The importance of true-false statements in mathematics teaching and learning

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Abstract: *The author suggests that true-false statements are a valuable tool in mathematical pedagogy, in moving students through the passive/active interface and nudging or directing them towards mathematical ideas of historical and contemporary importance.*

Introduction

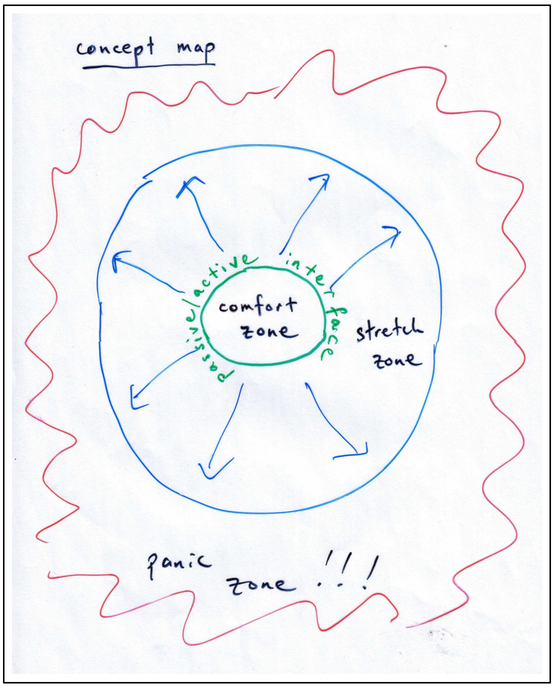
True-false statements, with only two possible outcomes, tend to be regarded as a poor man's multiple choice and relatively ineffective when used in assessment (see for example, Handbook for Graduate Teaching Assistants, University of Nebraska (2007)). Accumulating marks for isolated correct answers can be viewed as indiscriminating and placing emphasis on the score and not on its is composition: it does not matter what is correct, as long as there are enough of them (Biggs 2003). There is evidence that students are more likely to employ surface strategies and motives when preparing for multiple choice or short answer examinations (Scouller 2002), which provide insufficient incentive to assemble or integrate ideas into a coherent whole (Lohman 1993).

Nevertheless, despite apparent defects when used in summative assessment, particularly in the context of criterion-referenced allocation of grades, true-false statements do have a valuable role in formative assessment, provided they are used with care (Burton 2005). True-false statements do not need to be limited to specific, often trivial, factual details. It is possible to design questions and statements that not only test students' comprehension of broader principles but also their ability to apply them (Ebel 1979).

The author would go further, and assert, in the context of almost all of his postgraduate supervision involving original research in pure mathematics, that progress is most rapid when one focuses on resolving specific true-false statements, sometimes in isolation, but often in cascading series. When one begins a mathematical investigation, almost nothing is known. It is important simply to jump in, and make the boldest or silliest assertions as true-false statements. Either they are dealt with quickly, and then dispensed with, or one discovers some unforeseen subtlety at the boundary of one's knowledge or understanding. Like a binary search or bisection method, the investigator homes in on the substance of the topic with logarithmic rapidity.

Jogging the passive/active interface

The author has written previously about the passive/active interface, which is a moment of paralysis as a student switches from a passive role of watching others do mathematics, such as a lecturers, teachers, tutors or authors, to an active role of engaging in producing mathematics and solving problems himself or herself (Easdown 2006). This is related to the 'Challenge by choice' philosophy used by teambuilding facilitators (Rohnke 1989) and best described by *zones* (Pennsylvania State University (2002)). At the centre is the *comfort zone*, which is pleasant enough, but not where real learning or development takes place. It is the task of the facilitator to entice the student into the *stretch zone* (or *growth zone* in the parlance of teambuilders), but not as far as the *panic zone*, where irreversible physical or psychological damage may occur. Once in the stretch zone, the student finds and works at his or her own edge. If activities and associated assessment are properly aligned then the student is in an excellent position to construct his or her own learning and reach maximum potential.



The stretch zone is an immensely satisfying place to be, provided a high degree of awareness is maintained, to avoid slipping over the edge into the deleterious panic zone. However the other edge, between the comfort and stretch zones, which the author labels as *the passive/active interface*, can be the most problematic from the teacher's perspective. The author would like to suggest that a carefully chosen true/false statement can be the perfect medium or mechanism for teasing out the active from the passive. Just the act of making decisions about which mathematical techniques are appropriate is by no means passive and engages the student in a mental tug of war that develops an openmindedness and preparedness to think outside the square. A statement can seem absurd, but turn out to be true. A statement can seem perfectly reasonable and turn out to be false.

After making a choice about technique, then the student has to follow through and use it. At times just one technique can be used to resolve truth or falsity. At other times, completely different techniques are required, which depend on whether a counterexample exists, or some general argument is required. Choices may need to be made whether to search for the needle in the haystack, or provide convincing arguments that no such needle exists.

There is nothing 'wrong' with following blind alleys. These are thought experiments. No shiny expensive equipment will be wasted or shame incurred by being in mental error. Making and correcting mistakes develops a robustness that takes a student deeper towards understanding, and leads to strength of technique. Resolving true/false statements is exactly the process by which research mathematicians make discoveries. It can be equally exciting for the inexperienced learner under the guidance of an experienced teacher. The no man's land of uncertainty can add spice and subtle dynamics that enhance an appreciation of the long history and evolution of mathematical ideas.

Some examples

Consider the following statement (which turns out to be true):

> True or False: $2^{13} - 1 = 8191$ is a prime number.

The beginning student has to think about the meaning of prime number, and then some way of testing the statement. Since $\sqrt{8191} < 91$, and there are only a couple of dozen primes less than this to test as divisors, the question can be resolved fairly quickly. Primality testing is an interesting topic, and leads naturally into probabilistic techniques (how can one tell if a number is prime with very small risk of error?), and then the question of general factorisation of integers, which is an immensely difficult computational problem and the subject of intense research (in the light of the importance of protecting public key cryptosystems). The next statement is false:

True or False: $2^{15} - 1 = 32,767$ is a prime number.

The size of the number has increased the tedium of a simple search considerably, so the student may look for a conceptual means of resolving this quickly in the negative, say using the identity related to the geometric series:

$$x^n - 1 = (x-1)(1 + x + x^2 + \ldots + x^{n-1}) \ ,$$

taking $n = 3$ and $x = 2^5$. This identity can be used not only to dispose of the previous statement but any statement of the following form, for integers m and n larger than 1:

True or False: $2^{mn} - 1$ is a prime number.

We are left naturally to ponder the following, of which so far we have only resolved $n = 13$:

True or False: $2^n - 1$ is a prime number whenever n is prime.

This is an infinite number of statements and is false. The smallest counterexample takes a little perseverence: $2^{11} - 1$ is not prime. The question of when $2^n - 1$ is prime for n a prime has intrigued mathematicians for centuries, and leads to various conjectures, one of the most famous of which is

True or False: $2^n - 1$ is a prime number for infinitely many primes n.

Primes of the form $2^n - 1$ are called *Mersenne primes*, and these lead naturally into even *perfect* numbers. In just a few steps, judiciously chosen true-false statements direct the student towards fascinating topics in number theory with deep historical roots. A wonderful example of mathematical pedagogy was given recently by Terry Tao, recipient of the 2006 Fields Medal, in his address to the Australian Mathematical Society (Tao (2006)). He took the audience on a roller coaster ride, with various nudges, twists and turns, in explaining, in simple language, how he and his collaborator Ben Green developed stategies to famously resolve the following true-false statement affirmatively (Green and Tao (2004)):

True or False: There exist arbitrarily long arithmetic progressions of primes.

A mathematical proposition is either true or false (the so-called *Law of Excluded Middle*). But what is the student to make of the following self-contradictory statement?

True or False: $\{ X \mid X \notin X \} \ \notin \ \{ X \mid X \notin X \}$.

This is a symbolic version of *Russell's Paradox*, which Bertrand Russell sent to Gottlob Frege, causing the latter to abandon his attempt to formalise set theory, and the former to develop his theory of types (Whitehead and Russell (1927)). A single true-false statement causes the student to critically examine the validity of mathematical notation and the foundations of mathematics. Amazingly, no-one has resolved the following assertion about infinite cardinalities:

> True or False: There exists a set S of real numbers such that $|\mathbb{N}| < |S| < |\mathbb{R}|$.

To assert that this is false is the *Continuum Hypothesis.* But, perhaps it is true, no-one knows! The Law of Excluded Middle says that it must be one or the other, yet it is a celebrated theorem that both the Continuum Hypothesis *and* its negation are consistent with the present axioms of set theory (see, for example, Woodin 2001)!! Functioning like a koan, a well chosen true-false statement can confound and shock, and cause the student to reflect on the nature of mathematics.

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