \{y \in \mathbb{R}^N : \ 2 < |y| < 6\}. \tag{4.4}$$

For every $\beta \in (0, r_0/6)$, we define Ψ_β on Γ as follows

$$\Psi_\beta(\xi) := \frac{u(\beta\xi)}{g(\beta)} \quad \text{for } \xi \in \Gamma. \tag{4.5}$$

It follows that

$$\nabla u(x) = \frac{g(\beta)}{\beta}\nabla\Psi_\beta(x/\beta) \quad \text{for all } x \in \{\beta\xi : \ \xi \in \Gamma\}. \tag{4.6}$$

Since u is a solution of (1.1), we see that Ψ_β satisfies the equation (4.3) with $B = B_\beta$ given by

$$B_\beta(\xi) := \frac{\beta^p}{[g(\beta)]^{p-1}} b(\beta\xi)h(u(\beta\xi)) \quad \text{for } \xi \in \Gamma. \tag{4.7}$$

We prove that there exists a constant $\widetilde{C} > 0$, which is independent of $\beta \in (0, r_0/6)$, such that

$$\|\nabla\Psi_\beta\|_{C^{0,\alpha}(\Gamma^*)} \le \widetilde{C}. \tag{4.8}$$

To this end, we check that Ψ_β and B_β are in $L^\infty(\Gamma)$ with their L^∞-norms bounded above by a positive constant that is independent of $\beta \in (0, r_0/6)$. Using (4.1) and (4.5), we find

$$\Psi_\beta(\xi) \le C_1\frac{g(\beta|\xi|)}{g(\beta)} \quad \text{for every } \xi \in \Gamma \text{ and all } \beta \in (0, r_0/6). \tag{4.9}$$

Since $g \in RV_{-\delta}(0+)$, we can write it as $g(t) = t^{-\delta}\mathcal{L}(t)$ for some function $\mathcal{L}$ that is continuous on $(0, 2r_0)$ and slowly varying at zero. Using Proposition A.2, we have

$$\lim_{\beta\to 0} \frac{\mathcal{L}(\beta|\xi|)}{\mathcal{L}(\beta)} = 1 \quad \text{uniformly with respect to } \xi \in \Gamma.$$

Hence there exist positive constants $\widehat{c}$ and $\widehat{C}$, which depend on r_0, such that

$$\widehat{c}g(\beta) \le g(\beta|\xi|) \le \widehat{C}g(\beta) \quad \text{for every } \beta \in (0, r_0/6) \text{ and every } \xi \in \Gamma. \tag{4.10}$$

Using (4.9), we obtain that $\Psi_\beta \in L^\infty(\Gamma)$ and $\|\Psi_\beta\|_{L^\infty(\Gamma)} \le C_1\widehat{C}$ for every $\beta \in (0, r_0/6)$.

We now prove $B_\beta \in L^\infty(\Gamma)$. Since $h(t)/t^{p-1}$ is bounded for small $t > 0$, in view of Remark A.2, we can find two positive constants a_1 and a_2 such that

$$h(t) \le a_1 t^{p-1} + a_2 h_1(t/C_1) \quad \text{for every } t > 0.$$

This, combined with (4.1) and the properties of h_1, leads to

$$h(u(\beta\xi)) \le a_1 C_1^{p-1}[g(\beta|\xi|)]^{p-1} + a_2 h(g(\beta|\xi|)).$$

Using the above inequality and (4.7), we obtain

$$B_\beta(\xi) \le \left(\frac{g(\beta|\xi|)}{g(\beta)}\right)^{p-1} \left[a_1 C_1^{p-1}(\beta|\xi|)^p b(\beta\xi) + a_2(\beta|\xi|)^p b(\beta\xi)\frac{h(g(\beta|\xi|))}{[g(\beta|\xi|)]^{p-1}}\right], \quad \forall \xi \in \Gamma. \tag{4.11}$$

We claim that in the right-hand side of (4.11), the quantity in square brackets is bounded above by a constant independent of $\beta \in (0, r_0/6)$. By $\lim_{|x|\to 0} |x|^p b(x) = 0$ and $\limsup_{r\to 0} \frac{g(r)}{\Upsilon(r)} < \infty$, we deduce that there exist constants $c_* = c_*(r_0) > 0$ and $c^* = c^*(r_0) > 0$ such that

$$|x|^p b(x) \le c_*, \quad |x|^p b(x)\frac{h(g(|x|))}{[g(|x|)]^{p-1}} \le c^* \quad \text{for every } x \in \mathbb{R}^N \text{ with } 0 < |x| < 2r_0. \tag{4.12}$$

Using (4.10) and (4.12) in (4.11), we arrive at

$$B_\beta(\xi) \le \widehat{C}^{p-1}(a_1 C_1^{p-1} c_* + a_2 c^*) \quad \text{for every } \beta \in (0, r_0/6) \quad \text{and every } \xi \in \Gamma.$$

Hence, $B_\beta \in L^\infty(\Gamma)$ and $\|B_\beta\|_{L^\infty(\Gamma)}$ is bounded above by a constant independent of β. We can thus apply the above regularity result of Tolksdorf [15] to obtain (4.8).

We are now ready to prove the inequalities in (4.2), where it suffices to take $0 < |x| < r_0/2$. Hence we can find $\beta \in (0, r_0/6)$ such that x belongs to the set $\{\beta\xi :\ \xi \in \Gamma^* \text{ and } |\xi| \le 3\}$. For x in this set, $x/\beta \in \Gamma^*$ and (4.6) holds. Using (4.6), (4.8) and (4.10), we conclude the first inequality in (4.2). To prove the second inequality, we first assume that $0 < |x| \le |x'| < 2|x|$. Then x'/β also belongs to Γ^*. By (4.6) and (4.8), we obtain

$$\beta|\nabla u(x) - \nabla u(x')| = g(\beta)|\nabla\Psi_\beta(x/\beta) - \nabla\Psi_\beta(x'/\beta)| \le \widetilde{C}g(\beta)\beta^{-\alpha}|x - x'|^\alpha.$$

Hence by (4.10) we reach the second inequality in (4.2). Finally, if $2|x| \le |x'| < r_0$, then

$$|x' - x| \ge |x'| - |x| \ge |x|. \tag{4.13}$$

Since $g(t)/t$ belongs to $RV_{-\delta-1}(0+)$, by Proposition A.6 (see also Definition 1.2), $g(t)/t$ behaves near zero as a monotone decreasing function. By the first inequality of (4.2) and (4.13), we find

$$|\nabla u(x) - \nabla u(x')| \le C\left(\frac{g(|x|)}{|x|} + \frac{g(|x'|)}{|x'|}\right) \le C'\frac{g(|x|)}{|x|} \le C'\frac{g(|x|)}{|x|^{\alpha+1}}|x' - x|^\alpha,$$

where $C' > 0$ denotes a large constant. This completes the proof of (4.2).

5. Solutions without strong singularities at zero

Theorem 1.1 of Friedman and Véron [7] on (1.3) is extended below to equations like (1.1).

Theorem 5.1. *Let* (1.4) *and* (1.5) *hold for* $1 < p \leq N$ *and* $p - 1 < q < C_{N,p,\theta}$. *Assume that* u *is a positive solution of* (1.1) *such that* $\gamma := \limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} \neq \infty$. *Then we have:*

(a) *either* $u(x)$ *admits a finite limit at zero and* $u(x)$ *can be extended as a continuous solution of* (1.1) *in the whole* Ω*;*

(b) *or* $u(x)/\mu(x)$ *converges to* $\gamma \in (0, \infty)$ *as* $|x| \to 0$ *and*

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + b(x)h(u) = \gamma^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{5.1}$$

Proof. If $\gamma = 0$, then by Lemma 3.2 we conclude the alternative (a). We now assume that $\gamma \in (0, \infty)$ and prove that (b) occurs. We only give the details when $1 < p < N$, since the case $p = N$ follows a similar line of argument to Theorem 1.1 in [7]. Let $r_0 > 0$ be small such that $B_{2r_0}(0) \subset\subset \Omega$. Since $\gamma \in (0, \infty)$, there exists a positive constant $C_1 = C_1(r_0)$ such that

$$u(x) \leq C_1\mu(x) \quad \text{for every } 0 < |x| \leq 2r_0. \tag{5.2}$$

We take $g(|x|) := \mu(|x|)$ so that $g \in RV_{-\delta}(0+)$ with $\delta = \frac{N-p}{p-1}$. Since $1 < p < N$ and $q < C_{N,p,\theta}$, we find $0 < \delta < \frac{\theta+p}{q+1-p}$. By (5.2) and Remark 4.1, the assumptions of Lemma 4.1 are satisfied. Hence there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that for any x, x' with $0 < |x| \leq |x'| < r_0$,

$$|\nabla u(x)| \leq C\mu(1)|x|^{-\delta-1}, \quad |\nabla u(x) - \nabla u(x')| \leq C\mu(1)|x|^{-\delta-1-\alpha}|x - x'|^{\alpha}. \tag{5.3}$$

For $r \in (0, r_0)$ fixed, we now define the function

$$V_{(r)}(\xi) := \frac{u(r\xi)}{\mu(r)} \quad \text{for } 0 < |\xi| < \frac{r_0}{r}. \tag{5.4}$$

We shall prove below that

$$\lim_{r\to 0} V_{(r)}(\xi) = \frac{\gamma}{\mu(1)}\mu(\xi), \quad \lim_{r\to 0} \nabla V_{(r)}(\xi) = \frac{\gamma}{\mu(1)}\nabla\mu(\xi) \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.5}$$

To this end, we first show that $\lim_{r\to 0} \widetilde{\gamma}(r) = \gamma$, where

$$\widetilde{\gamma}(r) := \sup_{|x|=r} \frac{u(x)}{\mu(x)} \quad \text{for } r \in (0, r_0). \tag{5.6}$$

Since clearly $\limsup_{r\to 0} \widetilde{\gamma}(r) = \gamma$, it suffices to show that $\liminf_{r\to 0} \widetilde{\gamma}(r) = \gamma$. Assuming the contrary, there exists a decreasing sequence r_n that converges to 0 such that $\widetilde{\gamma}(r_n) \to \gamma_0 \in [0, \gamma)$. Let $\varepsilon > 0$ be small such that $\gamma_0 + \varepsilon < \gamma$. Hence there exists a large $n_0 \geq 1$ such that for every $n \geq n_0$, we have $\widetilde{\gamma}(r_n) \leq \gamma_0 + \varepsilon$. For each $n > n_0$, we define the set Q_n by

$$Q_n := \{x \in \mathbb{R}^N : r_n < |x| < r_{n_0}\}.$$

Since $\limsup_{r\to 0} \widetilde{\gamma}(r) = \gamma$, there exists $r_* > 0$ small such that $\gamma_0 + \varepsilon < \widetilde{\gamma}(r_*)$. Choose $n > n_0$ large such that $r_* \in Q_n$. Since $\widetilde{\gamma}(r_*)$ is greater than the maximum of $\widetilde{\gamma}$ over the boundary of Q_n, we find that u/μ achieves its maximum over Q_n in the interior of Q_n and $u/\mu \neq$ const. in Q_n. This is a contradiction to Remark A.3 in Section A.2. Hence, $\lim_{r\to 0}\widetilde{\gamma}(r) = \gamma$.

We now set to prove (5.5). This will involve an estimate, a compactness argument and the use of the strong maximum principle. It is easily seen that $V_{(r)}(\xi)$ in (5.4) satisfies the equation

$$-\mathrm{div}(|\nabla V_{(r)}(\xi)|^{p-2}\nabla V_{(r)}(\xi)) + [\mu(1)]^{1-p} r^N b(r\xi)h(u(r\xi)) = 0 \quad \text{for } 0 < |\xi| < r_0/r. \tag{5.7}$$

We start with an estimate for the second term in (5.7), namely

$$\lim_{r\to 0} r^N b(r\xi)h(u(r\xi)) = 0 \quad \text{for every fixed } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.8}$$

Using (1.5), Lemma A.7 and (5.2), we find that (5.8) holds if we can prove

$$\mathcal{T}(r) := r^N b_0(r|\xi|)h_2(C_1\mu(r\xi)) \to 0 \text{ as } r \to 0. \tag{5.9}$$

We observe that $r \longmapsto \mathcal{T}(r)$ is regularly varying at zero with index $N + \theta - \frac{q(N-p)}{p-1}$, and this index is positive by our assumption that $q < C_{N,p,\theta}$. Hence (5.9) holds, which proves (5.8).

Next we use a compactness argument to show that $V_{(r)}$ converges along a sequence $r_n \to 0$. From (5.2) and (5.3), it follows that for every fixed $r \in (0, r_0)$, we have

$$\begin{cases} 0 < V_{(r)}(\xi) \le C_1|\xi|^{-\delta}, \quad |\nabla V_{(r)}(\xi)| \le C|\xi|^{-\delta-1}, \\ |\nabla V_{(r)}(\xi) - \nabla V_{(r)}(\xi')| \le C|\xi - \xi'|^{\alpha}|\xi|^{-\delta-1-\alpha}, \end{cases} \tag{5.10}$$

for every ξ and ξ' in $\mathbb{R}^N$ satisfying $0 < |\xi| \le |\xi'| < r_0/r$.

From (5.7), (5.8) and (5.10), we find that for any sequence $\bar{r}_n$ decreasing to zero, there exists a subsequence r_n such that $V_{(r_n)} \to V$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$, and V satisfies the equation

$$-\mathrm{div}(|\nabla V|^{p-2}\nabla V) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \setminus \{0\}).$$

We now use the strong maximum principle to show that the limit function V is given by

$$V(\xi) = \frac{\gamma}{\mu(1)}\mu(\xi) \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{5.11}$$

From (5.6), we can choose ξ_{r_n} on the $(N-1)$-dimensional unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ such that

$$\widetilde{\gamma}(r_n) = \frac{u(r_n\xi_{r_n})}{\mu(r_n)}.$$

Using $\mu(\xi)\mu(r_n) = \mu(1)\mu(r_n\xi)$ and (5.6), we find that

$$\frac{V_{(r_n)}(\xi)}{\mu(\xi)} \le \frac{\widetilde{\gamma}(r_n|\xi|)}{\mu(1)} \quad \text{for } 0 < |\xi| < \frac{r_0}{r_n} \quad \text{and} \quad \frac{V_{(r_n)}(\xi_{r_n})}{\mu(\xi_{r_n})} = \frac{\widetilde{\gamma}(r_n)}{\mu(1)}.$$

We may assume $\xi_0 = \lim_{n\to\infty}\xi_{r_n}$. Then from $\lim_{r\to 0}\widetilde{\gamma}(r) = \gamma$ we deduce

$$\frac{V(\xi)}{\mu(\xi)} \le \frac{\gamma}{\mu(1)} \text{ for every } \xi \in \mathbb{R}^N \setminus \{0\} \text{ and } \frac{V(\xi_0)}{\mu(\xi_0)} = \frac{\gamma}{\mu(1)}.$$

By Lemma A.9, we conclude (5.11). Hence, using $V_{(r_n)} \to V$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$, we find

$$\lim_{n\to\infty} V_{(r_n)}(\xi) = \frac{\gamma}{\mu(1)}\mu(\xi), \qquad \lim_{n\to\infty} \nabla V_{(r_n)}(\xi) = \frac{\gamma}{\mu(1)}\nabla\mu(\xi), \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Since $\{\bar{r}_n\}$ is an arbitrary sequence decreasing to zero, the above implies (5.5). Taking $|\xi| = 1$ and $x = r\xi$ in (5.5), we obtain that $\lim_{|x|\to 0} u(x)/\mu(x) = \gamma$ and the following

$$\lim_{|x|\to 0} -\frac{x \cdot \nabla u(x)}{|x|^{(p-N)/(p-1)}} = \gamma (N\omega_N)^{-1/(p-1)} := C_0. \tag{5.12}$$

To complete the proof of the theorem, it remains to show (5.1). Thus we need to verify that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + \int_\Omega b(x) h(u) \varphi \, dx = \gamma^{p-1} \varphi(0), \quad \forall \varphi \in C_c^1(\Omega). \tag{5.13}$$

We fix $\varphi \in C_c^1(\Omega)$. For each $\varepsilon > 0$ small, let $w_\varepsilon(r)$ be a non-decreasing and smooth function on $(0, \infty)$ such that $w_\varepsilon(r) = 1$ for $r \geq 2\varepsilon$, $w_\varepsilon(r) = 0$ for $r \in (0, \varepsilon]$, and $0 < w_\varepsilon(r) < 1$ for $r \in (\varepsilon, 2\varepsilon)$. Since $\varphi(x) w_\varepsilon(|x|) \in C_c^1(\Omega^*)$ we can use φw_ε as a test function in Definition 1. Hence,

$$\int_\Omega |\nabla u|^{p-2} w_\varepsilon \nabla u \cdot \nabla \varphi \, dx + \int_\Omega b(x) h(u) \varphi \, w_\varepsilon \, dx = -\int_\Omega |\nabla u|^{p-2} \varphi \nabla u \cdot \nabla w_\varepsilon \, dx. \tag{5.14}$$

Let $RHS\,(\varepsilon)$ denote the right-hand side of (5.14), that is

$$RHS\,(\varepsilon) = -\int_\Omega |\nabla u|^{p-2} \varphi \nabla u \cdot \nabla w_\varepsilon \, dx = -\int_{\{\varepsilon < |x| < 2\varepsilon\}} |\nabla u|^{p-2} \varphi \, w_\varepsilon'(|x|) \nabla u \cdot \frac{x}{|x|} \, dx. \tag{5.15}$$

We prove that for every $\tau > 0$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, we have

$$(\varphi(0) C_0^{p-1} - \tau) N\omega_N \leq RHS\,(\varepsilon) \leq (\varphi(0) C_0^{p-1} + \tau) N\omega_N, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{5.16}$$

Indeed, from (5.12) we find

$$-|\nabla u|^{p-2} \varphi(x) |x|^{N-2} \nabla u \cdot x \to \varphi(0) C_0^{p-1} \quad \text{as } |x| \to 0.$$

Thus for every $\tau > 0$ there exists $\varepsilon_0 = \varepsilon_0(\tau) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$\varphi(0) C_0^{p-1} - \tau \leq -|\nabla u|^{p-2} \varphi(x) |x|^{N-2} \nabla u \cdot x \leq \varphi(0) C_0^{p-1} + \tau \tag{5.17}$$

for every $\varepsilon < |x| < 2\varepsilon$. We now use I_ε to denote

$$I_\varepsilon := \int_{\{\varepsilon < |x| < 2\varepsilon\}} |x|^{1-N} w_\varepsilon'(|x|) \, dx.$$

It follows that

$$I_\varepsilon = N\omega_N \int_\varepsilon^{2\varepsilon} w_\varepsilon'(r) \, dr = N\omega_N.$$

Hence, using (5.15) and (5.17), we arrive at (5.16). Since $\tau > 0$ is arbitrary, by (5.16) and (5.12) we conclude that $\lim_{\varepsilon \to 0} RHS\,(\varepsilon) = \gamma^{p-1} \varphi(0)$. Thus (5.13) follows by letting $\varepsilon \to 0$ in (5.14). This completes the proof of Theorem 5.1.

6. Analysis of the power model

For later applications, we give here several results for the equation (1.1) in the power case $b(x) = |x|^\theta$ and $h(t) = t^q$ for $t > 0$.

Lemma 6.1. *Let $1 < p \leq N$ and $\theta > -p$. Assume that $p - 1 \leq q < C_{N,p,\theta}$. Let $R > 0$ be any positive number. Then for any non-negative numbers λ and γ, there exists a unique non-negative function $\Psi = \Psi_{\gamma,\lambda}$ in $C^1(0, R]$ satisfying*

$$\begin{cases} -(r^{N-1}|\Psi_r|^{p-2}\Psi_r)_r + r^{N-1+\theta}\Psi^q = 0 \quad \text{in } (0, R), \\ \lim_{r\to 0} \dfrac{\Psi(r)}{\mu(r)} = \gamma, \quad \Psi(R) = \lambda. \end{cases} \tag{6.1}$$

Moreover, $\lim_{r\to 0} \frac{\Psi_r(r)}{\mu_r(r)} = \gamma$ *and the function $\gamma \longmapsto \Psi_{\gamma,\lambda}$ is non-decreasing in γ.*

Proof. If $p = N$, we let $\ell := N/(\theta + N)$ and define

$$w(r) := \ell^{\frac{p}{q-p+1}}\Psi\left(r^\ell\right) \quad \text{for } 0 < r < R^{1/\ell}. \tag{6.2}$$

Then (6.1) with $p = N$ holds if and only if w satisfies

$$\begin{cases} -(r^{N-1}|w_r|^{p-2}w_r)_r + r^{N-1}w^q(r) = 0 \quad \text{in } (0, R^{1/\ell}), \\ \lim_{r\to 0} \dfrac{w(r)}{\mu(r)} = \ell^{\frac{q+1}{q-p+1}}\gamma, \quad w(R^{1/\ell}) = \ell^{\frac{p}{q-p+1}}\lambda. \end{cases} \tag{6.3}$$

Lemma 1.4 in [7] shows that for any $q \geq p - 1$, the problem (6.3) admits a unique solution w in $C^1(0, R^{1/\ell}]$ that also satisfies $\lim_{r\to 0} \frac{w_r(r)}{\mu_r(r)} = \ell^{\frac{q+1}{q-p+1}}\gamma$. Using (6.2), we conclude the proof.

For $1 < p < N$ the arguments of Lemma 1.4 in [7] can be easily modified to our situation and therefore we omit the details.

Remark 6.1. The solution $\Psi_{\gamma,\lambda}$ is positive in $(0, R)$, unless both γ and λ are zero in which case $\Psi = 0$ on $[0, R]$. As in Remark 1.3 in [7], the solution $\Psi(r)$ of (6.1) solves the following singular Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla\Psi|^{p-2}\nabla\Psi) + |x|^\theta\Psi^q = \gamma^{p-1}\delta_0 \quad \text{in } \mathcal{D}'(B_R(0)), \\ \Psi(x) = \lambda \ \text{ for } x \in \partial B_R(0). \end{cases}$$

If in Lemma 6.1 we assume that $p - 1 < q < C_{N,p,\theta}$, then there also exist solutions for the problem (6.1) with $\gamma = \infty$. More precisely, we prove the following.

Lemma 6.2. *Let $1 < p \leq N$ and $\theta > -p$. Assume that $p - 1 < q < C_{N,p,\theta}$. Let $R > 0$ be any positive number. Then for any non-negative number λ, there exists a non-negative function $\Psi = \Psi_{\infty,\lambda}$ in $C^1(0, R]$, which is positive in $(0, R)$ and satisfies*

$$\begin{cases} -(r^{N-1}|\Psi_r|^{p-2}\Psi_r)_r + r^{N-1+\theta}\Psi^q = 0 \quad \text{in } (0, R), \\ \lim_{r\to 0} \dfrac{\Psi(r)}{\mu(r)} = \infty, \quad \Psi(R) = \lambda \geq 0. \end{cases} \tag{6.4}$$

Furthermore, for every such solution $\Psi \in C^1(0, R]$, we have

$$\liminf_{r\to 0} r^{\frac{\theta+p}{q-p+1}}\Psi(r) > 0. \tag{6.5}$$

Proof. For every constant $\gamma \geq 1$, by Lemma 6.1 and Remark 6.1, the problem (6.1) admits a unique solution $\Psi_\gamma = \Psi_{\gamma,\lambda} \in C^1(0,R]$ and Ψ_γ is positive in $(0,R)$. By Lemma 3.1 (a) and the weak maximum principle (for p-subharmonic functions), there exists a large constant $C > 0$ such that for every $\gamma \geq 1$, we have

$$\Psi_\gamma(r) \leq C r^{-\frac{\theta+p}{q-p+1}} \text{ for all } r \in (0, R/3], \quad \Psi_\gamma(r) \leq C(R/3)^{-\frac{\theta+p}{q-p+1}} \quad \text{for all } r \in [R/3, R].$$

By the comparison principle, $\gamma \to \Psi_\gamma$ is increasing. Using Lemma 4.1, we deduce that $\Psi_{\gamma,\lambda} \to \Psi_{\infty,\lambda}$ in C^1 in every compact subset of $(0,R]$ as $\gamma \to \infty$ and $\Psi_{\infty,\lambda}$ satisfies (6.4).

We now prove (6.5). We note that the case $\theta = 0$ in (6.4) is covered by Lemma 2.3 of Friedman and Véron [7]. When $p = N$ then (6.4) can be transformed to (6.3) (with $\gamma = \infty$) by using the change of variable in (6.2). By applying Lemma 2.3 of [7] to w, we conclude that

$$\lim_{r\to 0} r^{\frac{p}{q-p+1}} w(r) = \ell^{\frac{p}{q-p+1}} \lim_{r\to 0} r^{\frac{\theta+p}{q-p+1}} \Psi(r) = \text{Const.} > 0.$$

Suppose now that $1 < p < N$. We will use a simple variant of the argument of Step 1 in the proof of Lemma 2.3 in [7]. We make the change of variable $s = r^{(p-N)/(p-1)}$ and $\varphi(s) = \Psi(r)$. To prove (6.5), we need to show that

$$\liminf_{s\to\infty} s^{\frac{p-1-C_{N,p,\theta}}{q-p+1}} \varphi(s) > 0. \tag{6.6}$$

It is easily checked that

$$\Psi_r(r) = \left(\frac{p-N}{p-1}\right) r^{\frac{1-N}{p-1}} \varphi_s(s), \tag{6.7}$$

and φ satisfies the equation

$$-(|\varphi_s|^{p-2}\varphi_s)_s + \left(\frac{p-1}{N-p}\right)^p s^{-1-C_{N,p,\theta}} \varphi^q = 0 \quad \text{in } [R^{\frac{p-N}{p-1}}, \infty). \tag{6.8}$$

Hence, $|\varphi_s|^{p-2}\varphi_s$ is increasing in s and one of the following holds:

$$\text{(i) } \lim_{s\to\infty} \varphi_s(s) = \beta < \infty, \quad \text{(ii) } \lim_{s\to\infty} \varphi_s(s) = \infty.$$

Case (i), jointly with (6.7), implies that $\lim_{r\to 0} \frac{\Psi(r)}{\mu(r)} = \frac{\beta}{\mu(1)} < \infty$, which is a contradiction with (6.4). Hence (ii) holds. It follows that $\varphi_s(s) > 0$ and $\varphi(s) \leq s\varphi_s(s)$ for large s. Consequently,

$$(\varphi_s^{p-1})_s \leq \left(\frac{p-1}{N-p}\right)^p s^{q-1-C_{N,p,\theta}} \varphi_s^q \quad \text{for all } s \geq s_0,$$

where $s_0 > 0$ is sufficiently large. Substituting $a(s) = \varphi_s^{p-1}(s)$, we obtain

$$a_s \leq \left(\frac{p-1}{N-p}\right)^p s^{q-1-C_{N,p,\theta}} a^{\frac{q}{p-1}} \quad \text{for all } s \geq s_0. \tag{6.9}$$

By (ii), we have $a(s) \to \infty$ as $s \to \infty$. Hence from (6.9), it follows that

$$\left(\frac{p-1}{q+1-p}\right) a(s)^{\frac{p-1-q}{p-1}} \leq \left(\frac{p-1}{N-p}\right)^p \frac{s^{q-C_{N,p,\theta}}}{C_{N,p,\theta} - q} \quad \text{for every } s \geq s_0.$$

Since $\varphi_s(s) = a^{\frac{1}{p-1}}(s)$, we find $\varphi_s(s) \geq c\, s^{\frac{C_{N,p,\theta}-q}{q-p+1}}$ for $s \geq s_0$, where $c > 0$ is a constant. Hence

$$\varphi(s) - \frac{c(q-p+1)}{C_{N,p,\theta} - p + 1} s^{\frac{-p+1+C_{N,p,\theta}}{q-p+1}}$$

is a non-decreasing function for $s \geq s_0$. This proves (6.6), which completes the proof.

Corollary 6.3. *Let $1 < p \leq N$ and $\theta > -p$. Assume that $p - 1 < q < C_{N,p,\theta}$. Let $R > 0$ and u be a positive super-solution of the equation*

$$-\mathrm{div}(|\nabla v|^{p-2}\nabla v) + |x|^\theta v^q = 0 \quad \text{in } B_R(0) \setminus \{0\}. \tag{6.10}$$

If $\lim_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$, then we have

$$\liminf_{|x|\to 0} |x|^{\frac{\theta+p}{q-p+1}} u(x) > 0. \tag{6.11}$$

Proof. Let u be a positive super-solution of (6.10) such that $\lim_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$. Let $\Psi_{\infty,0} \in C^1(0, R]$ denote the unique positive solution of (6.4) with $\lambda = 0$. By the construction of $\Psi_{\infty,0}$ in Lemma 6.2 and the comparison principle, we infer that $u(x) \geq \Psi_{\infty,0}(|x|)$ for $|x| \in (0, R)$. Since $\Psi = \Psi_{\infty,0}$ satisfies (6.5), we conclude (6.11).

Our next result will be useful in the proof of Theorem 1.3.

Lemma 6.4. *Let $1 < p \leq N$. Assume that $\theta > -p$ and $q \geq p - 1$.*

(i) *If $R > 0$ and $\Psi \in C^1(0, R)$ is a positive solution of*

$$-(r^{N-1}|\Psi_r|^{p-2}\Psi_r)_r + r^{N-1+\theta}\Psi^q = 0 \quad \text{in } (0, R), \tag{6.12}$$

then there exists $\lim_{r\to 0} \Psi(r)/\mu(r) \in [0, \infty]$.

(ii) *If we assume in addition that $p \neq N$ and $q \geq C_{N,p,\theta}$, then any positive solution $\Psi \in C^1(0, R)$ of (6.12) must satisfy $\lim_{r\to 0} \Psi(r)/\mu(r) = 0$.*

Proof. (i) We argue by contradiction. If $\Psi(r)/\mu(r)$ does not admit a limit in $[0, \infty]$ as $r \to 0$, then there exists $M > 0$ such that

$$\liminf_{r\to 0} \frac{\Psi(r)}{\mu(r)} < M < \limsup_{r\to 0} \frac{\Psi(r)}{\mu(r)}. \tag{6.13}$$

Let $(r_n)_{n\geq 1}$ be a sequence of positive numbers decreasing to zero such that $\Psi(r_n)/\mu(r_n)$ converges to $\liminf_{r\to 0} \Psi(r)/\mu(r)$ as $n \to \infty$. We can assume that $r_n < R$ and $\Psi(r_n) \leq M\mu(r_n)$ for every $n \geq 1$. By the comparison principle in Lemma A.8, we find $\Psi(r) \leq M\mu(r)$ for any $r \in (r_n, r_1)$ and every $n \geq 2$. Since $\lim_{n\to\infty} r_n = 0$, we obtain that $\Psi(r) \leq M\mu(r)$ for every $r \in (0, r_1)$. This being a contradiction with (6.13), we conclude the proof of (i).

(ii) We assume that $1 < p < N$, which implies that $C_{N,p,\theta}$ in (1.6) is finite. Let Ψ be an arbitrary positive $C^1(0, R)$-solution of (6.12). Set $\gamma := \lim_{r\to 0} \Psi(r)/\mu(r)$. We need to show that $\gamma = 0$ whenever $q \geq C_{N,p,\theta}$. By Lemma 3.1, there exists a constant $C > 0$ such that

$$\Psi(r) \leq Cr^{-\frac{\theta+p}{q+1-p}} \quad \text{for every } r > 0 \text{ small.} \tag{6.14}$$

If $q > C_{N,p,\theta}$, then (6.14) implies that $\gamma = 0$. When $q = C_{N,p,\theta}$, then by (6.14) and (i), we find $\gamma \in [0,\infty)$. As in the proof of Lemma 6.2, we set $s = r^{(p-N)/(p-1)}$ and $\varphi(s) = \Psi(r)$. Hence, $\lim_{s\to\infty} \varphi(s)/s = \gamma\mu(1)$ and (6.8) holds with $q = C_{N,p,\theta}$, that is

$$(|\varphi_s|^{p-2}\varphi_s)_s = \left(\frac{p-1}{N-p}\right)^p s^{-1}\left(\frac{\varphi(s)}{s}\right)^{C_{N,p,\theta}} \qquad \text{for } s \in (R^{\frac{p-N}{p-1}}, \infty). \tag{6.15}$$

Thus φ_s is increasing for $s > R^{\frac{p-N}{p-1}}$. If we assume that $\gamma \in (0,\infty)$, then $\lim_{s\to\infty} \varphi_s(s) = \lim_{s\to\infty} \varphi(s)/s = \gamma\mu(1)$. By integrating the right-hand (respectively, left-hand) side of (6.15) over $(R^{\frac{p-N}{p-1}}, t)$ and letting $t \to \infty$, we obtain ∞ (respectively, a finite quantity). This contradiction shows that $\gamma = 0$, which concludes the proof.

7. Proof of Theorem 1.1

In this section we assume that (1.4) and (1.5) hold with $1 < p \le N$ and $p - 1 < q < C_{N,p,\theta}$. Let u be a positive solution of (1.1). We conclude either (i_2) or (i_3) of Theorem 1.1 by invoking Theorem 5.1 whenever $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} \ne \infty$. Assuming Facts 1 and 2, we proved in Theorem 1.4 that (i_1) of Theorem 1.1 holds when $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$. Since Fact 1 has been proved, to complete the proof of Theorem 1.4 we need only show that Fact 2 is valid.

Lemma 7.1 (Fact 2)**.** *If u is a positive solution of* (1.1) *with* $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$, *then*

$$\lim_{|x|\to 0} \frac{u(x)}{f(|x|)} = \infty \quad \textit{for every } f \in RV_\sigma(0+) \textit{ with } \sigma > -\frac{\theta+p}{q-p+1}. \tag{7.1}$$

Proof. Since $p - 1 < q < C_{N,p,\theta}$, we can choose θ_* and q_* (close to θ and q) such that

$$-p < \theta_* < \theta, \quad q < q_* < C_{N,p,\theta_*} \quad \text{and} \quad \sigma > -\frac{\theta_*+p}{q_*-p+1} > -\frac{\theta+p}{q-p+1}. \tag{7.2}$$

Using (7.2) and Proposition A.3 (ii), we see that to prove (7.1) it is enough to show that

$$\liminf_{|x|\to 0} |x|^{\frac{\theta_*+p}{q_*-p+1}} u(x) > 0. \tag{7.3}$$

Our choice of θ_* and q_* ensures that u is a super-solution for the equation

$$-\mathrm{div}(|\nabla v|^{p-2}\nabla v) + |x|^{\theta_*} v^{q_*} = 0 \quad \text{for } 0 < |x| < R, \tag{7.4}$$

where $R > 0$ is small enough. Indeed, using $b_0 \in RV_\theta(0+)$ with $\theta > \theta_*$ and $h \in RV_q$ with $q < q_*$, we get that $\lim_{t\to\infty} h(t)/t^{q_*} = 0$ and $\lim_{r\to 0} b_0(r)/r^{\theta_*} = 0$. Lemma 3.2 (i) shows that $\lim_{|x|\to 0} u(x) = \infty$. Hence, there exists $R > 0$ such that $B_R(0) \subset\subset \Omega$ and

$$b(x)h(u) \le |x|^{\theta_*} u^{q^*} \qquad \text{for } 0 < |x| \le R.$$

Thus u is a super-solution of (7.4). By our assumption, $\limsup_{|x|\to 0} \frac{u(x)}{\mu(x)} = \infty$, and hence there exists a sequence $\{x_n\}$ in $\mathbb{R}^N$ such that $|x_n| = r_n$ decreases to zero as $n \to \infty$ and $\lim_{n\to\infty} \frac{u(x_n)}{\mu(x_n)} = \infty$. Then by Lemma 3.1 (b), we obtain that

$$\lim_{n\to\infty} \min_{|x|=r_n} \frac{u(x)}{\mu(x)} = \infty. \tag{7.5}$$

For any $n \geq 1$, the equation (7.4), subject to $\lim_{|x|\to 0} \frac{v(x)}{\mu(x)} = n$ and $v|_{\partial B_R(0)} = 0$, admits a unique positive solution v_n, which is radial (by Lemma A.8 and Lemma 6.1). Using (7.5), we get $u(x) \geq v_n(|x|)$ on $|x| = r_n$ for large $n \geq 1$. Since also $u \geq v_n$ on $|x| = R$, by Lemma A.8

$$u(x) \geq v_n(|x|) \quad \text{if } r_n < |x| < R \tag{7.6}$$

for all large $n \geq 1$. As in the proof of Lemma 6.2, we have $v_n \to v_*$ in C^1 in every compact subset of $(0, R]$ as $n \to \infty$ and v_* is a positive solution of (7.4) such that $\lim_{r\to 0} \frac{v_*(r)}{\mu(r)} = \infty$. Letting $n \to \infty$ in (7.6), we obtain $u(x) \geq v_*(|x|)$ for every x with $0 < |x| < R$. Therefore $\lim_{|x|\to 0} u(x)/\mu(x) = \infty$. We now apply Corollary 6.3 to the super-solution u of (7.4) to obtain (7.3) (since $\theta_* > -p$ and $p - 1 < q_* < C_{N,p,\theta_*}$). This completes the proof of (7.1).

8. Proof of Theorem 1.2

(i) *Uniqueness.*

Let u_1, u_2 be two positive solutions of (1.8). We first prove uniqueness for $\gamma = 0$ in (1.8). By Lemma 3.2, both u_1 and u_2 belong to $W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and they can be extended as continuous solutions of (1.1) in the whole Ω. Hence, for every $\varphi \in C^1_c(\Omega)$, we have

$$\int_\Omega |\nabla u_i|^{p-2} \nabla u_i \cdot \nabla\varphi \, dx + \int_\Omega b(x) h(u_i) \varphi \, dx = 0 \qquad \text{with } i \in \{1, 2\}. \tag{8.1}$$

Using (1.5), the function b is locally in $L^{\frac{N}{p-\varepsilon}}(\Omega)$ for some $\varepsilon > 0$. It follows that (8.1) holds not only for functions φ in $C^1_c(\Omega)$, but in fact for any φ with strong derivatives in L^p and with compact support in Ω. This is deduced using the Hölder inequality and the Sobolev embedding theorem (see [13], p. 251). Since $(u_1 - u_2) \in W^{1,p}_0(\Omega)$, we let $\varphi = u_1 - u_2$ in (8.1) and find

$$\int_\Omega (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \cdot \nabla(u_1 - u_2) \, dx + \int_\Omega b(x)(h(u_1) - h(u_2))(u_1 - u_2) \, dx = 0.$$

Note that the integrand in the first integral is non-negative. Since $b(x) > 0$ in Ω^* and h is increasing, for the above equality to hold we must have $u_1 \equiv u_2$ in Ω.

We now assume that $\gamma \in (0, \infty) \cup \{+\infty\}$. We notice that $(u_1/u_2)(x) \to 1$ as $|x| \to 0$, where we apply Theorem 1.1 for $\gamma = \infty$. Let $\varepsilon > 0$ be arbitrary. Since $h(t)/t^{p-1}$ is non-decreasing on $(0, \infty)$, one can check that $(1 + \varepsilon)u_i$ is a super-solution of (1.1) for $i = 1, 2$. By the comparison principle, we find that $u_1 \leq (1 + \varepsilon)u_2$ in Ω^* and $u_2 \leq (1 + \varepsilon)u_1$ in Ω^*. By taking $\varepsilon \to 0$, we conclude that $u_1 = u_2$ in Ω^*.

(ii) *Existence.*

If $\gamma = 0$, then u is a regular solution of (1.1) in Ω (cf., Lemma 3.2). The existence assertion follows by a standard minimization argument. Assume that γ is any positive number. We prove that (1.8) admits at least one positive solution u_γ. Let $\theta_* \in (-p, \theta)$ and q_* be sufficiently close to θ and q, respectively such that $q < q_* < C_{N,p,\theta_*}$. We fix $C > 0$ large such that

$$C > \max_{x\in\partial\Omega} \vartheta(x) \quad \text{and } h(t) \leq t^{q_*} \quad \text{for every } t \geq C.$$

Let $r_* > 0$ be small enough such that $B_{r_*}(0) \subset\subset \Omega$ and

$$b(x) \leq |x|^{\theta_*} \quad \text{for every } 0 < |x| \leq r_*.$$

By Lemma 6.1, there exists a unique positive solution $\Psi_\gamma \in C^1(0, r_*]$ satisfying

$$\begin{cases} -(r^{N-1}|\Psi_r|^{p-2}\Psi_r)_r + r^{N-1+\theta_*}\Psi^{q_*} = 0 \quad \text{in } (0, r_*), \\ \lim_{r\to 0} \dfrac{\Psi(r)}{\mu(r)} = \gamma, \quad \Psi(r_*) = C. \end{cases}$$

Since $\Psi_\gamma(r)$ is decreasing in r, we have $\Psi_\gamma(r) \geq C$ for every $r \in (0, r_*)$. By the comparison principle, we obtain $\Psi_\gamma(|x|) \leq \gamma\mu(|x|) + C$ for $0 < |x| \leq r_*$. For every integer $n \geq 1$ satisfying $n > 1/r_*$, we consider the boundary value problem

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p-2}\nabla v) + b(x)h(v) = 0 \quad \text{for } x \in \Omega \setminus \overline{B_{1/n}(0)}, \\ v = \gamma\mu + C \ \text{ for } |x| = 1/n, \qquad v = \vartheta \ \text{ on } \partial\Omega. \end{cases} \tag{8.2}$$

Let v_n be the unique positive C^1-solution of (8.2). It follows that $v_{n+1} \leq v_n \leq \gamma\mu + C$ for $x \in \Omega \setminus B_{1/n}(0)$ and every $n > 1/r_*$. Since

$$-\operatorname{div}(|\nabla\Psi_\gamma|^{p-2}\nabla\Psi_\gamma) + b(x)h(\Psi_\gamma) \leq 0 \quad \text{for } 0 < |x| < r_*,$$

we deduce from the comparison principle that $\Psi_\gamma \leq v_n + C$ for $1/n < |x| < r_*$. By Lemma 4.1, we conclude that for a sequence $n_j \to \infty$ we have $v_{n_j} \to v_\infty$ in $C^1_{\text{loc}}(\Omega^*)$ and v_∞ is a positive solution of (1.1) such that $v_\infty = \vartheta$ on $\partial\Omega$. Moreover, we have $\Psi_\gamma \leq v_\infty + C \leq \gamma\mu + 2C$ for $0 < |x| < r_*$, which leads to $\lim_{|x|\to 0} v_\infty(x)/\mu(x) = \gamma$. Hence, v_∞ is a positive solution of (1.8).

Consequently, (1.8) admits a (unique) positive solution $u_\gamma \in C^1(\Omega^*)$ for every $\gamma \in [0, \infty)$. By Theorem 5.1, we know that u_γ satisfies (1.7). Applying Lemma 4.1 to u_γ with $g \equiv \text{Const.} > 0$ if $\gamma = 0$ and $g(|x|) = \mu(|x|)$ if $\gamma \in (0, \infty)$, we find that $u_\gamma \in C^{1,\alpha}_{\text{loc}}(\Omega^*)$ for some $\alpha \in (0, 1)$.

To construct a positive solution of (1.8) for $\gamma = \infty$, we proceed as follows. Let u_n be the unique positive solution of (1.8) with $\gamma = n \geq 1$. By the comparison principle, we find $u_n \leq u_{n+1}$ in Ω^*. By Remark 4.1 and Lemma 4.1, we see that, up to a subsequence, u_n converges in $C^1_{\text{loc}}(\Omega^*)$ to u_∞, which is a positive solution of (1.8) with $\gamma = \infty$. Moreover, $u_\infty \in C^{1,\alpha}_{\text{loc}}(\Omega^*)$ for some $\alpha \in (0, 1)$. This completes the proof.

9. Proof of Theorem 1.3

Let (1.4) and (1.5) hold with $1 < p < N$ and $q \geq C_{N,p,\theta}$. If $q = C_{N,p,\theta}$, then we further assume (1.9). Let u be any positive solution of (1.1). By Lemma 3.2, it is enough to show that

$$\lim_{|x|\to 0} \frac{u(x)}{\mu(x)} = 0. \tag{9.1}$$

We distinguish two cases. We first suppose that $q > C_{N,p,\theta}$. Then $\frac{p-N}{p-1}$ is *less* than $-\frac{\theta+p}{q-p+1}$. Since μ is regularly varying at zero of index $\frac{p-N}{p-1}$, by (2.1) we find (9.1).

We next consider the case $q = C_{N,p,\theta}$. Then Υ and μ vary regularly at zero with the same index, and we need condition (1.9) to prove (9.1). Set $\gamma := \limsup_{|x|\to 0} \frac{u(x)}{\mu(x)}$. It suffices to show that $\gamma = 0$. Arguing indirectly, we assume that $\gamma \neq 0$. We shall arrive at a contradiction with Lemma 6.4 (ii) as follows. By Lemma 3.2, we have $\lim_{|x|\to 0} u(x) = \infty$. Using (1.9), we find that u is a sub-solution of

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \varepsilon|x|^\theta v^{C_{N,p,\theta}} = 0 \quad \text{in } B_R(0) \setminus \{0\}, \tag{9.2}$$

where $\varepsilon > 0$ and $R > 0$ are small constants such that $B_R(0) \subset\subset \Omega$. By applying Lemma 3.1 (a) to (9.2), we conclude that $\gamma < \infty$. For every $\tau > 0$, the comparison principle leads to

$$u(x) \leq (\gamma + \tau)\mu(x) + \max_{|y|=R} u(y) \quad \text{for } 0 < |x| < R.$$

Letting $\tau \to 0$, we obtain

$$u(x) \leq \gamma\mu(x) + \max_{|y|=R} u(y) \quad \text{for } 0 < |x| < R.$$

For every large integer $n \geq 1$, we set $Q_n := \{x \in \mathbb{R}^N : \ 1/n < |x| < R\}$. Let v_n denote the unique positive solution of (9.2) considered in Q_n, subject to the boundary condition

$$v|_{\partial B_R(0)} = \max_{|x|=R} u(x) \quad \text{and} \quad v|_{\partial B_{1/n}(0)} = \max_{|x|=1/n} u(x). \tag{9.3}$$

From (9.3) and uniqueness of v_n, we must have that v_n is radially symmetric in Q_n. We notice that u is a sub-solution (respectively, $\gamma\mu(x) + \max_{|y|=R} u(y)$ is a super-solution) for (9.2) in Q_n, subject to (9.3). Using the comparison principle, we get

$$u(x) \leq v_n(|x|) \leq \gamma\mu(x) + \max_{|y|=R} u(y) \quad \text{in } Q_n. \tag{9.4}$$

Using Lemma 4.1, we find that for a sequence $n_k \to \infty$ we have $v_{n_k} \to v_\infty$ in $C^1_{\text{loc}}(0, R]$ and $V := \varepsilon^{\frac{1}{C_{N,p,\theta}-p+1}} v_\infty$ satisfies the following equation

$$-(r^{N-1}|V_r|^{p-2}V_r)_r + r^{N-1+\theta}V^{C_{N,p,\theta}} = 0 \quad \text{in } (0, R).$$

Letting $n \to \infty$ in (9.4) and using Lemma 6.4 (i), we find $\lim_{r\to 0} \frac{V(r)}{\mu(r)} = \varepsilon^{\frac{1}{C_{N,p,\theta}-p+1}}\gamma \in (0, \infty)$. But this is a contradiction with Lemma 6.4 (ii). This concludes the proof of (9.1).

A. Regular variation theory and related results

A.1. Properties of regularly varying functions

If h is a positive measurable function defined in a neighbourhood of infinity and the limit $\lim_{t\to\infty} h(\lambda t)/h(t)$ exists in $(0, \infty)$ for every $\lambda > 0$, then necessarily (1.4) holds for some $q \in \mathbb{R}$ (see [12]). Such functions were first introduced by Karamata [9] and are called *regularly varying functions* at ∞ with index q. Their theory, which was later extended and developed by many others, plays an important role in certain areas of probability theory such as in the theory of domains of attraction and max-stable distributions. For detailed accounts of the theory of regular variation, its extensions and many of its applications, we refer the interested reader to [12], [1] and [11].

For the reader's convenience, we include here some basic properties of regularly varying functions. We recall that a positive measurable function L defined on a neighbourhood of infinity is called slowly varying at ∞ if $\lim_{t\to\infty} L(\lambda t)/L(t) = 1$ for every $\lambda > 0$.

Notation. By $f_1(t) \sim f_2(t)$ as $t \to \infty$, we mean that $\lim_{t\to\infty} f_1(t)/f_2(t) = 1$. As in [11], let $f^{\leftarrow}$ denote the (left continuous) inverse of a non-decreasing function f on $\mathbb{R}$, namely

$$f^{\leftarrow}(t) = \inf\{s : \ f(s) \geq t\}.$$

Proposition A.1 (Representation Theorem)**.** *A function L is slowly varying at ∞ if and only if it can be written in the form*

$$L(t) = T(t)\exp\left\{\int_{t_0}^{t} \frac{\varphi(\xi)}{\xi}\,d\xi\right\} \quad (t \geq t_0 > 0) \tag{A.1}$$

where $\varphi \in C[t_0,\infty)$ satisfies $\lim_{t\to\infty}\varphi(t) = 0$ and T is measurable function on $[t_0,\infty)$ such that $\lim_{t\to\infty} T(t) := \widehat{T} \in (0,\infty)$.

Remark A.1. For any $f \in RV_\rho$ ($\rho \in \mathbb{R}$), there exists a C^1-function $\widehat{f} \in RV_\rho$ such that

$$\lim_{t\to\infty}\frac{\widehat{f}(t)}{f(t)} = 1 \quad \text{and} \quad \lim_{t\to\infty}\frac{t\widehat{f}'(t)}{\widehat{f}(t)} = \rho. \tag{A.2}$$

Indeed, if $L(t) := f(t)/t^\rho$, then L is slowly varying at ∞ and (A.1) holds. We define $\widehat{f}$ as follows

$$\widehat{f}(t) = \widehat{T}\,t^\rho \exp\left\{\int_{t_0}^{t}\frac{\varphi(\xi)}{\xi}\,d\xi\right\} \quad (t \geq t_0).$$

Hence, $\widehat{f}$ is a C^1-function that satisfies (A.2), since we have

$$\frac{\widehat{f}(t)}{f(t)} = \frac{\widehat{T}}{T(t)} \to 1 \quad \text{and} \quad \frac{t\widehat{f}'(t)}{\widehat{f}(t)} = \rho + \varphi(t) \to \rho \ \text{ as } t \to \infty.$$

Proposition A.2 (Uniform Convergence Theorem)**.** *If L is slowly varying at ∞, then $L(\lambda t)/L(t)$ converges to 1 as $t \to \infty$, uniformly on each compact λ-set in $(0,\infty)$.*

Proposition A.3 (Properties of slowly varying functions)**.** *Assume that L is slowly varying at ∞. The following hold:*

(i) $\log L(t)/\log t$ *converges to* 0 *as* $t \to \infty$*;*
(ii) *For any $j > 0$, we have $t^j L(t) \to \infty$ and $t^{-j}L(t) \to 0$ as $t \to \infty$;*
(iii) $(L(t))^j$ *varies slowly at ∞ for every $j \in \mathbb{R}$;*
(iv) *If L_1 varies slowly at ∞, so does the product (respectively the sum) of L and L_1.*

Proposition A.4 (Karamata's Theorem)**.** *If $f \in RV_\rho$ is locally bounded in $[A,\infty)$, then*

(i) $\displaystyle\lim_{t\to\infty}\frac{t^{j+1}f(t)}{\int_A^t \xi^j f(\xi)\,d\xi} = j+\rho+1$ *for any $j \geq -(\rho+1)$;*
(ii) *for any $j < -(\rho+1)$ (and for $j = -(\rho+1)$ if $\int^\infty \xi^{-(\rho+1)} f(\xi)\,d\xi < \infty$) we have*

$$\lim_{t\to\infty}\frac{t^{j+1}f(t)}{\int_t^\infty \xi^j f(\xi)\,d\xi} = -(j+\rho+1).$$

Proposition A.5 (see Proposition 0.8 in [11])**.** *We have*

(i) *If $f \in RV_\rho$, then* $\lim_{t\to\infty}\log f(t)/\log t = \rho$.
(ii) *If $f_1 \in RV_{\rho_1}$ and $f_2 \in RV_{\rho_2}$ with* $\lim_{t\to\infty} f_2(t) = \infty$*, then*

$$f_1 \circ f_2 \in RV_{\rho_1\rho_2}.$$

(iii) *Suppose f is non-decreasing, $f(\infty) = \infty$, and $f \in RV_\rho$ with $0 < \rho < \infty$. Then*

$$f^{\leftarrow} \in RV_{1/\rho}.$$

The next result shows that any function f varying regularly at ∞ with *positive* index is asymptotic to a monotone function.

Proposition A.6 (see Theorem 1.5.3 in [1])**.** *Let $f \in RV_\rho$ and choose $t_0 \geq 0$ so that f is locally bounded on $[t_0, \infty)$. If $\rho > 0$, then we have*

(a) $\overline{f}(t) := \sup\{f(s) :\ t_0 \leq s \leq t\} \sim f(t)$ *as* $t \to \infty$*;*

(b) $\underline{f}(t) := \inf\{f(s) :\ s \geq t\} \sim f(t)$ *as* $t \to \infty$.

A.2. Other results

Lemma A.7. *If $p > 1$, then there exist two functions h_1 and h_2 which have the properties of h stated in Assumption A in Section 1, as well as the following*

$$\begin{cases} h_1(t) \leq h(t) \leq h_2(t) \quad \textit{for } t \in [0, \infty), \\ \dfrac{h_1(t)}{t^{p-1}} \ \textit{ and } \ \dfrac{h_2(t)}{t^{p-1}} \ \textit{ are both increasing for } t \in (0, \infty). \end{cases} \tag{A.3}$$

Proof. Let $q > p - 1$. We set $g_*(t) := \inf_{s \geq t} g(s)$ for $t > 0$, where $g(t) := t^{-\frac{(q+p-1)}{2}} h(t)$. Hence, $g_* \leq g$ on $(0, \infty)$ and g_* is non-decreasing on $(0, \infty)$. We define h_1 on $[0, \infty)$ with

$$h_1(t) := t^{\frac{q+p-1}{2}} g_*(t) \quad \text{for any } t > 0 \ \text{ and } h_1(0) = 0. \tag{A.4}$$

Using the monotonicity of g_* and $q > p - 1$, we see that $h_1(t)/t^{p-1} = t^{\frac{q-p+1}{2}} g_*(t)$ is increasing for $t \in (0, \infty)$. Moreover, $h_1(t) \leq h(t)$ for any $t \geq 0$. We now construct h_2 on $[0, \infty)$ as follows

$$h_2(t) := t^{p-1} \left(\sup_{0 < s \leq t} \frac{h(s)}{s^{p-1}} + t^{\frac{q-p+1}{2}} \right) \ \text{ for any } t > 0 \ \text{ and } h_2(0) = 0. \tag{A.5}$$

Since $h(0) = 0$ and $h(t)/t^{p-1}$ is assumed to be bounded for small $t > 0$, we infer that h_2 is well-defined and satisfies the properties of h and (A.3).

Remark A.2. If in Lemma A.7 we assume, in addition, that $h \in RV_q$ for some $q > p-1$, then the functions h_1 and h_2 constructed in (A.4) and (A.5) are asymptotically equivalent to h at infinity, that is $\lim_{t \to \infty} h_i(t)/h(t) = 1$ for $i = 1, 2$. This follows by applying Proposition A.6.

The monotonicity of the functions h_1 and h_2 in Lemma A.7 allows us to use the following comparison principle (see, for example, Theorem 2.4.1 in [10]). For other versions, we refer to Theorem 10.7 in [8], or Proposition 2.2 in [5] (see also [6]).

Lemma A.8 (Comparison principle)**.** *Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 2$. Assume that $g : \Omega \times [0, \infty) \longmapsto [0, \infty)$ is in $L^\infty_{\text{loc}}(\Omega \times [0, \infty))$ and $g = g(x, z)$ is non-decreasing in z. Let $p > 1$ and u, v be positive C^1-functions on Ω such that*

$$-\text{div}\,(|\nabla u|^{p-2} \nabla u) + g(x, u) \leq 0 \leq -\text{div}\,(|\nabla v|^{p-2} \nabla v) + g(x, v) \quad \textit{in } \mathcal{D}'(\Omega).$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

The next result (see [7]) relies on the strong maximum principle in [8, Theorem 8.19].

Lemma A.9 (Lemma 1.3 in [7])**.** *Let $\mathcal{O}$ be a domain in $\mathbb{R}^N$ and $c \in L^\infty_{\mathrm{loc}}(\mathcal{O})$. Assume that $p > 1$ and u, v are C^1-functions on $\mathcal{O}$ such that*

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + cu \leq 0, \qquad -\mathrm{div}\,(|\nabla v|^{p-2}\nabla v) + cv \geq 0,$$

in the weak sense in $\mathcal{O}$, and $\nabla v \neq 0$ for every $x \in \mathcal{O}$. If $u \leq v$ in $\mathcal{O}$ and if there exists a point $x_0 \in \mathcal{O}$ such that $u(x_0) = v(x_0)$, then $u \equiv v$ in $\mathcal{O}$.

Remark A.3. Let $\mathcal{O}$ be a domain in $\mathbb{R}^N$ and $\overline{\mathcal{O}} \subset \Omega^*$. If u is a positive sub-solution of (1.1) in $\mathcal{O}$ and u/μ achieves a maximum β in $\mathcal{O}$, then $u/\mu \equiv \beta$ in $\mathcal{O}$. This follows by using Lemma A.9 with $v = \beta\mu$.

If (1.4) holds with $q > p - 1 > 0$ and $b_0 \in RV_\theta(0+)$ with $\theta > -p$, then we define $\mathcal{J}$ and $\mathcal{B}$ by

$$\mathcal{J}(t) := \frac{\int_t^\infty [h(s)]^{-\frac{1}{p-1}}\,ds}{t[h(t)]^{-\frac{1}{p-1}}} \quad \text{for } t > 0, \qquad \mathcal{B}(r) := \frac{[r^p b_0(r)]^{\frac{1}{p-1}}}{\int_0^r [sb_0(s)]^{\frac{1}{p-1}}\,ds} \quad \text{for small } r > 0. \tag{A.6}$$

Then by Proposition A.4, we have that

$$\lim_{t\to\infty} \mathcal{J}(t) = \frac{p-1}{q-p+1}, \qquad \lim_{r\to 0} \mathcal{B}(r) = \frac{\theta+p}{p-1}. \tag{A.7}$$

Remark A.4. In view of Remark A.1, in the definition of the function $\Upsilon(r)$ in (1.11), we can replace h and b_0 by asymptotically equivalent C^1-functions without affecting our proofs. With such a change, $\Upsilon(r)$ becomes a C^2-function on a small interval $(0, r_0)$, and we have

$$\lim_{r\to 0} \frac{rb_0'(r)}{b_0(r)} = \theta \quad \text{and} \quad \lim_{t\to\infty} \mathcal{F}(t) = q, \text{ where } \mathcal{F}(t) := \frac{th'(t)}{h(t)} \text{ for } t > 0 \text{ large.} \tag{A.8}$$

These conventions are used frequently in our proofs.

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