Proceedings of the 16th OCU
International Academic Symposium 2008
OCAMI Studies Volume 3
2009, pp.251–265

ASSOCIATIVE CONES IN THE IMAGINARY OCTONIONS

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Abstract. A curve in the nearly-Kähler 6-sphere is almost-complex precisely when the cone over it is an associative (singular) submanifold of $\mathrm{Im}\mathbb{O}$, and hence volume minimising in its homology class. Almost-complex curves in S^6 are either totally geodesic, pseudo-holomorphic or superconformal, the last case being generic and the subject of this paper. We begin by giving a geometric construction of a particularly natural G_2-framing for superconformal almost-complex curves. This framing can easily be shown to agree with that in [6]; the exposition here can be viewed as giving a geometric interpretation of and motivation for this framing together with a simpler proof that it indeed lies in G_2. We then focus our attention on superconformal almost-complex $f : \mathbb{C} \to S^6$ and use the above framing to construct a spectral curve for maps of finite type (which include all doubly-periodic examples). This curve is reducible, and we additionally obtain a linear flow in the Jacobian of the "main component" of the spectral curve. This linear flow is in fact restricted to the real slice of a sub-torus of this Jacobian and it is notable that the sub-torus is the intersection of two Prym varieties, rather than a single Prym variety as has arisen in spectral curve descriptions of other harmonic maps. This later part of the paper is a report on joint work with Erxiao Wang.

1. Introduction

The purpose of this manuscript is twofold. Firstly, we explore the relationship between the geometry of superconformal almost-complex curves $f : M^2 \to S^6$ and the exceptional Lie group G_2. In particular, we construct a canonical G_2-framing for f, chosen so as to include the principal directions of the second ellipse of curvature. The second part of the paper is a report on joint work with Erxiao Wang, the details of which will appear elsewhere [14]. The goal here is to give a bijective correspondence between spectral curve data and superconformal almost-complex $f : \mathbb{C} \to S^6$ satisfying a finite-type condition, which in particular is automatically satisfied by all doubly periodic examples, and hence all superconformal almost-complex tori. By "spectral curve data" is meant an algebraic curve, together with a rational function and a line bundle on this curve, all of which satisfy certain symmetries. The construction given here gives one side of this story: from superconformal almost-complex curves we obtain spectral curve data. It remains to reverse the construction; this is work in progress. The other types of almost-complex curves have already been classified by other means. Totally geodesic almost-complex curves are easily described as they are each given by the intersection of $S^6 \subset \mathrm{Im}\,\mathbb{O}$

Date: Received on July 22, 2009.
2000 *Mathematics Subject Classification.* Primary 53C43, 58E20 Secondary 53C45.
Key words and phrases. Riemann surfaces, Harmonic maps, integrable systems.

with an associative 3-plane, whilst Bryant has given a Weierstrass-representation for all pseudo-holomorphic almost-complex curves [7].

We now discuss some applications of the spectral curve approach in the general context of spectral curves for harmonic maps, of which almost-complex curves are an example. Firstly, an explicit one-to-one correspondence between harmonic maps and algebraic curve data allows one to describe the moduli-space of the relevant harmonic maps. In particular, the dimension of the sub-torus of the Jacobian of the spectral curve in which the linear flow lies is a very important invariant, because if for a harmonic map f this torus has dimension d, then f lies in a d-dimensional family of harmonic maps. It is easy to find harmonic maps of the complex plane, but difficult to find ones which are doubly-periodic and hence maps of genus one minimal surfaces. However for constant mean curvature tori in $\mathbb{R}^3$ [15, 28], minimal tori in S^3 [11, 12] and minimal Legendrian tori in S^5 [13] this periodicity problem has been solved, demonstrating the existence of doubly periodic examples coming in families of arbitrarily large dimension. It is not surprising that the dimension of the family in which a map lives is geometrically significant. Simple expressions in d (or equivalently in the spectral genus) give lower bounds on the area of constant mean curvature tori in $\mathbb{R}^3$ [16], and the area is an invariant measuring the geometric complexity of special Lagrangian T^2-cones [21]. Killian and Schmidt also made essential use of a spectral curve correspondence in their proof of the Pinkall-Sterling conjecture that the only embedded constant mean curvature tori in S^3 are tori of revolution [29].

Almost-complex curves $f : M^2 \to S^6$ are of particular interest, in part because of their relationship with associative 3-cycles, which are important in M-theory. The almost-complex condition is equivalent to requiring that the cone $C := \mathbb{R}^+ f(M^2)$ is associative. This means that its tangent space at each point is the imaginary part of an associative subalgebra of $\mathbb{O}$, or equivalently that C is calibrated with respect to the associative 3-form on $\operatorname{Im}\mathbb{O}$. If f has image in a totally geodesic $S^5 \subset S^6$, then we instead make contact with a different calibrated geometry: the cones over such surfaces are special Lagrangian, and play a special role in string theory and in particular the SYZ conjecture.

We now describe the contents of this manuscript in more detail. The 6-sphere may be considered as the space of unit length imaginary octonions, and Cayley multiplication induces on it an almost-complex structure J which is nearly-Kähler and optimal amongst almost-complex structures in the sense of having minimum volume [10]. An almost-complex curve is an immersion $f : M^2 \to S^6$ of a Riemann surface whose differential is complex linear

$$df \circ i = J \circ df. \tag{1.1}$$

Let $f : M^2 \to S^6$ be an almost-complex curve, which we are assuming to not be totally-geodesic. For each $p \in M^2$, the tangent space $T_{f(p)}S^6$ naturally decomposes into three 2-dimensional subspaces: $V_1 := f_*(T_pM^2)$, the first normal space V_2 (which for f not totally-geodesic is always 2-dimensional), and the orthogonal complement V_3 of $V_1 \oplus V_2$. Together with the line $V_0 = \mathbb{R}f(p)$, we have a natural orthogonal decomposition of $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$. Orthonormal vectors $e_1, \dots, e_7 \in \operatorname{Im}\mathbb{O}$ form a G_2-basis if $(e_1, \dots, e_7) \in G_2$, one way of expressing this is

$$e_3 = e_1 \cdot e_2, \quad e_5 = e_7 \cdot e_2, \quad e_6 = e_5 \cdot e_3, \quad e_4 = e_6 \cdot e_2.$$

Such a framing is clearly determined by the choice of e_1, e_2, e_7. For the spectral curve construction, we will need an intrinsic way of choosing a G_2-framing. This is done using ellipses of curvature. The first ellipse of curvature is the image of the unit circle in T_pM^2 under the second fundamental form. When f is totally geodesic this is a point; we have excluded this case and it is not difficult to show that then the first ellipse of curvature is a (nontrivial) circle and hence does not pick out any specific directions within V_2. The second ellipse of curvature is the image of the unit circle in T_pM^2 under the third fundamental form $\mathit{III}(u,v,w) = (\nabla_u \mathit{II}(v,w))^\perp$, where ∇ denotes the Levi–Civita connection. For f pseudo-holomorphic this ellipse is a circle, but in the superconformal case it is either a nontrivial ellipse (when f is linearly full) or a line segment (when f is orthogonal to a fixed unit vector $N \in S^6$. We assume hereafter that f is superconformal. Thus the second ellipse of curvature defines geometrically natural orthogonal directions within V_3. We show in Theorem 3 that **we may choose a local coordinate** $z = x + iy$ **on** M^2 **such that the choices** $e_1 = f$**,** $e_2 = f_x$**,** $e_7 = -\mathit{III}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial x})$ **determine a** G_2**-framing with the property that**

(1) if f **is linearly full, then** e_6 **and** e_7 **are, respectively, along the minor and major axes of the second ellipse of curvature**

(2) if the image of f **is contained in a totally geodesic** $S^5 \subset S^6$**, then** e_7 **lies along the 2nd ellipse of curvature, which is in this case a line segment.**

We also give simple expressions for each of $e_1, \ldots, e_7$.

The above G_2-framing F is primitive with respect to the 6th order involution τ which gives G_2/T^2 its usual 6-symmetric space structure (see section 4). For our purposes the importance of this is that it enables us to define a family of flat connections $\nabla_\zeta = \nabla^L + \varphi_\zeta$, $\zeta \in \mathbb{C} - \{0\}$, in a trivial rank 7 complex vector bundle V with trivial connection ∇^L. Parallel sections A_ζ of $\operatorname{End} V$ then satisfy an equation of Lax type, $dA = [A, \varphi_\zeta]$. The finite-type condition referred to above means that it is possible to find such parallel sections A_ζ which are Laurent polynomials in ζ and satisfy some additional conditions. These A_ζ are called polynomial Killing fields; see section 2. This fits into a general framework of descriptions for equations of Lax type in terms of spectral curve data, see for example [1, 2, 18, 3, 27]. The related spectral curve constructions for harmonic tori in several other target spaces [24, 34, 4, 17, 30, 31, 32, 33] are somewhat more complicated as the Lax pair equation does not arise directly.

Given a superconformal almost-complex $f : \mathbb{C} \to S^6$ of finite type, we construct an algebraic curve X, called the spectral curve, together with a degree 7 rational function $\lambda : X \to \mathbb{P}^1$. The curve X is reducible; it can be broken into a rational component and another curve Y, which we call the main component of the spectral curve. The restriction of λ to Y has degree 6. We show that Y is generically smooth, and construct a linear map from the domain $\mathbb{C}$ of f to the Jacobian of Y which assigns to each $z \in \mathbb{C}$ an eigenline bundle $\mathcal{E}_z$. In fact this map has image in a sub-torus of $\operatorname{Jac} Y$ and the most geometrically interesting part of the paper is describing how the geometry of G_2 determines what the sub-torus is. It is the real part of the intersection of two Prym varieties. The linear flow is restricted to one of these because the frame F is special orthogonal, and the further restriction to the other Prym variety arises because in fact $F \in G_2 \subset SO(7)$. The symmetries are the same as those found by Hitchin in his study of Langlands duality for G_2-Higgs bundles [26]. The degree of the map λ is equal to the size of the matrices in the group under consideration; in working with the main component Y it is necessary to exploit the relationship between $SO(7)$ and $Sp(6)$ which in that context is an explicit realisation of Langlands duality. We also compute various invariants, such as the genus of the spectral curve, the degree of the eigenline bundles and the dimension of the torus in which they lie.

The structure of this paper is as follows. In section 2 we collect standard facts regarding the octonions, the compact Lie group G_2 and almost-complex curves which we will use in what follows. Section 3 contains the construction of a canonical G_2-framing for a superconformal almost-complex curve using ellipses of curvature and in the last section is summarised the spectral curve construction for such maps $f : \mathbb{C} \to S^6$ of finite-type.

2. Preliminaries: G_2 and the Octonions

We use the natural cross product on the imaginary octonions to define an almost-complex structure on S^6, and to see how the compact group G_2 naturally appears in the study of almost-complex curves in S^6. We then explain the relationship between these almost-complex curves and calibrated geometry.

Beginning with the field of real numbers, the Cayley-Dickson construction yields an infinite sequence of nicely normed algebras, each having dimension twice that of the previous algebra. The first three algebras so constructed are the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$; at each step one takes pairs of elements and defines

$$(a, b)(c, d) := (ac - \bar{d}b, da + b\bar{c}),$$

where the conjugation operator is defined by $\overline{(a, b)} = (\bar{a}, -b)$ and the norm by $\|(a, b)\|^2 := (a, b)\overline{(a, b)}$. In the first few stages of this process, an important property of the algebra is lost in each step: $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ is not ordered, $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ is not commutative and $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l$ is not associative. The octonions, also called the Cayley numbers, are the last algebra in the Cayley-Dickson sequence which form a division algebra. Writing

$$\operatorname{Re} u := \frac{1}{2}(u + \bar{u}),\ \operatorname{Im} u := \frac{1}{2}(u - \bar{u}),$$

octonionic (or Cayley) multiplication defines on $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$ the usual Euclidean inner product by

$$\langle u, v\rangle := -\operatorname{Re}(uv) = -\frac{1}{2}(uv + vu), \tag{2.1}$$

and also yields on $\operatorname{Im} \mathbb{O}$ a cross product, in exactly the same way that a cross product on $\operatorname{Im} \mathbb{H} \cong \mathbb{R}^3$ may be defined using quaternionic multiplication, namely

$$u \times v := \operatorname{Im}(uv) = \frac{1}{2}(uv - vu). \tag{2.2}$$

A *cross product* on an inner product space V is a bilinear map $\times : V \times V \to V$ such that $u \times v$ is orthogonal to both u and v, and $|u \times v| = |u||v| \sin\theta$, where θ denotes the angle between u and v. Due to their relationship with normed division algebras, cross products exist only in 3 and 7 dimensions.

We define the exceptional Lie group G_2 as the automorphism group of the octonions. Any automorphism must preserve 1, and so G_2 may also be characterised as linear transformations of $\operatorname{Im}\mathbb{O}$ preserving Cayley multiplication. From (2.1) and (2.2) this is equivalent to preserving the inner product and the cross product but

$$\langle u, v\rangle = -\frac{1}{6}\operatorname{tr}\left(\mathrm{w} \mapsto \mathrm{u} \times (\mathrm{v} \times \mathrm{w})\right) \text{ for } \mathrm{u}, \mathrm{v}, \mathrm{w} \in \operatorname{Im}\mathbb{O} \tag{2.3}$$

so it suffices to preserve the cross product. Using (2.3), this is equivalent to preserving the trilinear functional

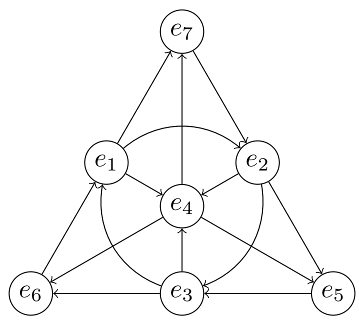
$$\begin{aligned} \phi : \operatorname{Im}\mathbb{O} \times \operatorname{Im}\mathbb{O} \times \operatorname{Im}\mathbb{O} &\to \mathbb{R} \\ (u, v, w) &\mapsto \langle u \times v, w\rangle \end{aligned}$$

which is the dual of the cross product with respect to the inner product. Note that from (2.1) and (2.2),

$$\begin{aligned} \phi(u, v, w) &= \langle uv + \langle u, v\rangle, w\rangle \\ &= \langle uv, w\rangle, \end{aligned}$$

in agreement with Bryant's characterisation of G_2 as the real-linear transformations of $\operatorname{Im}\mathbb{O}$ preserving the trilinear functional $(u, v, w) \mapsto \langle uv, w\rangle$ [19].

The Fano mnemonic below is a simple way of recording the multiplication table for the octonions. Writing $e_1 = i, e_2 = j, e_3 = k, e_4 = l, e_5 = il, e_6 = jl, e_7 = kl$, then if e_a, e_b, e_c lie along the seven lines/circle with a positive orientation, we have $e_a \cdot e_b = e_c$.



Fano mnemonic

For orthogonal vectors the cross product agrees with Cayley multiplication and writing $\omega_{abc} = e_a^* \wedge e_b^* \wedge e_c^*$, the associative form ϕ is given by the positively oriented lines in the Fano mnemonic,

$$\phi = \omega_{123} + \omega_{725} + \omega_{536} + \omega_{617} + \omega_{347} + \omega_{145} + \omega_{246}.$$

The cross product on $\operatorname{Im}\mathbb{O}$ allows us to define on $S^6 \subset \operatorname{Im}\mathbb{O}$ an almost-complex structure J, by

$$J_p(v) = p \times v \text{ where } p \in S^6, v \in T_pS^6.$$

Together with the standard metric g on the 6-sphere, J defines a nearly-Kähler structure, as straightforward calculations yield $(\nabla_v J)v = 0$, $g(Jv, Ju) = g(v, u)$, where ∇ denotes the Levi–Civita connection.

Let M be a Riemann surface with complex coordinate $z = x + iy$. An immersion $f : M^2 \to S^6$ is termed *almost-complex* if

$$df \circ i = J \circ df.$$

In particular, differentiating and permuting $f \times f_x = f_y$ we have

$$f \times (f_{xx} + f_{yy}) = 0$$

and so f is a branched minimal immersion.

We now explain the relationship between almost-complex curves in S^6 and calibrated geometry, which was introduced by Harvey and Lawson in [20]. A *calibration* on a Riemannian manifold X is a closed k-form φ such that

$$\varphi|_\xi \leq \mathrm{vol}|_\xi \text{ for all oriented tangent } k\text{-planes } \xi.$$

A k-submanifold N is *calibrated* with respect to φ if we have equality above for all of its oriented tangent planes. If N is compact, it must have minimum volume amongst all homologically equivalent submanifolds, since if $\tilde{N}$ is homologous to N then

$$\mathrm{vol} N = \int_N \varphi = \int_{\tilde{N}} \varphi \leq \mathrm{vol} \tilde{N}.$$

Example 1. The form $\phi(u, v, w) = \langle u \times v, w \rangle$ is a calibration on $\mathrm{Im}\,\mathbb{O}$, and a 3-fold $N^3 \subset \mathrm{Im}\,\mathbb{O}$ is termed *associative* if it is calibrated with respect to ϕ. This is clearly equivalent to requiring that for all $p \in N$, if u, v, w are an oriented orthonormal basis then $u \times v = w$. This holds if and only if $1 \oplus T_pN$ is an associative subalgebra of $\mathbb{O}$, justifying the terminology. Note that $f : M^2 \to S^6$ is almost-complex precisely when the cone C over $f(M^2)$ in $\mathrm{Im}\,\mathbb{O}$ is associative.

Example 2. The form $\mathrm{Re}\, dz = \mathrm{Re}(dz^1 \wedge \ldots \wedge dz^n)$ is a calibration on $\mathbb{C}^n$ and the calibrated submanifolds are called *special Lagrangian.* When $n = 3$, this arises as a special case of the previous example.

On $V_0 = \mathrm{span}\{e_1, e_2, e_3, e_5, e_6, e_7\}$, writing $z^1 = e_1 + ie_5, z^2 = e_2 + ie_6, z^3 = e_3 + ie_7$ then

$$\mathrm{Re}\, dz = \omega_{123} + \omega_{617} + \omega_{725} + \omega_{536} = \phi|_{V_0}.$$

We may use the action of G_2 to define complex coordinates on any other 6-plane $V \subset \mathrm{Im}\,\mathbb{O}$; since ϕ is invariant under this action we see that $\mathrm{Re}\, dz = \phi|_V$. Thus if an immersion $N^3 \to \mathrm{Im}\,\mathbb{O}$ lies in vector subspace $V^6 \subset \mathrm{Im}\,\mathbb{O}$, it is special Lagrangian with respect to this complex structure if and only if it is associative. In particular, if $f : M^2 \to S^6$ lies in a totally geodesic $S^5 = V \cap S^6$ then f is almost-complex if and only if the cone C over f in V is special Lagrangian.

3. Ellipses of Curvature and G_2-framing

In this section we explain how the geometry of the second ellipse of curvature of a superconformal almost-complex curve in S^6 gives rise to a canonical G_2-framing of f.

Let $f : M^2 \to S^6$ be an almost-complex curve. The second fundamental form of f is $I\!I(X, Y) = (\nabla_X Y)^\perp$, where $\perp$ denotes projection to the orthogonal complement of TM^2 in TS^6, or equivalently the orthogonal complement of the span of f, f_x, f_y in $\mathrm{Im}\,\mathbb{O}$. The *first ellipse of curvature* of f is the image

$$\{I\!I(v, v) : v \in T_pM^2,\ ||v|| = 1\}$$

of the unit circle in T_pM^2 under $I\!I$.

Lemma 1.

$$II\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big)=f\times II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)=-II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\Big).$$

Proof. Denoting derivatives by subscripts and writing $\kappa=\langle f_x,f_x\rangle=\langle f_y,f_y\rangle$ for the conformal factor,

$$\begin{aligned}
II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\Big)&=f_{xx}+\kappa f-\frac{\kappa_x}{2\kappa}f_y+\frac{\kappa_x}{2\kappa}f_x\\
II\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big)&=f_{yy}+\kappa f+\frac{\kappa_x}{2\kappa}f_x-\frac{\kappa_y}{2\kappa}f_y\\
II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)&=f_{xy}-\frac{\kappa_y}{2\kappa}f_x-\frac{\kappa_x}{2\kappa}f_y.
\end{aligned}$$

Hence

$$f\times II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)=f\times f_{xy}-\frac{\kappa_y}{2\kappa}f_y+\frac{\kappa_x}{2\kappa}f_x.$$

Differentiating $f\times f_x=f_y$ with respect to y rewrites the first term as $f\times f_{xy}=f_{yy}+\kappa f$ whilst differentiating $f\times f_y=-f_x$ with respect to x gives $f\times f_{xy}=-f_{xx}-\kappa f$ and hence the result. □

The unit circle in TM^2 consists of vectors of the form $v_\theta=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}$; the first ellipse of curvature is parametrised by

$$\begin{aligned}
II(v_\theta,v_\theta)&=\cos^2\theta II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\Big)+2\sin\theta\cos\theta II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)+\sin^2\theta II\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big)\\
&=\cos(2\theta)II\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\Big)+\sin(2\theta)II\Big(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big).
\end{aligned}$$

We thus observe (as did [6]) that the first ellipse of curvature is a point when f is totally geodesic and is otherwise a circle. We will assume we are not in the (trivial) totally geodesic case.

The third fundamental form is defined by

$$III(X,Y,Z)=\big(\nabla_X II(Y,Z)\big)^\perp,$$

where now $\perp$ now denotes projection to the orthogonal complement of $TM^2\oplus\mathrm{image}(II)$ in TS^6.

Lemma 2. *Write $III(v)=III(v,v,v)$. Then*

$$III(v_\theta)=2\cos(3\theta)III\Big(\frac{\partial}{\partial x}\Big)-2\sin(3\theta)III\Big(\frac{\partial}{\partial y}\Big).$$

Thus the image of the unit circle in TM^2 under the third fundamental form is an ellipse, which we call the second ellipse of curvature of f.

Proof. This is a straightforward computation:

$$
\begin{aligned}
III(v_\theta) &= \cos^3\theta\, III\Big(\frac{\partial}{\partial x}\Big) + 3\cos\theta\sin^2\theta\, III\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big)\\
&\quad + 3\cos^2\theta\sin\theta\, III\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big) + \sin^3\theta\, III\Big(\frac{\partial}{\partial y}\Big)\\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta)\, III\Big(\frac{\partial}{\partial x}\Big) + (\sin^3\theta - 3\cos^2\theta\sin\theta)\, III\Big(\frac{\partial}{\partial y}\Big)\\
&= 2\cos(3\theta)\, III\Big(\frac{\partial}{\partial x}\Big) - 2\sin(3\theta)\, III\Big(\frac{\partial}{\partial y}\Big),
\end{aligned}
$$

where in the second line we used that Lemma 1 implies $III(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial y}) = -III(\frac{\partial}{\partial x})$ and $III(\frac{\partial}{\partial x},\frac{\partial}{\partial x},\frac{\partial}{\partial y}) = -III(\frac{\partial}{\partial y})$. □

Elementary arguments [6] demonstrate that the second ellipse of curvature is a circle precisely when f is pseudo-holomorphic. Bryant has shown [7] that all pseudo-holomorphic almost-complex curves in S^6 can be constructed by a Weierstrass type representation, namely by integrating a certain holomorphic differential system on the Grassmannian $\widetilde{\mathrm{Gr}}(2,\operatorname{Im}\mathbb{O}) \subset \mathbb{P}(\mathbb{C}\otimes_{\mathbb{R}}\operatorname{Im}\mathbb{O})$ of oriented 2-planes in $\operatorname{Im}\mathbb{O}$ and projecting the resulting holomorphic curve to S^6. We assumed above that our almost-complex curves f are not totally-geodesic, and now further exclude the next easiest case of pseudo-holomorphic curves. We call the remaining almost-complex curves *superconformal* (this is consistent with the usual terminology, introduced in [5]) and restrict our attention to these; they have not been characterised algebraically.

By definition, for any local complex coordinate $z = x + iy$ on M^2 the spaces

$$V_0 = \mathbb{R}f,\ V_1 = TM^2,\ V_2 = \mathrm{image}(II),\ V_3 = (V_0\oplus V_1\oplus V_2)^\perp$$

give an orthogonal decomposition of $\operatorname{Im}\mathbb{O}$. In the general case, where away from isolated points the second ellipse of curvature is a non-trivial ellipse, then $V_3 = \mathrm{image}(III)$. If the second ellipse of curvature is everywhere a line segment then f has image in a totally geodesic $S^5 \subset S^6$ with unit normal N, then N is orthogonal to image(III), so that $V_3 = \mathrm{image}(III)\oplus\mathbb{R}N$. We shall specify a corresponding G_2 basis, that is an orthonormal basis $e_1,\dots,e_7$ of $\operatorname{Im}\mathbb{O}$ such that $(e_1,\dots,e_7)\in G_2$ and

$$V_0 = \mathbb{R}e_1,\ V_1 = \mathrm{span}_{\mathbb{R}}(e_2,e_3),\ V_2 = \mathrm{span}_{\mathbb{R}}(e_4,e_5),\ V_3 = \mathrm{span}_{\mathbb{R}}(e_6,e_7).$$

Corollary 1. *For any local complex coordinate $z = x + iy$ on M^2, by Lemma 1 the orthonormal framing defined by*

$$
\begin{gathered}
e_1 = f,\quad e_2 = \frac{f_x}{||f_x||},\quad e_3 = \frac{f_y}{||f_y||},\\
e_4 = \frac{II(\frac{\partial}{\partial x},\frac{\partial}{\partial x})}{||II(\frac{\partial}{\partial x},\frac{\partial}{\partial x})||},\quad e_5 = \frac{II(\frac{\partial}{\partial x},\frac{\partial}{\partial y})}{||II(\frac{\partial}{\partial x},\frac{\partial}{\partial y})||},\quad e_6 = e_2\times e_4,\quad e_7 = e_3\times e_4,
\end{gathered}
\tag{3.1}
$$

is a G_2 basis.

Proof. Recall that for orthogonal vectors the cross product agrees with the Cayley product, and we may express the property of being a G_2-frame in terms either of Cayley multiplication or the cross product. From the Fano mnemonic, the conditions

$$e_3 = e_1 \times e_2,\ e_5 = e_1 \times e_4,\ e_6 = e_2 \times e_4,\ e_7 = e_3 \times e_4$$

are equivalent to the requirement that $(e_1, \ldots, e_7) \in G_2$, and clearly the given basis satisfies these conditions. Note that this is an alternative characterisation to that specified in the introduction. □

Lemma 3. *For any local complex coordinate $z = x + iy$ on M^2, we have*

$$f \times \mathit{III}\Big(\frac{\partial}{\partial x}\Big) + f_x \times \mathit{II}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) = -\mathit{III}\Big(\frac{\partial}{\partial y}\Big)$$
$$f \times \mathit{III}\Big(\frac{\partial}{\partial y}\Big) - f_y \times \mathit{II}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) = \mathit{III}\Big(\frac{\partial}{\partial x}\Big).$$

Proof. From Lemma 1, $f \times \mathit{II}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = i\mathit{II}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$ and differentiating gives

$$f_z \times \mathit{II}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big) + f \times \frac{\partial}{\partial z}\mathit{II}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big) = i\frac{\partial}{\partial z}\mathit{II}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big).$$

Now $\mathit{III}(\frac{\partial}{\partial z})$ and $\frac{\partial}{\partial z}\mathit{II}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$ differ by a linear combination of $f, f_z, f_{\bar z}, \mathit{II}(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$ and $\mathit{II}(\frac{\partial}{\partial \bar z}, \frac{\partial}{\partial \bar z})$. Acting by $f\times$ on the first such term gives zero whilst on the other terms $f\times$ acts as multiplication by i. Hence we also have

$$f_z \times \mathit{II}\Big(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\Big) + f \times \mathit{III}\Big(\frac{\partial}{\partial z}\Big) = i\mathit{III}\Big(\frac{\partial}{\partial z}\Big).$$

Breaking this into real and imaginary parts yields

$$\frac{1}{2}\left(f_x \times \mathit{II}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) - f_y \times \mathit{II}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big)\right) + f \times \mathit{III}\Big(\frac{\partial}{\partial x}\Big) = -\mathit{III}\Big(\frac{\partial}{\partial y}\Big)$$
$$\frac{1}{2}\left(-f_y \times \mathit{II}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) - f_x \times \mathit{II}\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big)\right) + f \times \mathit{III}\Big(\frac{\partial}{\partial y}\Big) = \mathit{III}\Big(\frac{\partial}{\partial x}\Big).$$

From the invariance of the cross product under the action of G_2, we know that the G_2-basis of (3.1) satisfies $e_2 \times e_4 = -e_3 \times e_5$ and $e_3 \times e_4 = e_2 \times e_5$. Conformality of f tells us that $||f_x|| = ||f_y||$ and from Lemma 1 $||\mathit{II}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x})|| = ||\mathit{II}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})||$, which together yield the Lemma. □

To define a G_2-framing in a canonical and geometrically meaningful way, consider first the general case, in which the second ellipse of curvature is generically a non-trivial ellipse. This ellipse singles out two directions within V_3, namely its major and minor axes. We are motivated by Lemma 2 to seek a coordinate $z = x + iy$ such that $\mathit{III}(\frac{\partial}{\partial x})$ and $\mathit{III}(\frac{\partial}{\partial y})$ are orthogonal. To see that we can do this everywhere, we first extend the second and third fundamental forms complex-linearly so that they are defined on $T^{\mathbb{C}}M^2 = TM^2 \otimes \mathbb{C}$, and extend the first fundamental form $\langle \cdot, \cdot \rangle$ to the natural Hermitian inner product for which we use the same notation. Using Lemma 1 again,

$$\mathit{III}\Big(\frac{\partial}{\partial z}\Big) = \frac{1}{2}\Big(\mathit{III}\Big(\frac{\partial}{\partial x}\Big) + i\mathit{III}\Big(\frac{\partial}{\partial y}\Big)\Big),\ \mathit{III}\Big(\frac{\partial}{\partial \bar z}\Big) = \frac{1}{2}\Big(\mathit{III}\Big(\frac{\partial}{\partial x}\Big) - i\mathit{III}\Big(\frac{\partial}{\partial y}\Big)\Big),$$

and so $\langle \mathit{III}(\frac{\partial}{\partial x}), \mathit{III}(\frac{\partial}{\partial y})\rangle = 0$ if and only if $\langle \mathit{III}(\frac{\partial}{\partial z}), \mathit{III}(\frac{\partial}{\partial \bar z})\rangle \in \mathbb{R}$. But $\langle \mathit{III}(\frac{\partial}{\partial z}), \mathit{III}(\frac{\partial}{\partial \bar z})\rangle$ is holomorphic [6] and hence constant. Since the second ellipse of curvature was assumed to not be a circle this constant is non-zero so by rotating $z = x + iy$ we may fix

$$\Big\langle III\Big(\frac{\partial}{\partial x}\Big), III\Big(\frac{\partial}{\partial x}\Big)\Big\rangle - \Big\langle III\Big(\frac{\partial}{\partial y}\Big), III\Big(\frac{\partial}{\partial y}\Big)\Big\rangle = 4\Big\langle III\Big(\frac{\partial}{\partial z}\Big), III\Big(\frac{\partial}{\partial \bar{z}}\Big)\Big\rangle = a, \qquad (3.2)$$

for some $a \in \mathbb{R}^+$. Clearly then from Lemma 2, $III(\frac{\partial}{\partial x})$ and $III(\frac{\partial}{\partial y})$ define, respectively, the major and minor axes of the second ellipse of curvature.

Theorem 3. *(c.f. [6]) Let $f : M^2 \to S^6$ be a superconformal almost-complex curve, and choose a local complex coordinate $z = x+iy$ on M^2 as in (3.2). If f is linearly full, then (away from isolated points) the second ellipse of curvature is a non-trivial ellipse and $III(\frac{\partial}{\partial x})$ and $III(\frac{\partial}{\partial y})$ define the major and minor axes of the second ellipse of curvature. Moreover, taking unit vectors in the directions*

$$f,\ f_x,\ f_y,\ II\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big),\ II\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big),\ \pm III\Big(\frac{\partial}{\partial y}\Big),\ -III\Big(\frac{\partial}{\partial x}\Big)$$

gives a G_2-framing.

If the image of f is contained in a totally geodesic $S^5 \subset S^6$ with normal vector N, then the second ellipse of curvature is a line segment, $III(\frac{\partial}{\partial x})$ lies along this line segment and $III(\frac{\partial}{\partial y}) = 0$. In this case, replacing $III(\frac{\partial}{\partial y})$ above by $\pm N$ yields a G_2-framing.

Proof. Consider first the case when f is linearly full. For any local coordinate z, the action $f\times$ preserves V_3. Choosing z as in (3), $III(\frac{\partial}{\partial x})$ and $III(\frac{\partial}{\partial y})$ are orthogonal so

$$f \times III(\frac{\partial}{\partial y}) = bIII(\frac{\partial}{\partial x}) \text{ and } f \times III(\frac{\partial}{\partial x}) = -\frac{1}{b}III(\frac{\partial}{\partial y}), \text{ where } b = \pm\frac{||III(\frac{\partial}{\partial y})||}{||III(\frac{\partial}{\partial x})||}.$$

From Lemma 3 then

$$f_x \times II(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = -(1+\frac{1}{b})III(\frac{\partial}{\partial y}) \text{ and } f_y \times II(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = (b-1)III(\frac{\partial}{\partial x}).$$

By our choice of z, we have that $|b| < 1$, so $b - 1 < 0$, whilst the sign of $1 + \frac{1}{b}$ depends upon that of b. Hence the framing given above lies in G_2. The case when f is not linearly full is clear. □

4. Spectral Curves and Applications

To each superconformal almost-complex $f : T^2 \to S^6$ we shall associate a reducible algebraic curve X, called the spectral curve, and a linear flow in the intersection of two Prym varieties defined over the curve obtained by removing the rational "component" of X. The reader is referred to [14] for further details and proofs of the results in this section.

For any compact simple Lie group G, we may define an automorphism τ of order k, where k is the height of the highest root of the complexified linear algebra $\mathfrak{g}^{\mathbb{C}}$. Namely, choose simple roots $\beta^1, \ldots, \beta^r$ and write the highest root as a positive sum $\theta = \sum_{j=1}^r a_j\beta^j$, so that its height is given by $k = \sum_{j=1}^r \beta^j$. Now define $\tau : G \to G$ by $\tau = \mathrm{Ad}_{\mathrm{C}}$, where $C = \frac{1}{k}\sum_{j=1}^r t_j$ where each t_j lies in the toral subalgebra $\mathfrak{t}$ and is determined by $\beta^i(t_j) = \delta^i_j$. This then gives G/T the structure of a k-symmetric space, where T is the maximal torus of G. In particular, writing T^3 and T^2 for the maximal tori of $SO(7,\mathbb{R})$ and G_2 respectively, both $SO(7,\mathbb{R})/T^3$ and its submanifold G_2/T^2 are given a 6-symmetric space structure by this construction, where in both cases

$$C := (1,\, R_{\frac{\pi}{3}},\, R_{\frac{2\pi}{3}},\, R_{\pi}),$$

and R_θ denotes the matrix for rotation through the angle θ. Writing $\mathfrak{g}_2^{\mathbb{C}} := \mathfrak{g}_2 \otimes \mathbb{C}$, τ gives the decomposition

$$\mathfrak{g}_2^{\mathbb{C}} = \bigoplus_{j=0}^{5} \mathfrak{g}^j$$

where $\mathfrak{g}^j$ is the $\exp 2\pi\sqrt{-1}j/6$-eigenspace of τ, and hence

$$T(G_2/T)^{\mathbb{C}} = \bigoplus_{j=0}^{5} [\mathfrak{g}^j].$$

Let $f : M^2 \to S^6$ be superconformal almost-complex, with G_2 frame F as defined in the previous section. Then a computation [5] shows that

$$F^{-1}dF = (u_0 + u_1)dz + (\bar{u}_0 + \bar{u}_1)d\bar{z},\ u_j \in \mathfrak{g}_j,$$

and that $F^{-1}dF(T^{1,0}M^2)$ contains a cyclic element, that is an element expressible as $\sum_{j=1} c^j v_{\beta^j} + cv$, where v_{β^j}, v are in the root spaces of $\beta^j, -\theta$ respectively, and all coefficients are non-zero. We say then that F is τ-primitive.

For $\zeta \in \mathbb{C} - \{0\}$ define

$$\varphi_\zeta := (u_0 + u_1\zeta)dz + (\bar{u}_0 + \bar{u}_1\zeta^{-1})d\bar{z};$$

we make the standard observation that the connections

$$\nabla_\zeta = \nabla^L + \varphi_\zeta$$

defined by these forms in the trivial rank seven complex vector bundle V over the surface M^2 all have zero curvature. Here V may be thought of as (the pullback under F of) the tangent bundle to $G_2^{\mathbb{C}}$, and then $\nabla^L = \nabla - \frac{1}{2}F^{-1}dF$ is the connection which trivialises this bundle by left translation. Since $G_2^{\mathbb{C}}$ is the identity component of the subgroup of $GL(7,\mathbb{C})$ preserving a generic three-form (see e.g. [25]), we define a $G_2^{\mathbb{C}}$ structure on V by specifying a generic three-form α, by which we mean one for which

$$q(v,w) := -\frac{1}{6}(v \lrcorner \alpha') \wedge (w \lrcorner \alpha') \wedge \alpha' \tag{4.1}$$

satisfies $\det \neq 0$. Here $\det(q) \in (\Lambda^7 V^*)^9$ is taken when q is viewed as a map $V \to V^* \otimes \Lambda^7 V^*$.

We specialise now to the case when M^2 is the complex plane. Our connection forms satisfy

$$\tau(\varphi_\zeta) = \varphi_{\epsilon\zeta},\ \bar{\varphi}_\zeta = \varphi_{\bar{\zeta}^{-1}},$$

and we search for parallel sections of the endomorphism bundle $\operatorname{End} V$ with the same symmetries. We say that f is of *finite type* if there is a polynomial Killing field, i.e. a solution $A = \sum_{j=-d}^{d} A_j\zeta^j$ to

$$dA = [A, \varphi_\zeta]$$

with

$$\varphi_\zeta = (A_{d-1} + A_d\zeta)dz + (A_{1-d} + A_{-d}\zeta^{-1})d\bar{z}$$

and $\bar{A}_\zeta = A_{\bar{\zeta}^{-1}}$, $\tau(A_\zeta) = A_{\epsilon\zeta}$. All superconformal almost-complex $f : T^2 \to S^6$ are of finite type [8, 5], and the polynomial Killing fields form a commutative algebra [9]. We call d the *degree* of A_ζ; note that $d = 6k+1$ for some $k \in \mathbb{Z}^+ \cup \{0\}$.

Let $A_\zeta(z) \in H^0(\mathcal{O}(2(6k+1)) \otimes \mathrm{End}V)$ be a polynomial Killing field for f of minimal degree, and define an algebraic curve $\hat{X}$ by

$$\det(A_\zeta - \mu I) = 0.$$

The eigenvalues of A_ζ are of the form $0, \mu_1, \mu_2, \mu_3, -\mu_1, -\mu_2, -\mu_3$, and satisfy $\mu_1 + \mu_2 + \mu_3 = 0$. Hence $\hat{X}$ has the form

$$\mu\Big(\mu^6 - a_1(\zeta)\mu^4 + \frac{a_1(\zeta)^2}{4}\mu^2 - a_2(\zeta)\Big) = 0.$$

with

$$a_1(\zeta) = \mu_1^2 + \mu_2^2 + \mu_3^2, \quad a_2(\zeta) = (\mu_1\mu_2\mu_3)^2.$$

The τ-symmetry of A_ζ induces an order six involution $\tau : \zeta \mapsto \epsilon\zeta$; we call the quotient curve $X = \hat{X}/\tau$ the *spectral curve* of f. Writing $a_j(\zeta) = b_j(\lambda)|_{\lambda=\zeta^6}$, the *main component* Y of the spectral curve is given by

$$\mu^6 - b_1(\lambda)\mu^4 + \frac{b_1(\lambda)^2}{4}\mu^2 - b_2(\lambda) = 0.$$

The reality condition on the polynomial killing field A_ζ induces on the spectral curve an antiholomorphic involution

$$\rho : (\lambda, \mu) \mapsto (\bar{\lambda}^{-1}, \bar{\mu}).$$

Theorem 4. [14] *For a generic polynomial Killing field, the main component Y of the spectral curve is smooth.*

The spectral curve can be realised as the characteristic polynomial of a Killing field depending only on $\lambda = \zeta^6$. Let

$$C_\zeta = \mathrm{diagonal}(1, S_\zeta, S_{\zeta^2}, S_{\zeta^3})$$

where

$$S_\zeta = \begin{pmatrix} \frac{1}{2}(\zeta + \zeta^{-1}) & -\frac{1}{2i}(\zeta - \zeta^{-1}) \\ \frac{1}{2i}(\zeta - \zeta^{-1}) & \frac{1}{2}(\zeta + \zeta^{-1}) \end{pmatrix}$$

and define

$$A_\lambda(z) = \mathrm{Ad}_{C_{\zeta^{-1}}} A_\zeta(z).$$

The Laurent polynomial A_λ has degree $2(k+1)$ when A_ζ has degree $2(6k+1)$ and X is given by the characteristic polynomial of A_λ.

Let α be the generic three-form giving the trivial $\mathbb{C}^7$ bundle V on $\mathbb{C}$ its G_2 structure. It induces a metric g on V by

$$g = \frac{q}{(\det(q))^{1/9}}$$

where q was defined in (4.1). For each $z \in \mathbb{C}$, write V_z for the fibre of V over z and let $V^z = \mathbb{P}^1 \otimes V_z$, with metric g_z on V^z as above.

Lemma 4. [14] *Let V_0 be the line bundle defined by $V_0^z \subset \ker A_\zeta(z)$ and $E^z = V^z/V_0^z \otimes \mathcal{O}(-\frac{1}{6}\mathrm{deg}(\mathrm{V^z}/\mathrm{V_0^z}))$. Then*

$$\omega_z(v_1, v_2) = g^z(A_\zeta(z)v_1, v_2)$$

defines a symplectic form on E, and the restriction of the polynomial killing field $A_\zeta(z)$ acts as a symplectic endomorphism of E.

For each $z \in \mathbb{C}$ define $\mathcal{E}_z \to \hat{Y}$ to be the unique line bundle contained in the sub-sheaf $\ker(\mu \cdot \mathrm{id} - \zeta^* A_\zeta)(z) \subset \zeta^* E$. These bundles are preserved by the order six involution τ, and hence descend to *eigenline bundles* $\mathcal{E}_z$ on the main component Y of the spectral curve.

We have on Y the involution

$$\sigma : (\lambda, \mu) \mapsto (\lambda, -\mu);$$

and so may define $C_1 \simeq Y/\sigma$. Let C_2 be the hyperelliptic curve

$$z^2 = b_2(\lambda),$$

and observe that

$$\begin{aligned} \pi_2 : \quad & Y \to C_2 \\ (\lambda, \mu) \mapsto & \left(\lambda, \mu\left(\mu - \frac{b_1}{2}\right)\right) \end{aligned}$$

exhibits Y as a three-to-one cover of C_2.

$$\begin{tikzcd} & Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ C_1 \arrow[dr] & & C_2 \arrow[dl] \\ & P^1 & \end{tikzcd}$$

The *Prym variety* of the cover $\pi_i : Y \to C_i$ is defined to be the kernel of the norm map

$$\begin{aligned} \mathrm{Nm}_i : \quad & \mathrm{Jac}(Y) \to \mathrm{Jac}(C_i) \\ \Big[\sum c^j p_j\Big] \mapsto & \Big[\sum c^j \pi_i(p_j)\Big]. \end{aligned}$$

Hence in particular

$$P(Y, C_1) = \{\text{degree 0 line bundles on } Y \text{ satisfying } \sigma^* \mathcal{L} \cong \mathcal{L}^*\}$$

where σ is the involution of the 2–sheeted cover $Y \to C_1$. The intersection

$$\mathrm{Tur} := P(Y, C_1) \cap P(Y, C_2)$$

is connected.

Theorem 5. [14]

(1) *Let R denote the ramification divisor of $\lambda : Y \to \mathbb{P}^1$. The constant translate $\mathcal{E}_z^* \otimes \mathcal{O}(-\frac{1}{2}R)$of the eigenline bundles lies in the intersection* Tur *of the two Prym varieties, in fact in the real slice* $\mathrm{Tur}_{\mathbb{R}}(Y, \mathbb{P}^1)$ *given by* $\overline{\rho^* \mathcal{E}_z} \simeq \mathcal{E}_z$.

(2) *The map*

$$T^2 \to \mathrm{Tur}_{\mathbb{R}}(Y, \mathbb{P}^1)$$
$$z \mapsto \mathcal{E}_z^* \otimes \mathcal{O}(-\frac{1}{2}R)$$

defines a linear flow.

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