EMBEDDING 3-MANIFOLDS WITH CIRCLE ACTIONS IN 4-SPACE

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Abstract. We give constraints on the Seifert invariants of orientable 3-manifolds which admit fixed-point free circle actions and embed in $\mathbb{R}^4$. In particular, the generalized Euler invariant ε of the orbit fibration is determined up to sign by the base orbifold B unless $H_1(M;\mathbb{Z})$ is torsion free, in which case it can take at most one nonzero value (up to sign). No such manifold with base $B = S^2(\alpha_1, \ldots, \alpha_r)$ with r odd and $\varepsilon = 0$ embeds in $\mathbb{R}^4$.

The question of which closed 3-manifolds M embed in $\mathbb{R}^4$ has received surprisingly little attention. (The relevant papers known to us are [1–7].) In particular, it is not yet known which Seifert fibred 3-manifolds embed, although in many other respects this is a well-understood class of spaces, with natural parametrizations in terms of Seifert data. It was shown early that if M embeds in $\mathbb{R}^4$ it must be orientable, and the torsion subgroup $T(M)$ of $H_1(M;\mathbb{Z})$ must be a direct double: $T(M) \cong U \oplus U$ for some finite abelian group U [5]. Moreover, the linking pairing ℓ_M on $T(M)$ must be hyperbolic [7]. Most of the work to date has focused on smooth embeddings, but these conditions must also hold if M embeds as a TOP locally flat submanifold.

In §2 we shall observe that $T(M)$ being a direct double imposes strong constraints on the Seifert data of orientable 3-manifolds which admit fixed point free S^1-actions and which embed in $\mathbb{R}^4$. The present simple argument does not work for Seifert fibred 3-manifolds with nonorientable base orbifold; in [1] we use the $Z/2Z$-Index Theorem to constrain the Euler invariants in the latter case. In §3 and §4 we describe the linking pairing and the Blanchfield pairing (for infinite cyclic covers).

When the Seifert data is "skew-symmetric", i.e., is a set of complementary pairs the corresponding Seifert manifold embeds smoothly [1]. (Such manifolds have Euler invariant $\varepsilon = 0$ and an even number of exceptional fibres.) In §5 we show that no Seifert manifold with base

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$B = S^2(\alpha_1, \ldots, \alpha_r)$ with r odd and $\varepsilon = 0$ embeds in $\mathbb{R}^4$. Here we use a result of [6] on the Alexander polynomial associated to the canonical infinite cyclic cover of 3-manifolds M with $\beta_1(M) = 1$ and which embed in $\mathbb{R}^4$. It is not known whether skew-symmetry of the Seifert data is a necessary condition. In the final sections we consider further the cases with $r = 4$ or $\varepsilon \neq 0$ and $r = 3$.

1. Notation and some remarks on embeddings

An orientable 3-manifold admits a fixed-point free S^1-action if and only if it is Seifert fibred over an orientable base orbifold. Let T_g be the orientable surface of genus g. We shall assume henceforth that $M = M(g; S)$ is an orientable 3-manifold which is the total space of a Seifert bundle over the orbifold $B = T_g(\alpha_1, \ldots, \alpha_r)$, with Seifert invariants $S = \{(\alpha_i, \beta_i) \mid 1 \leq i \leq r\}$, where $1 < \alpha_i$ and $(\alpha_i, \beta_i) = 1$, for all $i \leq r$. (We do not assume that $0 < \beta_i < \alpha_i$). If $r = 1$ we allow also the possibility $\alpha_1 = 1$. Let $\varepsilon = -\Sigma_{i=1}^{i=r}(\beta_i/\alpha_i)$ be the generalized Euler invariant of the Seifert bundle, and let $\Pi = \Pi_{i=1}^{i=r}\alpha_i$. (Replacing each β_i by $\eta\beta_i + c_i\alpha_i$ where $\eta = \pm 1$ and $\Sigma c_i = 0$ gives a homeomorphic manifold.)

As B is the connected sum of T_g and $S^2(\alpha_1, \ldots, \alpha_r)$ we have $M = M(0; S)\sharp_f M(g; \emptyset) = M(0; S)\sharp_f (T_g \times S^1)$, where $\sharp_f$ denotes fibre sum. In particular, if M embeds in $\mathbb{R}^4$ then so does $M\sharp_f(T_g \times S^1)$, by Lemma 3.2 of [1]. In some of our arguments we shall need to assume that $g = 0$. Thus it would be very convenient to have a converse to this stabilization result. (The analogous implication in the case of nonorientable base orbifolds is not reversible. See [1]).

Let $L = L_+ \amalg L_-$ be a link which is partitioned into two disjoint slice links. If we attach 2-handles along L_+ to the unit ball D^4 in $\mathbb{R}^4$ and delete 2-handles embedded in D^4 along L_- the boundary of the resulting region of $\mathbb{R}^4$ is the result of 0-framed surgery on L. Thus 3-manifolds with such surgery descriptions embed smoothly in $\mathbb{R}^4$. It is easy to see that any Seifert manifold $M(0; S)$ may be obtained by Dehn surgery on a link whose components are fibres of the Hopf fibration, and with framings $\neq 1$. Are there natural surgery presentations for Seifert manifolds in terms of 0-framed surgery on links?

2. The torsion subgroup

In this section we shall describe the torsion subgroup of $H_1(M; \mathbb{Z})$ in terms of the Seifert invariants of M.

Theorem 1. *Let $M = M(g; S)$ be an orientable 3-manifold which is Seifert fibred over an orientable base orbifold $B = T_g(\alpha_1, \ldots, \alpha_r)$.*

Then $H_1(M;\mathbb{Z}) \cong Z^{2g} \oplus (\bigoplus_{i\geq 0}(Z/\lambda_i Z))$, where λ_i is determined by $\{\alpha_1,\dots,\alpha_r\}$ and is nonzero, for all $i>0$, while $|\varepsilon|\Pi = \lambda_0\Pi_{j\geq 1}\lambda_j$.

Proof. The fundamental group $\pi_1(M)$ has a presentation

$$\langle a_1,\dots b_g, q_1,\dots,q_r, h \mid (\Pi[a_i,b_i])(\Pi q_j)=1, q_i^{\alpha_i}h^{\beta_i}=1, h\ central\rangle.$$

Hence $H_1(M;\mathbb{Z}) \cong Z^{2g}\oplus Cok(A)$, where A is the matrix

$$A = \begin{pmatrix} 0 & 1 & \dots & 1 \\ \beta_1 & \alpha_1 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \beta_r & 0 & \dots & \alpha_r \end{pmatrix}$$

Let $E_i(A)$ be the ideal generated by the $(r+1-i)\times(r+1-i)$ subdeterminants of A, and let Δ_i be the positive generator of $E_i(A)$. Then $\Delta_0 = |det(A)| = |\varepsilon|\Pi$, while as the elements of each row are relatively prime Δ_i is the highest common factor of the $(r-i-1)$-fold products of distinct α_js, if $0<i<r$, and $\Delta_i = 1$ if $i\geq \max\{r-1,1\}$. (In particular, if $r>2$ then $\Delta_{r-2} = \mathrm{hcf}(\alpha_1,\dots,\alpha_r)$). Thus Δ_i depends only on $\{\alpha_1,\dots,\alpha_r\}$ and is nonzero, for all $i>0$. If we set $\lambda_i = \Delta_i/\Delta_{i+1}$, for $i\geq 0$, then $Cok(A)\cong \bigoplus_{i\geq 0}(Z/\lambda_i Z)$, by the Elementary Divisor Theorem. In particular, $|\varepsilon|\Pi = \lambda_0\Pi_{j\geq 1}\lambda_j$. □

Note that $T(M)\cong T(M(0;S))$ and $h\in T(M)$ if and only if $\varepsilon\neq 0$.

Corollary. *If $\Delta_1 = 1$ then $T(M)$ is cyclic, and $T(M)=0$ if and only if $\varepsilon = 0$ or $\pm 1/\Pi$. If $\Delta_1>1$ then $T(M)\neq 0$. Given $\{\alpha_1,\dots,\alpha_r\}$ such that $\Delta_1>1$, there is at most one value of $|\varepsilon|$ for which the group $T(M)$ is a direct double.*

Proof. As $\Delta_1 = \Pi_{i\geq 1}\lambda_i$ divides the order of $T(M)$, this group is nonzero unless $\Delta_1 = 1$. If $\varepsilon = 0$ then $T(M)\cong\bigoplus_{i\geq 1}(Z/\lambda_i Z)$, and so is a direct double if and only if $\lambda_{2i-1} = \lambda_{2i}$ for all $i>0$. If $\varepsilon\neq 0$ then $T(M)\cong \bigoplus_{i\geq 0}(Z/\lambda_i Z)$, and so is a direct double if and only if $\lambda_{2i} = \lambda_{2i+1}$ for all $i\geq 0$. In particular, $\varepsilon = (\Delta_1)^2/\Pi\Delta_2$. Clearly these two systems of equations can both be satisfied only if $\lambda_i = 1$ for all $i>0$ and $\lambda_0 = 0$ or 1, in which case $T(M)=0$. □

The elementary divisors λ_i may be determined more explicitly by localization. If p is a prime, an integer α has p-adic valuation v if $\alpha = p^v q$, where p does not divide q_i.

Corollary. *Let p be a prime and let $v_i\geq 0$ be the p-adic valuation of α_i. Assume that the indexing is such that $v_i\geq v_{i+1}$ for all i. If $\varepsilon = 0$ and $T(M)$ is a direct double then $v_{2j-1} = v_{2j}$ for all $j\geq 1$.*

Proof. The condition $v_1 = v_2$ follows immediately from the fact that $p^{v_2}\varepsilon$ is an integer. The p-adic valuation of λ_j is v_{j+2}, for all $j \geq 1$, and so $v_{2j-1} = v_{2j}$ for all $j \geq 2$, if $\bigoplus_{i\geq 1}(Z/\lambda_i Z)$ is a direct double. $\square$

If $S = \{(2,1),(3,-1),(6,-1)\}$ or $S = \{(2,1),(2,1),(2,-1),(3,-1),(6,-1)\}$ then $T(M(0;S)) = 0$ or $(Z/2Z)^2$, respectively, and so the hypotheses do not imply that r must be even. However in Theorem 2 we shall show that this must be so if $\varepsilon = 0$ and $M(0;S)$ embeds in $\mathbb{R}^4$.

Although the β_is only contribute to the structure of $T(M)$ via ε, they play a more substantial role in the linking pairing. (See §4 below).

It follows immediately that $M(g;S)$ is an homology 3-sphere if and only if $g = 0$ and $\varepsilon\Pi = \pm 1$. In particular, $\mathrm{hcf}\{\alpha_i, \alpha_j\} = 1$ for all $i < j \leq r$. If $M(0;S)$ is a $\mathbb{Z}$-homology sphere it embeds as a TOP locally flat submanifold of $\mathbb{R}^4$ [2]. Hence $M(g;S)$ embeds also.

Similarly, $M(g;S)$ is an homology $S^2 \times S^1$ if and only if $g = 0$, $\varepsilon = 0$ and $\mathrm{hcf}\{\alpha_i, \alpha_j, \alpha_k\} = 1$ for all $i < j < k \leq r$. If M is an homology $S^2 \times S^1$ there is an (essentially unique) degree-1 map $f : M \to S^2 \times S^1$. If, moreover, f induces an isomorphism on homology with local coefficients then M embeds in $\mathbb{R}^4$ [4]. However, this is not a necessary condition for embedding, and it shall follow from Theorem 2 below that if $r \geq 3$ no such map with domain $M(0;S)$ is ever a $\mathbb{Z}[Z]$-homology equivalence. It is not known in general which (Seifert fibred) homology $S^2 \times S^1$s embed.

When M is Seifert fibred over a nonorientable base orbifold $T(M)$ is again largely determined by the set $\{\alpha_1, \ldots, \alpha_r\}$, but ε is not constrained at all by the condition that $T(M)$ be a direct double [1].

3. Bilinear pairings

A *linking pairing* on a finite abelian group N is a symmetric bilinear function $\ell : N \times N \to \mathbb{Q}/\mathbb{Z}$ which is nonsingular in the sense that $\tilde{\ell} : n \mapsto \ell(-,n)$ defines an isomorphism from N to $Hom(N, \mathbb{Q}/\mathbb{Z})$. If L is a subgroup of N then $\tilde{\ell}$ induces an isomorphism $L^\perp = \{t \in N \mid \ell_M(t,l) = 0 \ \forall l \in L\} \cong N/L$. Such a pairing splits uniquely as the orthogonal sum (over primes p) of its restrictions to the p-primary subgroups of N. It is *metabolic* if there is a subgroup P with $P = P^\perp$, *split* [7] if also P is a direct summand and *hyperbolic* if N is the direct sum of two such subgroups. If ℓ is split N is a direct double.

If M is a closed oriented 3-manifold Poincaré duality determines a linking pairing $\ell_M : T(M) \times T(M) \to \mathbb{Q}/\mathbb{Z}$, which may be described as follows. Let w, z be disjoint 1-cycles representing elements of $T(M)$ and suppose that $mz = \partial C$ for some 2-chain C which is transverse to w and some nonzero $m \in \mathbb{Z}$. Then $\ell_M(w,z) = (w.C)/m \in \mathbb{Q}/\mathbb{Z}$. It follows easily from the Mayer-Vietoris theorem and duality that if M embeds in $\mathbb{R}^4$ then ℓ_M is hyperbolic. (If X and Y are the closures of the components of $\mathbb{R}^4 - M$ and T_X and T_Y are the kernels of the induced homomorphisms from $T(M)$ to $H_1(X;\mathbb{Z})$ and $H_1(Y;\mathbb{Z})$ (respectively) then $T(M) \cong T_X \oplus T_Y$ and the restriction of ℓ_M to each of these summands is trivial [7]).

There are analogous pairings on covering spaces of M. In particular, if $\phi : \pi_1(M) \to Z$ is an epimorphism with associated covering space M_ϕ the homology modules $H_*(M_\phi;R)$ are $R\Lambda$ modules, where $\Lambda = \mathbb{Z}[Z] = \mathbb{Z}[t,t^{-1}]$ and $R\Lambda = R \otimes \Lambda = R[t,t^{-1}]$, for any coefficient ring R. There is a *Blanchfield pairing* on the $\mathbb{Q}\Lambda$-torsion submodule of $H_1(M_\phi;\mathbb{Q})$ with values in $\mathbb{Q}(t)/\mathbb{Q}\Lambda$ which is nonsingular and hermitean with respect to the involution sending t to t^{-1}. Such a pairing is *neutral* (or *null-cobordant* [6]) if the underlying $\mathbb{Q}\Lambda$-torsion module has a submodule which is its own annihilator, and is hyperbolic if the underlying module is the direct sum of two such self-annihilating submodules. If M embeds in $\mathbb{R}^4$ then $H^1(M;\mathbb{Z}) \cong H^1(X;\mathbb{Z}) \oplus H^1(Y;\mathbb{Z})$. Thus if also $\beta_1(M) = 1$ the epimorphism ϕ is unique up to sign, and extends to an epimorphism on one of the complementary regions. The Blanchfield pairing of M is then neutral, by Theorem 4.2 of [6]. In particular, the characteristic polynomial of the automorphism t of the torsion submodule of $H_1(M_\phi;\mathbb{Q})$ is a product $g(t)g(t^{-1})$ for some $g \in \mathbb{Q}\Lambda - \{0\}$.

4. Pairings on Seifert manifolds

Assume now that $M = M(g;S)$. Then $T(M)$ is a subgroup of the group generated by the images of h and $q_1, \dots q_r$. (We shall use the same symbols to denote homology classes and representative cycles.) The Seifert structure gives natural 2-chains relating these 1-cycles. For let N_i be a torus neighborhood of the i^{th} singular fibre, with meridianal disc D_i, and let B_o be a section of the restriction of the Seifert fibration to $M - \cup_{i\geq 1} N_i$. Then $\partial D_i = \alpha_i q_i + \beta_i h$ and $\partial B_o = \Sigma q_i$. Moreover $h.B_o = 1$, $h.D_i = 0 = q_j.B_o$, $q_i.D_i = \beta_i$ and $q_i.D_j = 0$ if $i \neq j$, for all $1 \leq i, j, \leq r$. As these intersection numbers are independent of g, it is clear that $\ell_M \cong \ell_{M(0;S)}$.

If $g = 0$ and $\varepsilon = 0$ there is an essentially unique epimorphism $\phi : \pi \to Z$, and $h \notin T(M)$. The manifold M is the mapping torus $F \times_\theta S^1$ of a periodic self homeomorphism θ of a closed orientable surface F, and ϕ is the homorphism induced by the bundle projection. Then $M_\phi \cong F \times R$, and so $H_1(M_\phi;\mathbb{Q}) \cong H_1(F;\mathbb{Q})$ as a vector space. Hence it is a torsion $\mathbb{Q}\Lambda$-module. The Blanchfield pairing is determined by the intersection form I_F on $H_1(F;\mathbb{Q})$ and the isometry $\theta_* = H_1(\theta)$

[9]. In particular, it is neutral if and only if $H_1(F;\mathbb{Q}) = A \oplus B$ where $\beta_1(F) = 2\dim_{\mathbb{Q}} A$, $\theta_*(A) = A$ and $I_F(A,A) = 0$.

For small values of r we can compute ℓ_M explicitly. If $r = 1$ and $\varepsilon = 0$ then $T(M) = 0$, while if $\varepsilon \neq 0$ then $T(M) = (Z/\varepsilon\alpha Z)h$ and $\ell_M(h,h) = -\alpha/\beta$. If $r = 2$ let x, y be such that $x\alpha_1 - y\beta_1 = 1$, and let $k = yq_1 + xh$ and $k' = \alpha_1 q_1 + \beta_1 h = \partial D_1$. Then $q_1 = -\beta_1 k + xk'$ and $h = \alpha_1 k - yk'$, and $\varepsilon\alpha_1\alpha_2 k = \partial(\alpha_2 B_o - (x\alpha_2 + y\beta_2)D_1 - D_2)$. If $\varepsilon = 0$ then $T(M) = 0$, while if $\varepsilon \neq 0$ then $T(M) = (Z/\varepsilon\alpha_1\alpha_2 Z)k$ and $\ell_M(k,k) = (x\alpha_2 + y\beta_2)/\varepsilon\alpha_1\alpha_2$.

In particular, if $r \leq 2$ and $T(M)$ is a direct double $\varepsilon\Pi = 0$ or ± 1, and so $M(0;S) \cong S^3$ or $S^2 \times S^1$. (This also follows immediately from Theorem 1.) More generally, if $r \leq 2$ then $M = M(g;S)$ embeds in $\mathbb{R}^4$ if and only if $T(M) = 0$, in which case M is a circle bundle over T_g with Euler invariant 0 or ± 1.

5. $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds with $g = 0$ and r odd

If $S = \{(\alpha_i, \beta_i), (\alpha_i, -\beta_i) \mid 1 \leq i \leq t\}$ (so that the Seifert data occurs in opposite pairs) a fibre sum construction shows that $M(0;S)$ embeds smoothly in $\mathbb{R}^4$, by Lemma 3.1 of [1]. In this case $\varepsilon = 0$ and $r = 2t$ is even. In this section we shall show that when $\varepsilon = 0$ and r is odd $M(0 : S)$ does not embed. It remains an open question whether "skew-symmetry" of the Seifert data is also necessary for embedding when $\varepsilon = 0$. Our argument uses the fact that if $\varepsilon = 0$ then $M(0;S)$ has an essentially unique infinite cyclic cover, and does not appear to extend easily to the case $g > 0$.

If $r \geq 3$ and $\varepsilon = 0$ then $M(g;S)$ is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold, with three exceptions: $M(0;(2,1),(4,-1),(4,-1))$, $M(0;(2,1),(3,-1),(6,-1))$ and $M(0;(2,1),(2,1),(2,-1),(2,-1))$ are flat manifolds.

Theorem 2. *Let $M = M(0;S)$, where $S = \{(\alpha_i, \beta_i) \mid 1 \leq i \leq r\}$ is such that $\varepsilon = -\Sigma_{i=1}^{i=r}(\beta_i/\alpha_i) = 0$. If M embeds in $\mathbb{R}^4$ then r is even.*

Proof. We may assume that $r > 2$, and hence that $\alpha_i > 1$ for all $i \leq r$. The group $\pi = \pi_1(M)$ has a presentation

$$\langle q_1, \dots, q_r, h \mid q_1 \dots q_r = 1, q_i^{\alpha_i} h^{\beta_i} = 1, h \textit{ central}\rangle,$$

and $H_1(M;\mathbb{Z})/T(M) \cong Z$, by Theorem 1, since $g = 0$ and $\varepsilon = 0$. Hence there is an essentially unique epimorphism $\phi : \pi \to Z$. Let $n_i = \phi(q_i)$, for $i \leq r$, and $\mu = \phi(h)$. Choose $t \in \pi$ such that $\phi(t) = 1$. Then $t = wh^e$ for some exponent e and some word w in the q_is. After modifying our choice of t, if necessary, we may assume that $w = \Pi_{i<r} {q_i}^{f_i}$ for some exponents f_i. Let $r_i = q_i t^{-n_i}$, for $i \leq r$ and $s = ht^{-\mu}$. Then

$\phi(r_i) = \phi(s) = 0$ and π has an equivalent presentation

$$\langle r_1, \ldots, r_r, s, t \mid \Pi_{i \leq r} r_i t^{n_i} = 1, s^e(\Pi_{i<r}(r_i t^{n_i})^{f_i}) t^{e\mu - 1} = 1,$$
$$(r_i t^{n_i})^{\alpha_i}(st^\mu)^{\beta_i} = 1, st^\mu r_i = r_i st^\mu, ts = st\rangle.$$

On applying the homomorphism ϕ we see that

(1) $\Sigma_{i \leq r} n_i = 0$;
(2) $e\mu + \Sigma_{i<r} n_i f_i = 1$;
(3) $n_i\alpha_i + \mu\beta_i = 0$ for $i \leq r$.

Hence $\mathrm{hcf}(n_i f_i, \mu) < \mu$ for some $i < r$, $\mathrm{hcf}(n_i, \mu) < \mu$ for all $i \leq r$ and $\mu = \mathrm{lcm}\{\alpha_1 \ldots, \alpha_r\}$. Let $m_i = \mu/\alpha_i$, for $i \leq r$. We see also that $T(M)$ is generated by the images of the r_i and s, and $\Sigma r_i = 0$, $\alpha_i r_i + \beta_i s = 0$ and $es + \Sigma f_i r_i = 0$. (We may choose e and the f_i subject only to the condition (2). In particular, we may assume that $0 \leq f_i < \alpha_i$ for $i < r$. The f_i cannot all be 0, since $\mu > 1$. On the other hand, if $\mu = \alpha_1$, say, then we may assume $f_i = 0$ for $i > 1$.)

We may use the free differential calculus to find a $(2r + 3) \times (r + 1)$ presentation matrix for the Λ-module $H_1(M_\phi; \mathbb{Z})$. Let $\nu_0(y) = 0$ and $\nu_k(y) = (y^k - 1)/(y - 1)$ for $k > 0$. Then $H_1(M_\phi; \mathbb{Z}) = \mathrm{Cok}(P)$, where

$$P = \begin{pmatrix} 1 & t^{n_1} & \ldots & t^{\Sigma_{k<r} n_k} & 0 \\ \nu_{f_1}(t^{n_1}) & t^{n_1 f_1}\nu_{f_2}(t^{n_2}) & \ldots & 0 & e \\ \nu_{\alpha_1}(t^{n_1}) & 0 & \ldots & 0 & t^{n_1\alpha_1}\nu_{\beta_1}(t^\mu) \\ \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & \ldots & \nu_{\alpha_r}(t^{n_r}) & t^{n_r\alpha_r}\nu_{\beta_r}(t^\mu) \\ t^\mu - 1 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & \ldots & t^\mu - 1 & 0 \\ 0 & 0 & \ldots & 0 & t - 1 \end{pmatrix}.$$

(Here the columns correspond to the generators $r_1, \ldots, r_r, s$ and the rows to the relations. This can be simplified by row operations, corresponding to Tietze moves on the presentation).

This matrix clearly has maximal rank, and so $H_1(M_\phi; \mathbb{Z})$ is a torsion Λ-module. We may tensor over Λ with $\mathbb{Q}[\zeta_\mu] = \mathbb{Q}[t, t^{-1}]/(\phi_\mu(t))$, the field of μ^{th} roots of unity, via the homomorphism sending $t \in \Lambda$ to the primitive root ζ_μ. The powers $\zeta_\mu^{n_i}$ are α_i^{th} roots of unity, and are not 1, since $\mathrm{hcf}(n_i, \mu) = m_i < \mu$. Therefore $\nu_{\alpha_i}(\zeta_\mu^{n_i}) = 0$ for all i. On the other hand, if $\mathrm{hcf}(n_i f_i, \mu) < \mu$ then $\nu_{f_i}(\zeta_\mu^{n_i}) \neq 0$. The resulting matrix has rank 3, and so $\mathbb{Q}(\zeta_\mu) \otimes_\Lambda H_1(M_\phi; \mathbb{Q})$ has dimension $r - 2$ over the field $\mathbb{Q}(\zeta_\mu)$. Thus if r is odd the characteristic polynomial of the automorphism t of $H_1(M_\phi; \mathbb{Q})$ is not a product $g(t)g(t^{-1})$ for any $g \in \mathbb{Q}\Lambda - \{0\}$, and so no such manifold can embed in $\mathbb{R}^4$. $\square$

If M, ϕ, h, s, t, μ are as in the theorem then $\phi^{-1}(\mu Z) \cong \mathrm{Ker}(\phi) \times Z$, since $h = st^{\mu}$ is central. Therefore the μ-fold covering space associated to the subgroup $\phi^{-1}(\mu Z)$ is a product $F \times S^1$, where F is a closed surface, and $M_\phi \cong F \times \mathbb{R}$. Moreover, $M \cong F \times_\theta S^1$, where θ has order μ, and so the base orbifold $B = S^2(\alpha_1, \dots, \alpha_r)$ is the quotient of F by an effective action of $Z/\mu Z$. Hence $\chi(F) = \mu\chi^{orb}(B) = (2-r)\mu + \Sigma_{i\leq r} m_i$.

In particular, $H_1(M_\phi; \mathbb{Z})$ is never 0, if $r \geq 3$. (This is also clear from the proof of Theorem 2.) Hence there are no $\mathbb{Z}[Z]$-homology equivalences $f : M(0; S) \to S^2 \times S^1$.

6. $\mathbb{H}^2 \times \mathbb{E}^1$-manifolds with $g = 0$ and $r = 4$

If $M = M(0; S)$ with r even and $\varepsilon = 0$, and M embeds in $\mathbb{R}^4$, must the Seifert invariants occur in complementary pairs? In this section we shall examine the simplest nontrivial case, when $r = 4$. Let $\delta = \Delta_2 = \mathrm{hcf}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and let $\alpha_i' = \alpha_i/\delta$, for $i \leq r$. Since ε is an integer it follows that each α_i' divides the product of the other three. We assume henceforth that $T(M)$ is a direct double. Then $\lambda_1 = \lambda_2 = \delta$, since $\Delta_i = 1$ for $i > 2$. Hence $T(M) \cong (Z/\delta)^2$ and $\Delta_1 = \delta^2$. It follows that no three of the α_i's have a common factor > 1. Hence $\alpha_i = \delta\Pi_{j\neq i} a_{ij}$, where $a_{ij} = a_{ji} = \mathrm{hcf}(\alpha_i', \alpha_j')$, and the a_{ij} are otherwise pairwise relatively prime.

The second corollary of Theorem 1 implies that if $\alpha_1 = \alpha_2$ then $\alpha_3 = \alpha_4$. If moreover $\beta_1 = -\beta_2$ then $\beta_3 = -\beta_4$, since $\varepsilon = 0$. This is the case if $\alpha_1 = \alpha_2$ and $\delta = 1$ (i.e., $\mathrm{hcf}(\alpha_1, \alpha_3) = 1$) or if $\alpha_1 = \alpha_2 = 2$ or if the α_is are all 3 or 6. To go further we need to consider invariants beyond the group $T(M)$.

The easiest case to consider first is when $\alpha_i = \delta$ for all i. Here we may assume that $e\delta - f_1\beta_1 = 1$ and $f_2 = f_3 = 0$. Then

$$r_1 = q_1(q_1^{f_1}h^e)^{\beta_1} = q_1^{1+f_1\beta_1}h^{e\beta_1} = (q_1^\delta h^{\beta_1})^e = 1.$$

It follows that $s^e = 1$ and $s^{\beta_1} = 1$, and so $s = 1$. Hence the presentation for $\pi_1(M)$ used in Theorem 2 simplifies to

$$\langle r_2, r_3, t \mid t^\delta r_2 = r_2 t^\delta, t^\delta r_3 = r_3 t^\delta, t^{\beta_2\delta} = (t^{\beta_2} r_2^{-1})^\delta, t^{\beta_3\delta} = (t^{\beta_3} r_3^{-1})^\delta,$$
$$t^{\beta_4\delta} = (t^{-\beta_1} r_2 t^{-\beta_2} r_3 t^{-\beta_3})^\delta \rangle.$$

The torsion group $T(M)$ is generated by the images of r_2 and r_3. If D_i is the meridianal disc for a regular neighbourhood of the i^{th} exceptional fibre (as in §4) then $\delta r_i = \partial(D_i + \beta_i f_1 D_1)$ for $i = 2, 3$. Hence the matrix of ℓ_M with respect to these generators is

$$\delta^{-1}\begin{pmatrix} \beta_2 - \beta_2^2 f_1 & -\beta_2\beta_3 f_1 \\ -\beta_2\beta_3 f_1 & \beta_3 - \beta_3^2 f_1 \end{pmatrix}.$$

The first case not already settled has $S = \{(4,1),(4,1),(4,1),(4,-3)\}$ and $\delta = 4$. In this case ℓ_M has matrix $\frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and is not hyperbolic.

If δ is an odd prime ℓ_M is hyperbolic if and only if the "discriminant"

$$(\delta)^2(\ell_M(r_2,r_3)^2 - \ell_M(r_2,r_2)\ell_M(r_3,r_3)) = \beta_2\beta_3((\beta_2+\beta_3)f_1 - 1)$$

is a square *mod* (δ). Since $e\delta - f_1\beta_1 = 1$ this reduces *mod* (δ) to $f_1^2\beta_1\beta_2\beta_3\beta_4$, and so $\beta_1\beta_2\beta_3\beta_4$ must be a square *mod* (δ). Although this a rather weak criterion, it is enough to confirm that if $\delta \leq 6$, $\alpha_i = \delta$ for all i and ℓ_M is hyperbolic then the Seifert data is skew-symmetric, and so $M(0;S)$ embeds smoothly.

The next invariant to consider is the homology of the infinite cyclic cover M_ϕ, considered as a Λ-module. The above presentation gives a 5×2 presentation matrix

$$\begin{pmatrix} t^\delta - 1 & 0 \\ 0 & t^\delta - 1 \\ \nu_\delta(t^{\beta_2}) & 0 \\ 0 & \nu_\delta(t^{\beta_3}) \\ \nu_\delta(t^{\beta_4}) & \nu_\delta(t^{\beta_4}) \end{pmatrix}.$$

It is easy to see that if $\mathrm{hcf}(\beta,\delta) = 1$ then $\nu_\delta(X^\beta) - \nu_\delta(X)$ is divisible by $X^\delta - 1$, while $X^\delta - 1 = (X-1)\nu_\delta(X)$. Thus we may further simplify this matrix, and we find that $H_1(M_\phi;\mathbb{Z}) \cong (\Lambda/(\nu_\delta(t)))^2$. In particular, $\chi(M_\phi) = 4 - 2\delta = 4 - 2\mu$, as observed after Theorem 2.

How does the Blanchfield pairing depend on the β_i?

7. $\widetilde{\mathbb{SL}}$-manifolds

The situation is less clear when $\varepsilon \neq 0$. The manifold $M = M(0;S)$ is then a $\mathbb{Q}$-homology 3-sphere, and is a $\widetilde{\mathbb{SL}}$-manifold unless $\Sigma\frac{1}{\alpha_i} \geq r-2$, in which case $r \leq 4$. The group $T(M) = H_1(M;\mathbb{Z})$ is generated by the images of $q_1, \dots q_r$ and h. If $\varepsilon\Pi = 1$ then M is an homology 3-sphere and so $M(g;S)$ embeds in $\mathbb{R}^4$ for all $g \geq 0$. It is easy to find examples with $\varepsilon\Pi = 1$ for any $r \geq 1$ and $g \geq 0$. Thus there is no reason to expect a parity constraint on r for embedding such manifolds. However, the question of which such homology spheres embed smoothly is still open. (For instance, the Poincaré homology 3-sphere $S^3/I^* = M(0;(2,1),(3,-1),(5,-1))$ does not. See Problem 4.2 of [8]).

Let $\delta = \mathrm{hcf}(\alpha_1,\dots,\alpha_r)$ and let $\alpha_i' = \alpha_i/\delta$, for $i \leq r$. Then

$$\varepsilon\Pi\alpha_i q_i = \partial(\varepsilon\Pi D_i + \beta_i(\Sigma(\Pi/\alpha_j)D_j) - \beta_i\Pi B_o)$$

and

$$\varepsilon\Pi h = \partial(\Pi B_o - \Sigma(\Pi/\alpha_i)D_i),$$

where D_i and B_o are the 2-cycles defined in §4. Hence

$$\ell_M(q_i,q_i)=\beta_i(\alpha_i\varepsilon\Pi+\beta_i\Pi)/\varepsilon\Pi\alpha_i^2,$$
$$\ell_M(q_i,q_j)=\beta_i\beta_j\Pi/\varepsilon\Pi\alpha_i\alpha_j,\quad \text{if } i\neq j,$$
$$\ell_M(h,q_i)=-\beta_i\Pi/\varepsilon\Pi\alpha_i \quad\text{and}\quad \ell_M(h,h)=\Pi/\varepsilon\Pi.$$

Since ℓ_M is nonsingular $h=0$ if and only if $\varepsilon\Pi$ divides Π and each $\beta_i\Pi/\alpha_i$, i.e., $\varepsilon\Pi=\mathrm{hcf}(\Pi/\alpha_1,\dots,\Pi/\alpha_r)$. In this case $T(M)$ is generated by any $r-1$ elements of $\{q_1,\dots q_r\}$. In particular, $T(M)\cong(Z/\delta Z)^{r-1}$ if and only if $\varepsilon\Pi=\delta^{r-1}$. It may not always be so easy to find a minimal generating set for $T(M)$.

The simplest nontrivial case is when $r=3$. We then have $\Delta_0=|\varepsilon|\Pi$, $\Delta_1=\delta=\mathrm{hcf}(\alpha_1,\alpha_2,\alpha_3)$ and $\Delta_j=1$ for $j>1$. If $T(M)$ is a direct double then $T(M)\cong(Z/\delta Z)^2$ and $\varepsilon\Pi=\eta\delta^2$, where $\eta=\pm1$. Therefore $\alpha_1'\alpha_2'\beta_3+\alpha_1'\beta_2\alpha_3'+\beta_1\alpha_2'\alpha_3'=-\eta$, and so the α_i's must be pairwise coprime. (That is, δ is also the highest common factor of any two of the α_i's). The discriminant of ℓ_M is $\eta\beta_1\beta_2\beta_3$. Thus if δ is an odd prime ℓ_M is hyperbolic if and only if $\varepsilon\Pi=\eta\delta^2$ and $\eta\beta_1\beta_2\beta_3$ is a square *mod* (δ).

If $\alpha_i=\delta$ for all i then $\beta_1+\beta_2+\beta_3=\pm1$. When $\delta\leq4$ or $\delta=6$ we have $\{\beta_1,\beta_2,\beta_3\}=\{1,-1\}$, and there are no further restrictions on embeddability. For $M(0;(\delta,1),(\delta,1),(\delta,-1))$ embeds smoothly in $\mathbb{R}^4$, since it may be obtained by 0-framed surgery on the $(2,2\delta)$-torus link. (See the Appendix to [1].) The cases $\delta=2$ and 3 give the quaternionic space $S^3/Q(8)$ and a $\mathbb{N}il^3$-manifold, respectively. (If $\delta>3$ then M is an $\widetilde{\mathbb{SL}}$-manifold.) The one case with $\delta\leq6$ not covered by this construction and not excluded by the discriminant condition is when $M=M(0;(5,1),(5,2),(5,-2))$. (Can Theorem 2.1 of [3] be applied here to show that M does not embed? On the other hand this manifold has a fairly simple Dehn surgery model, as in Fig. A2 of [1]. Is there a corresponding simple 0-framed surgery model?)

If $r\geq4$ then $M(g;S)$ is an $\widetilde{\mathbb{SL}}$-manifold, unless $g=0$, $r=4$ and $\alpha_i=2$ for all i, in which case M is a $\mathbb{N}il^3$-manifold and does not embed. (The only 3-manifolds not already mentioned which are Seifert fibred over orientable bases, have virtually solvable fundamental group and embed in $\mathbb{R}^4$ are the flat manifolds $M(0;(2,1),(2,1),(2,-1),(2,-1))$ and $M(1;(1,0))=S^1\times S^1\times S^1$, and the Nil^3-coset space $M(1;(1,1))$, which each embed smoothly [1].)

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