An Introduction to Volatility Models with Indices

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Abstract

This paper considers a class of volatility models generated by autoregressive (AR) type models with indices. Some results associated with the autocorrelation function (acf) of this class are given and the spectral density is obtained in terms of the kurtosis of the error distribution and the model parameters.

Key words: Time series, Frequency, Spectrum, Autoregression, Correlation, Index, Moving average, Kurtosis, Moments, ARCH, GARCH

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1   Introduction

It is well known that time series have their own frequency behaviour. This is a very common phenomenon in practice, especially in financial time series data (see, for instance, Priestley (1981), Peiris (2003) and Peiris et al (2005) for details) and the series cannot be identified using the existing standard time series techniques. In other words the acf, the pacf, and the spectrum are similar for many series and one may propose the same classical model for all of these cases. Obviously, this may lead to poor forecast values leading to serious consequences in managerial decisions. One way of handling this problem in practice is to introduce a new class of time series models with an additional parameter (or an index) $\delta(> 0)$. Therefore, this paper considers a class of time series models satisfying

$$(I - \alpha B)^{\delta} X_t = e_t; \; -1 < \alpha < 1; \; \delta > 0 , \tag{1.1}$$

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where e_t is a white noise sequence and B is the backshift operator such that $B^j X_t = X_{t-j}$; $j \geq 0$ with $B^0 X_t = X_t$.

This class of models covers the traditional AR(1) family when $\delta = 1$. Peiris (2003) and Peiris et. al. (2005) have discussed some useful properties of (1.1). It is clear that when $\alpha = 1$ and $0 < \delta < \frac{1}{2}$, (1.1) reduces to the well-known class of fractionally integrated white noise processes (see, for example, Hosking (1981), Beran (1994) and Chen et al (1994) for details). Therefore, (1.1) constitutes a new family of $AR(1)$ type models and can be applied many standard time series in practice. This class of time series models generated by (1.1) is called **'Power Integrated AR(1)'** or **'PIAR(1)'**.

This paper attempts to generalise the class in (1.1) to incorporate ARCH and GARCH type models. With that view in mind Section 2 reviews the class of of GARCH process and give some examples of calculating the kurtosis.

2   GARCH(p, q) Process

Consider the general class of GARCH (p, q) models for the time series y_t satisfying

$$y_t = \sqrt{h_t} Z_t, \tag{2.1}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \tag{2.2}$$

where $\omega > 0; \alpha_i \geq 0; \beta_j \geq 0$ and Z_t is a sequence of independent and identically distributed (iid) random variables with zero mean and unit variance.

It is well known that $u_t = y_t^2 - h_t$ is a martingale difference and let $\sigma_u^2 = var(u_t)$. Now (2.1) and (2.2) can be written as

$$\phi(B) y_t^2 = \omega + \beta(B) u_t, \tag{2.3}$$

where $\phi(B) = 1 - \sum_{i=1}^{r} \phi_i B^i$, $\phi_i = (\alpha_i + \beta_i)$, $\beta(B) = 1 - \sum_{j=1}^{q} \beta_j B^j, r = \max(p, q)$ and $\mu' = E(y_t^2) = \frac{\omega}{1 - \phi_1 - /phi_2 - \cdots - phi_r}$.

Suppose that the following assumptions are hold:

(A.1). all the zeroes of the polynomial $\phi(B)$ lie outside of the unit circle.

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(A.2). $\sum_{i=0}^{\infty} \psi_i^2 < \infty$, where the $\psi_i's$ are obtained from the relation $\psi(B)\ \phi(B) = \beta(B)$ satisfying $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$.

Let $K^{(X)}$ be the kurtosis of any random variable X given by

$$K^{(X)} = \frac{E[(X - \mu)^4]}{[Var(X)]^2},$$

where $\mu = E(X)$.

Below we give two examples of calculating the kurtosis based on two specific cases:

Example 2.1

For the GARCH (1,1) model $(1 - \phi_1 B)y_t^2 = \omega + (1 - \beta_1 B)u_t$, $(\phi_1 = \alpha_1 + \beta_1)$ and $\psi_1 = \alpha_1$, $\psi_2 = \alpha_1(\alpha_1 + \beta_1)$, $\psi_3 = \alpha_1(\alpha_1 + \beta_1)^2, \ldots \psi_j = \alpha_1(\alpha_1 + \beta_1)^{j-1}$. Clearly, $\sum_{j=1}^{\infty} \psi_j^2 = \alpha_1^2 + \alpha_1^2(\alpha_1 + \beta_1)^2 + \ldots = \frac{\alpha_1^2}{1 - (\alpha_1 + \beta_1)^2}$. The kurtosis $K^{(y)}$ of $\{y_t\}$ is

$$K^{(y)} = \frac{3}{1 - 2\sum_{j=1}^{\infty} \psi_j^2} = \frac{3}{1 - \frac{2\alpha_1^2}{1 - (\alpha_1 + \beta_1)^2}} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2},$$

and it turns out to be the same as the one given in Bollerslev (1986). Moreover, $\sigma_u^2 = \frac{\mu^2(K^{(y)} - 1)}{1 + \frac{\alpha_1^2}{1 - (\alpha_1 + \beta_1)^2}}$.

Example 2.2

For the ARCH (1) model of the form $y_t = \sqrt{h_t} Z_t$, $h_t = \omega + \alpha y_{t-1}^2$ can be obtained by setting $\beta_1 = 0$ in Example 2.1. The corresponding $K^{(y)}$ turns out to be $\frac{3(1 - \alpha_1^2)}{1 - 3\alpha_1^2}$ and $\sigma_u^2 = \mu^2(K^{(y)} - 1)(1 - \alpha_1^2)$.

Although ARCH and GARCH type models are very popular in volatility modelling in finance, we still have room to accomodate additional components in order to understand the volatility process without violating the priciple of parsimony. With that view in mind, Section 3 introduces a class of volatility models with indices.

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3 AR Type Volatility Models with Indices

Suppose that $\{y_t^2\}$ is an AR(p) process as in (2.3) with $h_t = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2$, where p is a large positive integer. In this case $\{y_t^2\}$ follows a model of the form

$$y_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + u_t, \tag{3.1}$$

where u_t is as defined before.

Since the model (3.1) involves p number of parameters $\alpha_1, \alpha_2, \cdots, \alpha_p$, we consider the corresponding analog of (1.1) for $\{y_t^2\}$ is given by

$$(I - \alpha B)^\delta y_t^2 = \omega + u_t, \tag{3.2}$$

where $|\alpha| < 1$, $\delta > 0$ and u_t is a suitable martingale difference sequence.

Clearly when $\delta = 1$, (3.2) reduces to (3.1) with $p = 1$. However, when $\delta > 0$ equation (3.2) can be approximated by a pth order polynomial.

Let

$$I - \alpha B)^\delta = \sum_{j=0}^{\infty} \pi_j \alpha^j B^j, \tag{3.3}$$

where $B^0 = I$, $\pi_0 = 1$ and

$$\pi_j = (-1)^j \binom{\delta}{j} = \frac{(-\delta)(-\delta+1)\cdots(-\delta+j-1)}{j!}; \ j \geq 1.$$

Notes:

1. If δ is a positive integer, then $\pi_j = 0$ for $j \geq \delta + 1$. For any non-integer $\delta > 0$, it is known that,

$$\pi_j = \frac{\Gamma(j-\delta)}{\Gamma(j+1)\Gamma(-\delta)}, \tag{3.4}$$

where $\Gamma(\cdot)$ is the gamma function.

2. The model in (3.2) can be thought of as a model incorporating all the past volatilities in a parsimonius way using only one additional parameter δ. This model can be used to forecast future volatilities using most of the available

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information than any other ARCH model. In practice we may use the approximation

$$(I - \alpha B)^\delta \approx \sum_{j=0}^{m} \pi_j \alpha^j B^j$$

where m is a suitably chosen large integer.

It is easy to see that the series $\sum_{j=0}^{\infty} \pi_j \alpha^j$ converges for all δ since $|\alpha| < 1$. Thus y_t^2 in (3.2) has a valid AR representation of the form

$$\sum_{j=0}^{\infty} \pi_j \alpha^j y_{t-j}^2 = \omega + u_t \tag{3.5}$$

with $\sum |\pi_j \alpha^j|^2 < \infty$. Now we state and prove the following theorem for a stationary solution of (3.2).

Theorem 3.1: For all $\delta > 0$ and $|\alpha| < 1$, the infinite series.

$$y_t^2 - \mu' = \sum_{j=0}^{\infty} \Psi_j \alpha^j u_{t-j} \tag{3.6}$$

converges absolutely with probability one provided $E(u_t^2) < C$, $C > 0$, where $\psi_j = \dfrac{\Gamma(j + \delta)}{\Gamma(j + 1)\,\Gamma(\delta)}$ and $\mu' = E(y_t^2)$.

Proof: Let $(I - \alpha B)^{-\delta} = \sum_{j=0}^{\infty} \Psi_j \alpha^j B^j$, where

$$\psi_j = (-1)^j \binom{-\delta}{j} = \frac{\Gamma(j + \delta)}{\Gamma(j + 1)\,\Gamma(\delta)}; \ j \geq 0.$$

Now $E\left(\sum_{j=0}^{\infty} |\Psi_j \alpha^j u_{t-j}|\right)^2 = \sum_{j=0}^{\infty} |\Psi_j \alpha^j|^2 \, E\{|u_{t-j}|^2\}$.

Since $\sum_{j=0}^{\infty} |\Psi_j \alpha^j|^2 < \infty$, the result follows. Thus (3.6) gives a stationary solution for the process in (3.2). For $\alpha = 1$, (3.6) converges for all $0 < \delta < 1/2$.

Let $\gamma_k = Cov(y_t^2, y_{t-k}^2)$ be the autocovariance function at lag k of $\{y_t^2\}$ satisfying the conditions of Theorem 3.1. It is clear from (3.5) that $\{\gamma_k\}$ satisfy a Yule-Walker type recursion

$$\sum_{j=0}^{\infty} \pi_j \, \alpha^j \, \gamma_{k-j} = 0 \,; \ k > 0 \tag{3.7}$$

and the corresponding autocorrelation function (acf), ρ_k, at lag k is given by

$$\sum_{j=0}^{\infty} \pi_j \, \alpha^j \rho_{k-j} = 0 \, ; \; k > 0 \, . \tag{3.8}$$

It is interesting to note that $\rho_k = \alpha^k$ is a solution of (3.8), since $\sum_{j=0}^{\infty} \pi_j = 0$ for any $\delta > 0$. However, the general solution for ρ_k may be expressed as

$$\rho_k = g(k, \alpha, \delta) \alpha^k,$$

where $g(.)$ is a suitably chosen function of k, α, and δ. To find this function g, we use the following approach:

The spectrum of $\{y_t^2\}$ in (3.2) is

$$\begin{aligned}
f_{y_t^2}(\omega) &= |1 - \alpha \, e^{-i\omega}|^{-2\delta} \frac{\sigma_u^2}{2\pi}; \; -\pi \leq \omega \leq \pi \\
&= (1 - 2\alpha \, Cos \, \omega + \alpha^2)^{-\delta} \frac{\sigma_u^2}{2\pi} \, .
\end{aligned} \tag{3.9}$$

In a neighbourhood of $\omega = 0$, $f_{y_t^2}^g \sim \frac{\sigma_u^2}{2\pi}(1 - \alpha)^{-2\delta}$, where g stands for generalized process in (3.2). Now the exact form of γ_k (or ρ_k) can be obtained from

$$\begin{aligned}
\gamma_k &= \int_{-\pi}^{\pi} e^{ik\omega} f_{y_t^2}(\omega) d\omega \\
&= \frac{\sigma_u^2}{\pi} \int_0^{\pi} \frac{Cos(k\omega)}{(1 - 2\alpha \, Cos \, \omega + \alpha^2)^{\delta}} \, d\omega \, .
\end{aligned} \tag{3.10}$$

In order to obtain the variance of the volatility process y_t^2, we evaluate the integral in (3.10) for $k = 0$. Section 4 reports this result.

4   Main Results

This section is devoted to report some results associated with AR type process given in (3.2). Since the variance plays a significant role in financial modelling, the following theorem gives an expression for γ_k.

Theorem 4.1: For the process given in (3.2),

(a) the variance,

$$\gamma_0 = Var(y_t^2) = \sigma_u^2 F(\delta, \, \delta; \, 1; \, \alpha^2), \tag{4.1}$$

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(b) the autocovariance function of y_t^2 is

$$\gamma_k = \frac{\sigma_u^2 \, \alpha^k \Gamma(k+\delta) \, F(\delta, \, k+\delta; \, k+1; \, \alpha^2)}{\Gamma(\delta) \, \Gamma(k+1)}; \; k \geq 0,$$

(4.2)

where $F(\theta_1, \theta_2; \; \theta_3; \; \theta) = \sum_{j=0}^{\infty} \frac{\Gamma(\theta_1+j) \, \Gamma(\theta_2+j)\Gamma(\theta_3)\theta^j}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3+j)\Gamma(j+1)}$ is the hypogeometric function (see, Gradsteyn and Ryzhik (1980) (GR), p. 1039) for details).

Proof: We first evaluate (3.10) at $k = 0$ and then for any k. From GR p.384, we have

$$\gamma_0 = \frac{\sigma_u^2}{\pi} \int_0^\pi \frac{d\omega}{(1 - 2\alpha \, Cos\omega + \alpha^2)^\delta} = B(\frac{1}{2}, \frac{1}{2}) F(\delta, \delta; \; 1; \; \alpha^2),$$

(4.3)

where $B(x, y) = \frac{\Gamma(x) \, \Gamma(y)}{\Gamma(x+y)}$ is the Beta function and $B(\frac{1}{2}, \frac{1}{2}) = \pi$, and hence (a) follows.

(b) follows by writing $y_t^2 - \mu' = \sum_{j=0}^{\infty} \psi_j \, \alpha^j \, u_{t-j}$ and using

$$\gamma_k = \sigma_u^2 \, \alpha^k \sum_{j=0}^{\infty} \psi_j \, \psi_{j+k} \, \alpha^{2j} = \sigma_u^2 \alpha^k \sum_{j=0}^{\infty} \frac{\Gamma(j+\delta) \, \Gamma(j+k+\delta)(\alpha^2)^j}{\Gamma^2(\delta)\Gamma(j+1)\Gamma(j+k+1)},$$

where $\mu' = E(y_t^2) = \frac{\omega}{(1-\alpha)^\delta}$.

From p.556 of Abramovitz & Stegun (1965), we have

$$\sum_{j=0}^{\infty} \frac{\Gamma(\delta + j) \, \Gamma(k+\delta+j)(\alpha^2)^j}{\Gamma(k+1+j) \, \Gamma(j+1)} = \frac{\Gamma(\delta) \, \Gamma(k+\delta) \, F(\delta, \, k+\delta; \, k+1; \, \alpha^2)}{\Gamma(k+1)}$$

and hence (4.2) follows.

Note: Using the properties of $F()$, it is easy to see that for $\delta = 1$ one has $F(1, 1; \, 1; \, \alpha^2) = (1 - \alpha^2)^{-1}$ (Compare with GR, p.1040). That is (4.1) turns out to be the variance of an ARCH(1) process:

$$\gamma_0 = \; Var(y_t^2) = \frac{\sigma_u^2}{1 - \alpha^2} \; \text{ for } |\alpha| < 1.$$

It is known that for $\theta_3 - \theta_1 - \theta_2 > 0$, $F(\theta_1, \theta_2; \theta_3; 1) = \frac{\Gamma(\theta_3) \, \Gamma(\theta_3 - \theta_1 - \theta_2)}{\Gamma(\theta_3 - \theta_1) \, \Gamma(\theta_3 - \theta_2)}$ and hence part (a) of the Theorem 4.1 reduces to the $Var(y_t^2)$ for a fractionally differenced (long memory) volatility process satisfying $(I - B)^\delta y_t^2 = u_t$ when $0 < \delta < \frac{1}{2}$. That is, when $\alpha = 1$ and $0 < \delta < \frac{1}{2}$, (4.1) gives $\gamma_0 = \frac{\sigma_u^2 \Gamma(1-2\delta_1)}{\Gamma^2(1-\delta)}$.

5 Conclusions

We have introduced a new class of models for modelling volatility and obtained the autocorrelation function of the underlying process. It is shown that the proposed class of models provides a noval approach to incorporate additional components in volatility modelling. The new result in (4.1) and (4.2) are particularly useful in theoretical developments of power integrated ARMA (PIARMA) and power integrated GARCH (PIGARCH) processes with indices and thse will be discussed in a future paper. Moreover the forecasting with this type of volatility models will be discussed in a forthcoming paper following Thavaneswaran et al (2005).

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