A quantum Sylvester theorem and skew representations of twisted Yangians

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Abstract

Analogs of the classical Sylvester theorem have been known for matrices with entries in noncommutative algebras including the quantized algebra of functions on GL_N and the Yangian for $\mathfrak{gl}_N$. We prove a version of this theorem for the twisted Yangians $\mathrm{Y}(\mathfrak{g}_N)$ associated with the orthogonal and symplectic Lie algebras $\mathfrak{g}_N = \mathfrak{o}_N$ or $\mathfrak{sp}_N$. This gives rise to representations of the twisted Yangian $\mathrm{Y}(\mathfrak{g}_{N-M})$ on the space of homomorphisms $\mathrm{Hom}_{\mathfrak{g}_M}(W, V)$, where W and V are finite-dimensional irreducible modules over $\mathfrak{g}_M$ and $\mathfrak{g}_N$, respectively. In the symplectic case these representations turn out to be irreducible and we identify them by calculating the corresponding Drinfeld polynomials. We also apply the quantum Sylvester theorem to realize the twisted Yangian as a projective limit of certain centralizers in universal enveloping algebras.

1 Introduction

Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$ a reductive subalgebra. Suppose that V is a finite-dimensional irreducible $\mathfrak{g}$-module and consider its restriction to the subalgebra $\mathfrak{a}$. This restriction is isomorphic to a direct sum of irreducible finite-dimensional $\mathfrak{a}$-modules W_μ with certain multiplicities m_μ,

$$V|_{\mathfrak{a}} \cong \bigoplus_{\mu} m_\mu W_\mu.$$

If each W_μ is provided with a basis and the decomposition is multiplicity-free (i.e., $m_\mu \leqslant 1$ for all μ) then it can be used to get a basis of V as the union of the bases of the spaces W_μ which occur in the decomposition. This observation played a key role in the construction of the Gelfand–Tsetlin bases for the representations of the general linear and orthogonal Lie algebras. Although the restriction of an irreducible finite-dimensional representation of the symplectic Lie algebra $\mathfrak{sp}_{2n}$ to the subalgebra $\mathfrak{sp}_{2n-2}$ is not multiplicity-free in general, this approach can be extended to the symplectic case with the use of the isomorphism

$$V \cong \bigoplus_{\mu} U_\mu \otimes W_\mu, \tag{1.1}$$

where

$$U_\mu = \mathrm{Hom}_{\mathfrak{a}}(W_\mu, V), \qquad \dim U_\mu = m_\mu.$$

The space U_μ is an irreducible module over the algebra $\mathrm{C}(\mathfrak{g}, \mathfrak{a}) = \mathrm{U}(\mathfrak{g})^{\mathfrak{a}}$, the centralizer of $\mathfrak{a}$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$; see e.g. Dixmier [1, Section 9.1]. Now, if some bases of the spaces U_μ and W_μ are given then the decomposition (1.1) yields the natural tensor product basis of V. The general difficulty of this approach is the complicated structure of the algebra $\mathrm{C}(\mathfrak{g}, \mathfrak{a})$. For each pair of the classical Lie algebras

$$(\mathfrak{g}, \mathfrak{a}) \quad = \quad (\mathfrak{gl}_N, \mathfrak{gl}_M), \quad (\mathfrak{o}_N, \mathfrak{o}_M), \quad (\mathfrak{sp}_N, \mathfrak{sp}_M),$$

(with even N and M in the symplectic case), the centralizer $\mathrm{C}(\mathfrak{g}, \mathfrak{a})$ and its representations can be studied with the use of the quantum algebras called Yangians and twisted Yangians. The *Yangian* $\mathrm{Y}(\mathfrak{gl}_N)$ for the general linear Lie algebra $\mathfrak{gl}_N$ is a deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N \otimes \mathbb{C}[x])$ in the class of Hopf algebras; see e.g. Drinfeld [2]. The *twisted Yangian* $\mathrm{Y}(\mathfrak{g}_N)$ for the orthogonal or symplectic Lie algebra ($\mathfrak{g}_N = \mathfrak{o}_N$ or $\mathfrak{g}_N = \mathfrak{sp}_N$) was introduced by Olshanski [18]. This is a subalgebra of $\mathrm{Y}(\mathfrak{gl}_N)$ and it can also be presented by generators and defining relations; see also [13]. Finite-dimensional irreducible representations of the algebras $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}(\mathfrak{g}_N)$ admit a complete parametrization; see Drinfeld [3] and Tarasov [19] for the

Yangian case, and the author's work [9] for the twisted Yangian case. The Olshanski *centralizer construction* [17, 18] provides 'almost surjective' algebra homomorphisms

$$\mathrm{Y}(\mathfrak{gl}_{N-M}) \to \mathrm{C}(\mathfrak{gl}_N, \mathfrak{gl}_M), \qquad \mathrm{Y}(\mathfrak{g}_{N-M}) \to \mathrm{C}(\mathfrak{g}_N, \mathfrak{g}_M) \tag{1.2}$$

which allow one to equip the corresponding $\mathrm{C}(\mathfrak{g}, \mathfrak{a})$-module U_μ in (1.1) with the structure of a representation of the Yangian or twisted Yangian, respectively. In particular, in the case $N-M=2$ this module over the twisted Yangian $\mathrm{Y}(\mathfrak{g}_2)$ admits a natural basis which leads to a construction of weight bases of Gelfand–Tsetlin type for the representations of the orthogonal and symplectic Lie algebras; see [12] for a review of these results.

In this paper we exploit the relationship between the (twisted) Yangians and the classical Lie algebras in the reverse direction: we use the weight bases constructed in [12] to investigate the representations of the twisted Yangians $\mathrm{Y}(\mathfrak{g}_{N-M})$ emerging from the homomorphisms (1.2).

By the results of [3] and [9] the isomorphism class of each finite-dimensional irreducible representation V of the (twisted) Yangian is determined by its highest weight which is a tuple of formal series over $\mathbb{C}$ in a formal parameter. Moreover, simultaneous multiplication of all components of the highest weight by a fixed invertible formal series corresponds to a representation obtained from V by the composition with a simple automorphism of the (twisted) Yangian. It is natural to combine these representations into a single *similarity class*. In the case of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ these similarity classes correspond to finite-dimensional irreducible representations of the Yangian for the special linear Lie algebra $\mathfrak{sl}_N$. Both in the case of the Yangian and the twisted Yangian the similarity classes are parameterized by families of the *Drinfeld polynomials* $(P_1(u), \dots, P_r(u))$ with some additional data in the twisted case. Each $P_i(u)$ is a monic polynomial in u, and r is the rank of the corresponding Lie algebra.

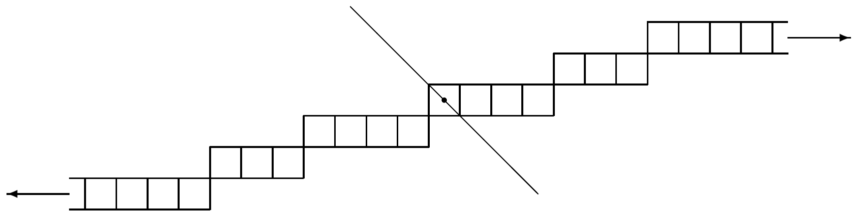
Given partitions $\lambda = (\lambda_1, \dots, \lambda_N)$ and $\mu = (\mu_1, \dots, \mu_M)$, let $V(\lambda)$ and $V(\mu)$ be the finite-dimensional irreducible representation of $\mathfrak{gl}_N$ and $\mathfrak{gl}_M$ with the highest weights λ and μ, respectively. The space $\mathrm{Hom}_{\mathfrak{gl}_M}(V(\mu), V(\lambda))$ is then an irreducible representation of the Yangian $\mathrm{Y}(\mathfrak{gl}_{N-M})$. Its Drinfeld polynomials were calculated by Nazarov and Tarasov [16]. The result is a simple combinatorial rule which allows one to 'read off' each polynomial $P_i(u)$ from the contents of the cells of the skew diagram λ/μ. These *skew* representations of the Yangian (they were called *elementary* in [16]), may be regarded as building blocks for the class of *tame* representations. This class is characterized by the property that the action of a natural commutative subalgebra of the Yangian in such a representation is semisimple. By [16], each tame representation is isomorphic to a tensor product of skew representations.

A different way to define the homomorphism (1.2) in the case of $\mathfrak{gl}_N$ is provided by the *quantum Sylvester theorem*. Recall that the classical Sylvester theorem is the following identity for a numerical $N\times N$ matrix $A=(a_{ij})$:

$$\det B=\det A\cdot\Big(a^{\,m+1\cdots N}_{\,m+1\cdots N}\Big)^{m-1},$$

where $B=(b_{ij})$ is the $m\times m$ matrix formed by the minors $b_{ij}=a^{\,i,m+1\cdots N}_{\,j,m+1\cdots N}$ of A. The sequences of top and bottom indices indicate the row and column numbers of the minor, respectively. The most general noncommutative analog of this identity was given by Gelfand and Retakh in the context of the theory of *quasideterminants* originated in their work [5]; see also [4] for a review of this theory. 'Quantum' versions of this identity apply to the matrices formed by the generators of certain quantum algebras, and the determinants are replaced by appropriate quantum determinants. In particular, such a version was given by Krob and Leclerc [7] for the quantized algebra of functions on GL_N. Their approach is also applicable to the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. A different proof for the Yangian case is given in [11] where the corresponding quantum Sylvester theorem was used to give a modified version of the Olshanski centralizer construction. This provided a new definition of the skew representations of the Yangian and the calculation of their Drinfeld polynomials.

In this paper we produce a quantum Sylvester theorem for the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ with the use of the Sklyanin minors of the matrix of generators of $\mathrm{Y}(\mathfrak{g}_N)$. We first obtain the theorem for the *extended twisted Yangian* $\mathrm{X}(\mathfrak{g}_N)$ (Section 2), following the approach of [7]. The twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ is a quotient of $\mathrm{X}(\mathfrak{g}_N)$ which yields the corresponding result for $\mathrm{Y}(\mathfrak{g}_N)$ (Section 3). In Section 4 we apply the quantum Sylvester theorem to construct a new homomorphism (1.2) for the twisted Yangian and introduce the corresponding skew representations. We show that in the symplectic case each skew representation is irreducible and calculate its highest weight and the Drinfeld polynomials. The Drinfeld polynomials are found by the following simple combinatorial rule somewhat analogous to the Yangian case [16] (see Section 4 below for a detailed formulation). Given a partition $\nu=(\nu_1,\dots,\nu_n)$ we draw its *diagram* $\Gamma(\nu)$ as follows. First, place the row with ν_n unit cells on the plane in such a way that the center of the leftmost cell coincides with the origin. Then place the second row with $\nu_{n-1}-\nu_n$ cells in such a way that the southwest corner of this row coincides with the northeast corner of the first row. Continuing in this manner, we complete this procedure by placing an infinite row of cells in such a way that its southwest corner coincides with the northeast corner of the row with $\nu_1-\nu_2$ cells. The diagram $\Gamma(\nu)$ is obtained as the union of the rows just placed and their images under the central symmetry with respect to the southwest corner of the first row. The figure below represents the diagram for the partition $\nu=(7,4)$, where the dot indicates the origin.



To each cell of the diagram we attach its diagonal number, where by diagonals we mean the lines passing northwest-southeast through the integer points of the plane. The line on the figure indicates the 0-th diagonal and the diagonal numbers are consecutive integers increasing from right to left. For any nonnegative integer p denote by $\Gamma(\lambda)^{(p)}$ the diagram $\Gamma(\lambda)$ lifted p units up.

Suppose now that $V(\lambda)$ and $V(\mu)$ are the irreducible finite-dimensional representations of $\mathfrak{sp}_{2n}$ and $\mathfrak{sp}_{2m}$ corresponding to partitions λ and μ having n and m parts, respectively. Then the Drinfeld polynomials $P_1(u), \dots, P_{n-m}(u)$ for the skew representation $\mathrm{Hom}_{\mathfrak{sp}_{2m}}(V(\mu), V(\lambda))$ of the twisted Yangian $\mathrm{Y}(\mathfrak{sp}_{2n-2m})$ can be calculated by the following rule: all roots of the polynomial $P_k(u)$ are simple and they coincide with the diagonal numbers decreased by $1/2$ of the cells of the intersection $\Gamma(\mu) \cap \Gamma(\lambda)^{(k-1)}$ (see Theorem 4.9 and Example 4.10 below).

Finally, in Section 5 we give a realization of the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ as a projective limit of centralizers in the universal enveloping algebras. This is a new version of the centralizer construction (cf. [18, 14]) which is based on the quantum Sylvester theorem.

The recent work of Nazarov [15] is also devoted to the skew representations of the twisted Yangians although from a different perspective. He uses the classical Weyl's approach and gives a realization of the skew representations in the tensor powers of the vector representation by applying certain generalized Young symmetrizers.

2 Extended twisted Yangian

We start by stating and proving some auxiliary results about the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_N)$; see [13] for more details.

2.1 Preliminaries

We shall be considering the orthogonal and symplectic cases simultaneously, unless otherwise stated. Given a positive integer N, we number the rows and columns of $N \times N$ matrices by the indices $\{-n, \dots, -1, 0, 1, \dots, n\}$ if $N = 2n + 1$, and by

$\{-n,\dots,-1,1,\dots,n\}$ if $N=2n$. Similarly, in the latter case the range of summation indices $-n\leqslant i,j\leqslant n$ will usually exclude 0. It will be convenient to use the symbol θ_{ij} which is defined by

$$\theta_{ij}=\begin{cases}1 & \text{in the orthogonal case,}\\ \operatorname{sgn} i\cdot\operatorname{sgn} j & \text{in the symplectic case.}\end{cases}$$

Throughout the paper, whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the orthogonal case and the lower sign to the symplectic case. By $A\mapsto A^t$ we will denote the matrix transposition such that $(A^t)_{ij}=\theta_{ij}\,A_{-j,-i}$. Let the E_{ij} denote the standard basis vectors of the general linear Lie algebra $\mathfrak{gl}_N$. These vectors may be also regarded as elements of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. For this reason we want to distinguish the E_{ij} from the standard matrix units e_{ij} which are considered as basis elements of the endomorphism algebra $\operatorname{End}\mathbb{C}^N$. Introduce the following elements of the Lie algebra $\mathfrak{gl}_N$:

$$F_{ij}=E_{ij}-\theta_{ij}E_{-j,-i},\qquad -n\leqslant i,j\leqslant n.$$

The Lie subalgebra $\mathfrak{g}_N$ of $\mathfrak{gl}_N$ spanned by the elements F_{ij} is isomorphic to the orthogonal Lie algebra $\mathfrak{o}_N$ or the symplectic Lie algebra $\mathfrak{sp}_N$ (in the latter case N is even).

The *extended twisted Yangian* $\mathrm{X}(\mathfrak{g}_N)$ corresponding to the Lie algebra $\mathfrak{g}_N$ is the associative algebra with generators $s_{ij}^{(1)},\ s_{ij}^{(2)},\dots$ where $-n\leqslant i,j\leqslant n$, subject to the defining relations written in terms of the generating series

$$s_{ij}(u)=\delta_{ij}+s_{ij}^{(1)}u^{-1}+s_{ij}^{(2)}u^{-2}+\dots\in\mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]$$

as follows

$$\begin{aligned}(u^2-v^2)\,[s_{ij}(u),s_{kl}(v)]&=(u+v)\,\big(s_{kj}(u)s_{il}(v)-s_{kj}(v)s_{il}(u)\big)\\&\qquad-(u-v)\,\big(\theta_{k,-j}s_{i,-k}(u)s_{-j,l}(v)-\theta_{i,-l}s_{k,-i}(v)s_{-l,j}(u)\big)\\&\qquad\quad+\theta_{i,-j}\big(s_{k,-i}(u)s_{-j,l}(v)-s_{k,-i}(v)s_{-j,l}(u)\big),\end{aligned}$$

where u and v denote formal variables. The defining relations can also be presented in a convenient matrix form. Denote by $S(u)$ the $N\times N$ matrix whose ij-th entry is $s_{ij}(u)$. We may regard $S(u)$ as an element of the algebra $\mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]\otimes\operatorname{End}\mathbb{C}^N$ given by

$$S(u)=\sum_{i,j}s_{ij}(u)\otimes e_{ij},$$

where the e_{ij} denote the standard matrix units. For any positive integer m we shall be using the algebras of the form

$$\mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]\otimes\operatorname{End}\mathbb{C}^N\otimes\dots\otimes\operatorname{End}\mathbb{C}^N,\tag{2.1}$$

with m copies of $\operatorname{End}\mathbb{C}^N$. For any $a \in \{1,\dots,m\}$ we denote by $S_a(u)$ the matrix $S(u)$ which acts on the a-th copy of $\operatorname{End}\mathbb{C}^N$. That is, $S_a(u)$ is an element of the algebra (2.1) of the form

$$S_a(u) = \sum_{i,j} s_{ij}(u) \otimes 1 \otimes \dots \otimes 1 \otimes e_{ij} \otimes 1 \otimes \dots \otimes 1,$$

where the e_{ij} belong to the a-th copy of $\operatorname{End}\mathbb{C}^N$ and 1 is the identity matrix. Similarly, if

$$C = \sum_{i,j,k,l} c_{ijkl}\, e_{ij} \otimes e_{kl} \in \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N,$$

then for distinct indices $a, b \in \{1,\dots,m\}$ we introduce the element C_{ab} of the algebra (2.1) by

$$C_{ab} = \sum_{i,j,k,l} c_{ijkl}\, 1 \otimes 1 \otimes \dots \otimes 1 \otimes e_{ij} \otimes 1 \otimes \dots \otimes 1 \otimes e_{kl} \otimes 1 \otimes \dots \otimes 1,$$

where the e_{ij} and e_{kl} belong to the a-th and b-th copies of $\operatorname{End}\mathbb{C}^N$, respectively. Consider now the permutation operator

$$P = \sum_{i,j} e_{ij} \otimes e_{ji} \in \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N.$$

The rational function $R(u) = 1 - Pu^{-1}$ with values in the tensor product algebra $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$ is called the *Yang R-matrix*. Introduce its transposed $R^t(u)$ by

$$R^t(u) = 1 - Q\, u^{-1}, \qquad Q = \sum_{i,j} \theta_{ij}\, e_{-j,-i} \otimes e_{ji}. \tag{2.2}$$

The defining relations for the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_N)$ are equivalent to the *quaternary relation*

$$R(u-v)S_1(u)R^t(-u-v)S_2(v) = S_2(v)R^t(-u-v)S_1(u)R(u-v). \tag{2.3}$$

The matrix $S(u)$ is invertible and we shall denote by $S^{-1}(u)$ the inverse matrix. The mapping

$$S(u) \mapsto S^{-1}\big(-u - N/2\big) \tag{2.4}$$

defines an involutive automorphism of the algebra $\mathrm{X}(\mathfrak{g}_N)$; see [13, Proposition 6.5].

Let $u_1,\dots,u_k$ be independent variables. For $k \geqslant 2$ consider the rational function $R(u_1,\dots,u_k)$ with values in $(\operatorname{End}\mathbb{C}^N)^{\otimes k}$ defined by

$$R(u_1,\dots,u_k) = (R_{k-1,k})(R_{k-2,k}R_{k-2,k-1})\cdots(R_{1k}\cdots R_{12}),$$

where we abbreviate $R_{ij} = R_{ij}(u_i - u_j)$. Set

$$S_i = S_i(u_i), \quad 1 \leqslant i \leqslant k \qquad \text{and} \qquad R^t_{ij} = R^t_{ji} = R^t_{ij}(-u_i - u_j), \quad 1 \leqslant i < j \leqslant k.$$

For an arbitrary permutation $(p_1, \dots, p_k)$ of the numbers $1, \dots, k$, we abbreviate

$$\langle S_{p_1}, \dots, S_{p_k} \rangle = S_{p_1} (R^t_{p_1 p_2} \cdots R^t_{p_1 p_k}) S_{p_2} (R^t_{p_2 p_3} \cdots R^t_{p_2 p_k}) \cdots S_{p_k}.$$

The identity

$$R(u_1, \dots, u_k) \langle S_1, \dots, S_k \rangle = \langle S_k, \dots, S_1 \rangle R(u_1, \dots, u_k) \tag{2.5}$$

can be deduced from the quaternary relation (2.3); see [13, Proposition 4.2]. Now specialize the variables u_i by setting

$$u_i = u - i + 1, \qquad i = 1, \dots, k. \tag{2.6}$$

It is well known that under this specialization, $R(u_1, \dots, u_k)$ coincides with the anti-symmetrization operator A_k on $(\mathbb{C}^N)^{\otimes k}$, where

$$A_k = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot P_\sigma,$$

and P_σ denotes the image of $\sigma \in \mathfrak{S}_k$ under the natural action of $\mathfrak{S}_k$ on $(\mathbb{C}^N)^{\otimes k}$; see e.g. [13, Proposition 2.3]. Hence specializing the variables in (2.5) we get

$$A_k \, \langle S_1, \dots, S_k \rangle = \langle S_k, \dots, S_1 \rangle \, A_k.$$

This element of the tensor product $\mathrm{X}(\mathfrak{g}_N)[[u^{-1}]] \otimes (\operatorname{End} \mathbb{C}^N)^{\otimes k}$ can be written as

$$\sum s^{\, a_1 \cdots a_k}_{\, b_1 \cdots b_k}(u) \otimes e_{a_1 b_1} \otimes \cdots \otimes e_{a_k b_k},$$

summed over the indices $a_i, b_i \in \{-n, \dots, n\}$. We also set $s^{\, a}_{\, b}(u) = s_{ab}(u)$. We call the elements $s^{\, a_1 \cdots a_k}_{\, b_1 \cdots b_k}(u)$ of $\mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]$ the *Sklyanin minors* of the matrix $S(u)$. Clearly, the Sklyanin minors are skew-symmetric with respect to permutations of the upper indices and of the lower indices:

$$s^{\, a_{\sigma(1)} \cdots a_{\sigma(k)}}_{\, b_1 \cdots b_k}(u) = \operatorname{sgn} \sigma \cdot s^{\, a_1 \cdots a_k}_{\, b_1 \cdots b_k}(u) \quad \text{and} \quad s^{\, a_1 \cdots a_k}_{\, b_{\sigma(1)} \cdots b_{\sigma(k)}}(u) = \operatorname{sgn} \sigma \cdot s^{\, a_1 \cdots a_k}_{\, b_1 \cdots b_k}(u)$$

for any $\sigma \in \mathfrak{S}_k$.

Proposition 2.1. *We have the relations*

$$
\begin{aligned}
&(u^2-v^2)\,[s_{pq}(u), s\,{}^{a_1\cdots a_k}_{b_1\cdots b_k}(v)]\\
&\qquad = (u+v)\sum_{i=1}^{k}\Big(s_{a_iq}(u)\, s\,{}^{a_1\cdots p\cdots a_k}_{b_1\ \cdots\ \ b_k}(v) - s\,{}^{a_1\ \cdots\ \ a_k}_{b_1\cdots q\cdots b_k}(v)\, s_{pb_i}(u)\Big)\\
&\qquad - (u-v)\sum_{i=1}^{k}\Big(\theta_{a_i,-q}\, s_{p,-a_i}(u)\, s\,{}^{a_1\cdots -q\cdots a_k}_{b_1\ \cdots\ \ b_k}(v) - \theta_{p,-b_i}\, s\,{}^{a_1\ \cdots\ \ a_k}_{b_1\cdots -p\cdots b_k}(v)\, s_{-b_i,q}(u)\Big)\\
&\qquad + \theta_{p,-q}\sum_{i=1}^{k}\Big(s_{a_i,-p}(u)\, s\,{}^{a_1\cdots -q\cdots a_k}_{b_1\ \cdots\ \ b_k}(v) - s\,{}^{a_1\ \cdots\ \ a_k}_{b_1\cdots -p\cdots b_k}(v)\, s_{-q,b_i}(u)\Big)\\
&\qquad + \sum_{i\neq j}\Big(\theta_{a_j,-q}\, s_{a_i,-a_j}(u)\, s\,{}^{a_1\cdots p\cdots -q\cdots a_k}_{b_1\ \cdots\ \ b_k}(v) - \theta_{p,-b_i}\, s\,{}^{a_1\ \cdots\ \ a_k}_{b_1\cdots -p\cdots q\cdots b_k}(v)\, s_{-b_i,b_j}(u)\Big),
\end{aligned}
$$

where in the Sklyanin minors the indices p and q replace a_i and b_i, respectively, in the first sum; the indices $-q$ and $-p$ replace a_i and b_i, respectively, in the second and third sums; in the fourth sum p and $-q$ replace a_i and a_j, respectively, and $-p$ and q replace b_i and b_j, respectively.

Proof. By (2.5), we have the relation

$$
\begin{aligned}
&R(u,v,v-1,\dots,v-k+1)\,\langle S_0,\dots,S_k\rangle\\
&\qquad\qquad = \langle S_k,\dots,S_0\rangle\, R(u,v,v-1,\dots,v-m+1), \qquad (2.7)
\end{aligned}
$$

where we have used an extra copy of the algebra $\mathrm{End}\,\mathbb{C}^N$ labelled by 0 and the parameters are specialized as follows

$$
u_0 = u, \quad \text{and} \quad u_i = v-i+1 \quad \text{for} \quad i=1,\dots,k.
$$

Then one easily verifies (see e.g. [11]) that the product of R-matrices in (2.7) simplifies to

$$
R(u,v,v-1,\dots,v-k+1) = A_k\Big(1-\frac{1}{u-v}(P_{01}+\dots+P_{0k})\Big).
$$

Applying the transposition over the zeroth copy of $\mathrm{End}\,\mathbb{C}^N$ and replacing u by $-u$ we also deduce that

$$
A_k\, R^t_{01}\cdots R^t_{0k} = A_k\Big(1+\frac{1}{u+v}(Q_{01}+\dots+Q_{0k})\Big). \qquad (2.8)
$$

Hence (2.7) takes the form

$$
\begin{aligned}
&\Big(1-\frac{1}{u-v}(P_{01}+\dots+P_{0k})\Big)\, S_0(u)\, \Big(1+\frac{1}{u+v}(Q_{01}+\dots+Q_{0k})\Big)\, A_k\,\langle S_1,\dots,S_k\rangle\\
&= \langle S_k,\dots,S_1\rangle\, A_k\Big(1+\frac{1}{u+v}(Q_{01}+\dots+Q_{0k})\Big)\, S_0(u)\, \Big(1-\frac{1}{u-v}(P_{01}+\dots+P_{0k})\Big).
\end{aligned}
$$

It remains to apply both sides to the vector $e_q \otimes e_{b_1} \otimes \cdots \otimes e_{b_k}$ and compare the coefficients at the vector $e_p \otimes e_{a_1} \otimes \cdots \otimes e_{a_k}$, where the e_i denote the canonical basis vectors of $\mathbb{C}^N$. $\square$

Corollary 2.2. *Suppose that for some indices $i, j, l, m \in \{1, \dots, k\}$ we have $a_i = -b_l$ and $b_j = -a_m$. Then*

$$[s_{a_i b_j}(u), s^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(v)] = 0.$$

Proof. By the skew-symmetry property, the Sklyanin minor is zero if it has two repeated upper or lower indices. Hence we may assume that $i = m$ if and only if $j = l$. Suppose first that $i \neq m$. Then using the skew-symmetry of Sklyanin minors, we derive from Proposition 2.1 that

$$(u - v - 1)(u + v + 1)\, [s_{a_i b_j}(u), s^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(v)] = \theta_{a_i, -b_j}\, [s_{-b_j, -a_i}(u), s^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(v)].$$

The same relation holds with i and j replaced by m and l, respectively, which proves the claim in the case under consideration. If $a_i = -b_j$ then Proposition 2.1 immediately gives

$$(u - v - 1)(u + v + 1)\, [s_{a_i b_j}(u), s^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(v)] = 0,$$

completing the proof. $\square$

The series

$$\operatorname{sdet} S(u) = s^{-n \cdots n}_{-n \cdots n}(u) \in \mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]$$

is called the *Sklyanin determinant* of the matrix $S(u)$. Corollary 2.2 implies that all the coefficients of this series belong to the center of the algebra $\mathrm{X}(\mathfrak{g}_N)$; see also [13, Theorem 4.8] for a slightly different proof. The *Sklyanin comatrix* $\widehat{S}(u)$ is defined by the relation

$$\widehat{S}(u)\, S(u - N + 1) = \operatorname{sdet} S(u). \tag{2.9}$$

Due to (2.4), the mapping

$$S(u) \mapsto \widehat{S}(-u + N/2 - 1) \tag{2.10}$$

defines a homomorphism of $\mathrm{X}(\mathfrak{g}_N)$ into itself. Multiplying both sides of (2.9) by the inverse to $S(u - N + 1)$ and taking the nn-th entry we get

$$\widehat{s}_{nn}(u) = \operatorname{sdet} S(u)\, \big(S^{-1}(u - N + 1)\big)_{nn}.$$

Therefore,

$$\operatorname{sdet} S(u) = \widehat{s}_{nn}(u) \cdot \big|S(u - N + 1)\big|_{nn}, \tag{2.11}$$

where we have used the notation $|A|_{ij}$ for the ij-th *quasideterminant* of a matrix A over a ring; see [5, 6]. By definition, $|A|_{ij}$ equals the inverse of the matrix element $(A^{-1})_{ji}$ provided that the matrix A and the element $(A^{-1})_{ji}$ are invertible.

We shall also use the *auxiliary minors* $\check{s}\,^{a_1\cdots a_k}_{b_1\cdots b_{k-1},c}(u) \in \mathrm{X}(\mathfrak{g}_N)[[u^{-1}]]$ defined by

$$
\begin{aligned}
A_k \,\langle S_1, \dots, S_{k-1}\rangle \, R^t_{1k} \cdots R^t_{k-1,k}& \\
&= \sum \check{s}\,^{a_1\cdots a_k}_{b_1\cdots b_{k-1},c}(u) \otimes e_{a_1 b_1} \otimes \cdots \otimes e_{a_{k-1} b_{k-1}} \otimes e_{a_k c},
\end{aligned}
\tag{2.12}
$$

summed over $a_i, b_i, c \in \{-n, \dots, n\}$. Since

$$
A_k \,\langle S_1, \dots, S_{k-1}\rangle \, R^t_{1k} \cdots R^t_{k-1,k}\, S_k = A_k \,\langle S_1, \dots, S_k\rangle,
$$

we immediately obtain the relation

$$
s\,^{a_1\cdots a_k}_{b_1\cdots b_k}(u) = \sum_{c=-n}^{n} \check{s}\,^{a_1\cdots a_k}_{b_1\cdots b_{k-1},c}(u)\, s_{c b_k}(u-k+1). \tag{2.13}
$$

We obviously have

$$
\check{s}\,^{a_{\sigma(1)}\cdots a_{\sigma(k)}}_{b_1\cdots b_{k-1},c}(u) = \operatorname{sgn}\sigma \cdot \check{s}\,^{a_1\cdots a_k}_{b_1\cdots b_{k-1},c}(u)
$$

for any $\sigma \in \mathfrak{S}_k$. Also,

$$
\check{s}\,^{a_1\cdots a_k}_{b_{\sigma(1)}\cdots b_{\sigma(k-1)},c}(u) = \operatorname{sgn}\sigma \cdot \check{s}\,^{a_1\cdots a_k}_{b_1\cdots b_{k-1},c}(u)
$$

for any $\sigma \in \mathfrak{S}_{k-1}$; see [8]. Furthermore, it is straightforward to obtain the following property of the auxiliary minors from their definition (cf. [8, Proposition 4.4]): if $c \notin \{a_1, \dots, a_{k-1}\}$ and $c \notin \{-b_1, \dots, -b_{k-1}\}$ then

$$
\check{s}\,^{a_1\cdots a_k}_{b_1\cdots b_{k-1},c}(u) = 0 \tag{2.14}
$$

if $c \neq a_k$, while

$$
\check{s}\,^{a_1\cdots a_{k-1},\,c}_{b_1\cdots b_{k-1},\,c}(u) = s\,^{a_1\cdots a_{k-1}}_{b_1\cdots b_{k-1}}(u). \tag{2.15}
$$

Set $(a_1, \dots, a_N) = (-n, \dots, n)$. Then the matrix elements $\widehat{s}_{a_i a_j}(u)$ of the Sklyanin comatrix $\widehat{S}(u)$ are given by

$$
\widehat{s}_{a_i a_j}(u) = (-1)^{N-i}\, \check{s}\,^{a_1\cdots a_N}_{a_1\cdots \widehat{a}_i \cdots a_N,\, a_j}(u), \tag{2.16}
$$

where the hat on the right hand side indicates the index to be omitted; see [8, Section 6].

2.2 Sylvester theorem

Fix a nonnegative integer $m < n$ and set $M = 2m$ or $M = 2m + 1$ if $N = 2n$ or $N = 2n + 1$, respectively, so that $N - M = 2n - 2m$. Set $\mathcal{B} = \{-m, \dots, m\}$ and denote by $S_{\mathcal{B}\mathcal{B}}(u)$ the submatrix of $S(u)$ whose rows and columns are numbered by the elements of $\mathcal{B}$. It follows from the defining relations for the extended twisted Yangian that the subalgebra of $\mathrm{X}(\mathfrak{g}_N)$ generated by the elements $s_{ij}^{(r)}$ with $i, j \in \mathcal{B}$ can be regarded as a natural homomorphic image of the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_M)$. The homomorphism takes the generators $s_{ij}^{(r)}$ of $\mathrm{X}(\mathfrak{g}_M)$ to the elements of $\mathrm{X}(\mathfrak{g}_N)$ with the same name. This makes the Sklyanin determinant of the matrix $S_{\mathcal{B}\mathcal{B}}(u)$ well-defined. Denote by $\mathcal{A}$ the complement of the subset $\mathcal{B}$ in the set $\{-n, \dots, n\}$ and let $\mathrm{X}(\mathfrak{g}_{N-M})$ denote the extended twisted Yangian whose generator series $s_{ab}(u)$ are enumerated by elements $a, b \in \mathcal{A}$. For any such a, b set

$$\widetilde{s}_{ab}(u) = s_{-m\cdots m,b}^{-m\cdots m,a}\,(u + M/2) \tag{2.17}$$

and denote by $\widetilde{S}(u)$ the $(N - M) \times (N - M)$ matrix whose ab-entry is $\widetilde{s}_{ab}(u)$.

Theorem 2.3. *The mapping*

$$s_{ab}(u) \mapsto \widetilde{s}_{ab}(u) \tag{2.18}$$

defines an algebra homomorphism $\mathrm{X}(\mathfrak{g}_{N-M}) \to \mathrm{X}(\mathfrak{g}_N)$. *Moreover,*

$$\begin{aligned} \operatorname{sdet} \widetilde{S}(u) &= \operatorname{sdet} S(u + M/2) \\ &\qquad \times \operatorname{sdet} S_{\mathcal{B}\mathcal{B}}(u + M/2 - 1) \cdots \operatorname{sdet} S_{\mathcal{B}\mathcal{B}}(u + M/2 - N + M + 1). \end{aligned}$$

Proof. We follow the approach of [7] based on complimentary minor identities. By the definition of the Sklyanin determinant,

$$A_N \langle S_1, \dots, S_N \rangle = A_N \operatorname{sdet} S(u).$$

This implies the relation

$$\begin{aligned} &A_N \langle S_1, \dots, S_{M+1} \rangle \prod_{i=1,\dots,M}^{\rightarrow} (R^t_{i,M+2} \cdots R^t_{iN}) \\ &= A_N \operatorname{sdet} S(u)\, S_N^{-1} (R^t_{N-1,N})^{-1}\, S_{N-1}^{-1} \cdots\, S_{M+2}^{-1} (R^t_{M+1,N})^{-1} \cdots (R^t_{M+1,M+2})^{-1}. \end{aligned} \tag{2.19}$$

Note that since $Q^2 = NQ$ we find from (2.2) that

$$R^t(u)^{-1} = R^t(-u + N).$$

Therefore, the right hand side of (2.19) can be written as

$$A_N \operatorname{sdet} S(u)\, S_N^{\circ} R_{N-1,N}^{\circ}\, S_{N-1}^{\circ} \cdots\, S_{M+2}^{\circ} R_{M+1,N}^{\circ} \cdots R_{M+1,M+2}^{\circ},$$

where we have used the notation $S^{\circ}(u) = S^{-1}(-u - N/2)$ and

$$S_i^{\circ} = S_i^{\circ}(u_i^{\circ}), \qquad R_{ij}^{\circ} = R_{ij}^t(-u_i^{\circ} - u_j^{\circ}), \qquad u_i^{\circ} = -u_i - N/2 \tag{2.20}$$

with the u_i are specialized as in (2.6). Let us set

$$(a_1, \dots, a_{N-M}) = (-n, \dots, -m-1, m+1, \dots, n)$$

and apply both sides of (2.19) to the vector

$$v_{ij} = e_{-m} \otimes \dots \otimes e_m \otimes e_{a_j} \otimes e_{a_1} \otimes \dots \otimes \widehat{e}_{a_i} \otimes \dots \otimes e_{a_{N-M}},$$

where the e_r denote the canonical basis vectors of $\mathbb{C}^N$ and $i, j \in \{1, \dots, N-M\}$. Comparing the coefficients at the vector $A_N\, v_{11}$ we come to the relation

$$s^{-m \cdots m,\, a_i}_{-m \cdots m,\, a_j}(u) = (-1)^{N-M-i}\, \mathrm{sdet}\, S(u) \cdot \check{s}^{\circ}{}^{\,a_1 \cdots a_{N-M}}_{\,a_1 \cdots \widehat{a}_i \cdots a_{N-M},\, a_j}(u_N^{\circ}), \tag{2.21}$$

where the auxiliary minors on the right hand side correspond to the matrix $S^{\circ}(u)$. They are well-defined, since by (2.4) $S^{\circ}(u)$ satisfies the quaternary relation (2.3). Now observe that we may regard these auxiliary minors as those for the submatrix $S^{\circ}_{\mathcal{A}\mathcal{A}}(u)$ of $S^{\circ}(u)$ whose row and column indices belong to the set $\mathcal{A} = \{a_1, \dots, a_{N-M}\}$. Indeed, we have the following identity analogous to (2.8) which is verified in the same way:

$$A_{k-1}\, R_{1k}^t \cdots R_{k-1,k}^t = A_{k-1}\Big(1 + \frac{Q_{1k} + \dots + Q_{k-1,k}}{2u - k + 1}\Big). \tag{2.22}$$

Therefore the left hand side of (2.12) with $S(u)$ replaced by $S^{\circ}(u)$ can be written as

$$A_k\, S_1^{\circ} \Big(1 + \frac{Q_{12}}{2u-1}\Big)\, S_2^{\circ} \Big(1 + \frac{Q_{13} + Q_{23}}{2u-2}\Big) \cdots S_{k-1}^{\circ} \Big(1 + \frac{Q_{1k} + \dots + Q_{k-1,k}}{2u - k + 1}\Big).$$

If we now put $k = N - M$ and apply this operator to the vector

$$e_{a_1} \otimes \dots \otimes \widehat{e}_{a_i} \otimes \dots \otimes e_{a_{N-M}} \otimes e_{a_j}$$

then the coefficient at the vector $e_{a_1} \otimes \dots \otimes e_{a_{N-M}}$ will be an expression involving only the entries of the submatrix $S^{\circ}_{\mathcal{A}\mathcal{A}}(u)$ of $S^{\circ}(u)$. Furthermore, by (2.16),

$$(-1)^{N-M-i}\, \check{s}^{\circ}{}^{\,a_1 \cdots a_{N-M}}_{\,a_1 \cdots \widehat{a}_i \cdots a_{N-M},\, a_j}(u_N^{\circ}) = \widehat{s}^{\circ}_{a_i a_j}(u_N^{\circ}),$$

where by $\widehat{s}^{\circ}_{ab}(u)$ we denote the entries of the Sklyanin comatrix corresponding to the matrix $S^{\circ}_{\mathcal{A}\mathcal{A}}(u)$. Recalling that $u_N^{\circ} = -u + N/2 - 1$ and $N - M = 2n - 2m$, we derive from (2.21) that

$$s^{-m \cdots m,\, a_i}_{-m \cdots m,\, a_j}(u + M/2) = \mathrm{sdet}\, S(u + M/2) \cdot \widehat{s}^{\circ}_{a_i a_j}(-u + n - m - 1). \tag{2.23}$$

By (2.10), the mapping $s_{a_ia_j}(u) \mapsto \widehat{s}^{\circ}_{a_ia_j}(-u+n-m-1)$ defines a homomorphism $\mathrm{X}(\mathfrak{g}_{N-M}) \to \mathrm{X}(\mathfrak{g}_N)$, and hence so does the mapping (2.18) since the coefficients of the Sklyanin determinant are central in $\mathrm{X}(\mathfrak{g}_N)$. This proves the first part of the theorem.

Applying the relation (2.23) to the matrix $S^{\circ}(u)$ we obtain for any elements $a, b \in \mathcal{A}$

$$s^{\circ}{}^{-m\cdots m,\,a}_{-m\cdots m,\,b}(u+M/2) = \operatorname{sdet} S^{\circ}(u+M/2)\cdot \widehat{s}^{\,\prime}_{ab}(-u+n-m-1), \tag{2.24}$$

where the $\widehat{s}^{\,\prime}_{ab}(u)$ denote the entries of the Sklyanin comatrix corresponding to the matrix $S_{\mathcal{A}\mathcal{A}}(u)$. Denote by $S^{\sharp}(u)$ the inverse matrix to $S_{\mathcal{A}\mathcal{A}}(-u-n+m)$. Due to (2.9) we have

$$\widehat{S}_{\mathcal{A}\mathcal{A}}(-u+n-m-1) = \operatorname{sdet} S_{\mathcal{A}\mathcal{A}}(-u+n-m-1)\cdot S^{\sharp}(u).$$

Combining this with (2.24) and using the notation (2.17) for the matrix $S^{\circ}(u)$ we come to

$$\widetilde{s}^{\circ}_{ab}(u) = \operatorname{sdet} S^{\circ}(u+M/2)\cdot \operatorname{sdet} S_{\mathcal{A}\mathcal{A}}(-u+n-m-1)\cdot s^{\sharp}_{ab}(u).$$

By [13, Theorem 7.6] applied to the matrix $S^{\circ}(u)$ we have

$$\operatorname{sdet} S^{\circ}(u)\cdot \operatorname{sdet} S_{\mathcal{A}\mathcal{A}}\big(-u+N/2-1\big) = \operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}(u). \tag{2.25}$$

Together with the previous relation this gives

$$s^{\sharp}_{ab}(u) = \operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}\big(u+M/2\big)^{-1}\cdot \widetilde{s}^{\circ}_{ab}(u). \tag{2.26}$$

Note that $\operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}(u)$ commutes with $\widetilde{s}^{\circ}_{ab}(v)$. Indeed, this follows from Corollary 2.2 and the fact that the expansion of $\operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}(u)$ in terms of the matrix elements of $S^{\circ}(u)$ only involves the series $s^{\circ}_{ij}(u)$ with $i, j \in \mathcal{B}$. The latter is easily deduced from the definition of the Sklyanin minors and with the assistance of (2.22).

On the other hand, the particular case of (2.25) with $\mathcal{B} = \emptyset$ yields

$$\operatorname{sdet} S^{\circ}(u)\cdot \operatorname{sdet} S\big(-u+N/2-1\big) = 1.$$

Hence,

$$\operatorname{sdet} S^{\sharp}(u)\cdot \operatorname{sdet} S_{\mathcal{A}\mathcal{A}}(-u+n-m-1) = 1$$

and so by (2.25),

$$\operatorname{sdet} S^{\sharp}(u) = \operatorname{sdet} S^{\circ}\big(u+M/2\big)\cdot \operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}\big(u+M/2\big)^{-1}. \tag{2.27}$$

Finally, write down the expansion of sdet $S^\sharp(u)$ in terms of the matrix elements $s^\sharp_{ab}(u)$ which is implied by the definition of the Sklyanin determinant. The expansion has the form of a linear combination of products

$$s^\sharp_{a_1b_1}(u)\, s^\sharp_{a_2b_2}(u-1)\cdots s^\sharp_{a_{N-M}b_{N-M}}(u-N+M+1),$$

the coefficients being rational functions in u. Using the relation (2.26) we obtain

$$\mathrm{sdet}\, \widetilde{S}^{\circ}(u) = \mathrm{sdet}\, S^\sharp(u)\cdot \mathrm{sdet}\, S^{\circ}_{\mathcal{B}\mathcal{B}}(u+M/2)\cdots \mathrm{sdet}\, S^{\circ}_{\mathcal{B}\mathcal{B}}(u+M/2-N+M+1).$$

Taking into account (2.27) we derive the desired formula for the matrix $S^{\circ}(u)$. Since $S(u)\mapsto S^{\circ}(u)$ is an involutive automorphism of $\mathrm{X}(\mathfrak{g}_N)$, the formula holds for the matrix $S(u)$ as well. □

Denote by $\widehat{\sigma}_{ab}(u)$ the entries of the Sklyanin comatrix corresponding to the matrix $\widetilde{S}(u)$.

Proposition 2.4. *For any $a,b\in\mathcal{A}$ we have the relation*

$$\begin{aligned}\widehat{\sigma}_{ab}(u) = \widehat{s}_{ab}&(u+M/2)\\ &\times \mathrm{sdet}\, S_{\mathcal{B}\mathcal{B}}(u+M/2-1)\cdots \mathrm{sdet}\, S_{\mathcal{B}\mathcal{B}}(u+M/2-N+M+2).\end{aligned}$$

Proof. The defining relations for $\mathrm{X}(\mathfrak{g}_N)$ imply that

$$[s^{(1)}_{ab}, s_{ij}(u)] = \delta_{ib}\, s_{aj}(u) - \delta_{aj}\, s_{ib}(u) - \theta_{i,-b}\,\delta_{a,-i}\, s_{-b,j}(u) + \theta_{a,-j}\,\delta_{-j,b}\, s_{i,-a}(u). \tag{2.28}$$

Hence, the elements $s^{(1)}_{ab}$ with $a,b\in\mathcal{A}$ commute with sdet $S_{\mathcal{B}\mathcal{B}}(v)$ (this also follows from Proposition 2.1). On the other hand, it is straightforward to verify with the use of (2.22) that $s^{(1)}_{ab}$ with $a\neq b$ is stable under both the homomorphism (2.10) and the homomorphism which takes $S_{\mathcal{A}\mathcal{A}}(u)$ to the Sklyanin comatrix $\big[\widehat{\sigma}_{ab}(-u+n-m-1)\big]$. Therefore, it is sufficient to prove the relation in the particular case $a=b=n$. Taking subsequent commutators with appropriate elements $s^{(1)}_{ab}$ will yield the relation in the general case.

Using the notation of the proof of Theorem 2.3 we can write the following relations implied by (2.11),

$$\mathrm{sdet}\, S^{\circ}(u+M/2) = \widehat{s}^{\,\circ}_{nn}(u+M/2)\cdot \big|S^{\circ}(u+M/2-N+1)\big|_{nn}$$

and

$$\mathrm{sdet}\, S^\sharp(u) = \widehat{s}^{\,\sharp}_{nn}(u)\cdot \big|S^\sharp(u-N+M+1)\big|_{nn},$$

where the $\widehat{s}^{\,\sharp}_{ij}(u)$ denote the entries of the Sklyanin comatrix corresponding to the matrix $S^\sharp(u)$. The definition of quasideterminants implies that

$$\big|S^{\circ}(u+M/2-N+1)\big|_{nn} = \big|S^\sharp(u-N+M+1)\big|_{nn}.$$

Therefore, the above relations together with (2.27) yield

$$\widehat{s}^{\sharp}_{nn}(u) = \widehat{s}^{\circ}_{nn}\big(u + M/2\big) \cdot \operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}\big(u + M/2\big)^{-1}. \tag{2.29}$$

Now we conclude the argument in the same way as in the proof of Theorem 2.3. Using (2.16) write down the expansion of $\widehat{s}^{\sharp}_{nn}(u)$ in terms of the matrix elements $s^{\sharp}_{ab}(u)$. It has the form of a linear combination of products

$$s^{\sharp}_{a_1 b_1}(u)\, s^{\sharp}_{a_2 b_2}(u-1) \cdots s^{\sharp}_{a_{N-M} b_{N-M}}(u - N + M + 2),$$

the coefficients being rational functions in u. Now (2.26) implies

$$\widehat{\sigma}^{\circ}_{nn}(u) = \widehat{s}^{\sharp}_{nn}(u) \cdot \operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}(u + M/2) \cdots \operatorname{sdet} S^{\circ}_{\mathcal{B}\mathcal{B}}(u + M/2 - N + M + 2),$$

where by $\widehat{\sigma}^{\circ}_{ab}(u)$ we denote the entries of the Sklyanin comatrix corresponding to the matrix $\widetilde{S}^{\circ}(u)$. Using (2.29) we derive the desired formula for the matrix $S^{\circ}(u)$ and hence it holds for the matrix $S(u)$ as well. □

Interchanging the roles of the sets $\mathcal{A}$ and $\mathcal{B}$ in the above arguments one can easily derive the corresponding dual versions of Theorem 2.3 and Proposition 2.4. Here we only record the counterpart of the first part of Theorem 2.3 which will be used below.

Proposition 2.5. *The mapping*

$$s_{ij}(u) \mapsto s^{-n \cdots -m-1,\, i,\, m+1 \cdots n}_{-n \cdots -m-1,\, j,\, m+1 \cdots n}(u + n - m), \qquad -m \leqslant i, j \leqslant m$$

defines an algebra homomorphism $\mathrm{X}(\mathfrak{g}_M) \to \mathrm{X}(\mathfrak{g}_N)$.

3 Sylvester theorem for the twisted Yangian

The *twisted Yangian* $\mathrm{Y}(\mathfrak{g}_N)$ corresponding to the Lie algebra $\mathfrak{g}_N$ is the quotient of the extended twisted Yangian $\mathrm{X}(\mathfrak{g}_N)$ by the following *symmetry relation*

$$\theta_{ij} s_{-j,-i}(-u) = s_{ij}(u) \pm \frac{s_{ij}(u) - s_{ij}(-u)}{2u}, \tag{3.1}$$

or, in the matrix form,

$$S^t(-u) = S(u) \pm \frac{S(u) - S(-u)}{2u}.$$

From now on, we shall mainly work with the twisted Yangian and so we keep the same notation $s^{(r)}_{ij}$ for the generators of the algebra $\mathrm{Y}(\mathfrak{g}_N)$. Note that for any even series $g(u) \in 1 + \mathbb{C}[[u^{-2}]]\, u^{-2}$ the mapping

$$s_{ij}(u) \mapsto g(u)\, s_{ij}(u) \tag{3.2}$$

defines an automorphism of $\mathrm{Y}(\mathfrak{g}_N)$.

As we shall see below, the homomorphism of Theorem 2.3 respects the symmetry relation in the orthogonal case, while in the symplectic case a minor correction is needed to obtain a corresponding homomorphism of the twisted Yangians. In order to treat both cases simultaneously, introduce the following notation

$$\alpha_p(u) = \begin{cases} 1 & \text{in the orthogonal case} \\ \dfrac{u+1/2}{u-p+1/2} & \text{in the symplectic case.} \end{cases}$$

The image of the Sklyanin determinant in the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ acquires the following symmetry property

$$\alpha_n(u)^{-1} \cdot \operatorname{sdet} S(u) = \alpha_n(-u+N-1)^{-1} \cdot \operatorname{sdet} S(-u+N-1), \tag{3.3}$$

see [13, Section 4.11]. Moreover, the mapping

$$S(u) \mapsto \alpha_n(u) \cdot \widehat{S}(-u+N/2-1) \tag{3.4}$$

defines a homomorphism of $\mathrm{Y}(\mathfrak{g}_N)$ into itself; see [9, Proposition 2.1].

We can now prove a quantum Sylvester theorem for the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$. We use the notation of the previous section. In particular, recall that $\widetilde{s}_{ab}(u)$ denotes the quantum minor as in (2.17) for any $a, b \in \mathcal{A}$.

Theorem 3.1. *The mapping*

$$s_{ab}(u) \mapsto \alpha_{-m}(u)\, \widetilde{s}_{ab}(u) \tag{3.5}$$

defines an algebra homomorphism $\mathrm{Y}(\mathfrak{g}_{N-M}) \to \mathrm{Y}(\mathfrak{g}_N)$. *Moreover,*

$$\begin{aligned} \operatorname{sdet}\big[\alpha_{-m}(u)\, \widetilde{S}(u)\big] &= \alpha(u) \cdot \operatorname{sdet} S(u+M/2) \\ &\quad \times \operatorname{sdet} S_{\mathcal{BB}}(u+M/2-1) \cdots \operatorname{sdet} S_{\mathcal{BB}}(u+M/2-N+M+1), \end{aligned}$$

where

$$\alpha(u) = \alpha_{-m}(u)\, \alpha_{-m}(u-1) \cdots \alpha_{-m}(u-N+M+1).$$

Proof. Denote by $S^*(u)$ the matrix which occurs on the right hand side of (3.4). Then by (3.3) we have

$$S^{\circ}(u) = \frac{\alpha_n(u+N/2)}{c\,(u+N/2)} \cdot S^*(u),$$

where we have put $c(u) = \operatorname{sdet} S(u)$ for brevity. Due to (2.16) and the definition of the auxiliary minors, the relation (2.23) can be written as

$$s^{-m\cdots m,\, a_i}_{-m\cdots m,\, a_j}(u+M/2) = c\,(u+M/2) \\ \times \prod_{i=1}^{N-M-1} \frac{\alpha_n(-u+n-m+N/2-i)}{c\,(-u+n-m+N/2-i)} \cdot \widehat{s}^{\,*}_{a_i a_j}(-u+n-m-1), \quad (3.6)$$

where we have also used the fact that the coefficients of the Sklyanin determinant are central in the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$, and by $\widehat{s}^{\,*}_{ab}(u)$ we denote the entries of the Sklyanin comatrix corresponding to the matrix $S^*_{\mathcal{A}\mathcal{A}}(u)$. Observe that by (3.3) we have

$$c\,(u+M/2) \cdot \frac{\alpha_n(-u+n-m+N/2-1)}{c\,(-u+n-m+N/2-1)} = \alpha_n(u+M/2).$$

Therefore, (3.6) takes the form

$$s^{-m\cdots m,\, a_i}_{-m\cdots m,\, a_j}(u+M/2) = \alpha_n(u+M/2) \cdot \varphi(u) \cdot \widehat{s}^{\,*}_{a_i a_j}(-u+n-m-1),$$

where

$$\varphi(u) = \prod_{i=2}^{N-M-1} \frac{\alpha_n(-u+n-m+N/2-i)}{c\,(-u+n-m+N/2-i)}.$$

By the symmetry property (3.3) we have $\varphi(u) = \varphi(-u)$, and so the multiplication of the generator series $s_{ij}(u)$ by $\varphi(u)$ preserves the twisted Yangian defining relations. Furthermore, by (3.4), the mapping

$$s_{ab}(u) \mapsto \alpha_{n-m}(u)\, \widehat{s}^{\,*}_{ab}(-u+n-m-1), \qquad a, b \in \mathcal{A}$$

defines a homomorphism $\mathrm{Y}(\mathfrak{g}_{N-M}) \mapsto \mathrm{Y}(\mathfrak{g}_{N-M})$. Thus, we may conclude that the mapping

$$s_{ab}(u) \mapsto \alpha_{n-m}(u)\, \alpha_n(u+M/2)^{-1}\, s^{-m\cdots m,\, a_i}_{-m\cdots m,\, a_j}(u+M/2)$$

defines a homomorphism $\mathrm{Y}(\mathfrak{g}_{N-M}) \mapsto \mathrm{Y}(\mathfrak{g}_{N-M})$. To complete the proof, observe that $\alpha_{n-m}(u)\, \alpha_n(u+M/2)^{-1} = \alpha_{-m}(u)$.

The formula for the Sklyanin determinant of the matrix $\alpha_{-m}(u)\widetilde{S}(u)$ is immediate from Theorem 2.3 and the definition of $\operatorname{sdet} \widetilde{S}(u)$. □

The corresponding version of Proposition 2.5 for the twisted Yangian has the following form.

Proposition 3.2. *The mapping*

$$s_{ij}(u) \mapsto \alpha_{m-n}(u) \cdot s^{-n\cdots -m-1,\, i,\, m+1\cdots n}_{-n\cdots -m-1,\, j,\, m+1\cdots n}\,(u+n-m), \qquad -m \leqslant i, j \leqslant m$$

defines an algebra homomorphism $\mathrm{Y}(\mathfrak{g}_M) \to \mathrm{Y}(\mathfrak{g}_N)$. □

Now we shall demonstrate that in the case of the twisted Yangian the entries of the Sklyanin comatrix can be expressed in terms of Sklyanin minors; cf. (2.16). We need the following lemma.

Lemma 3.3. *For the twisted Yangian* $\mathrm{Y}(\mathfrak{g}_N)$ *we have*

$$A_N\, S_1(u)\, R^t_{12}(-2u+1)\cdots R^t_{1N}(-2u+N-1) = \frac{2u+1}{2u\pm 1}\, A_N\, S^t_1(-u). \tag{3.7}$$

Proof. We have $(N-1)!\, A_N = A_N\, A'_{N-1}$, where A'_{N-1} denotes the anti-symmetrizer corresponding to the subset of indices $\{2,\dots,N\}$. By (2.8),

$$A'_{N-1}\, R^t_{12}(-2u+1)\cdots R^t_{1N}(-2u+N-1) = A'_{N-1}\Big(1+\frac{Q_{12}+\dots+Q_{1N}}{2u-1}\Big).$$

Therefore, the left hand side of (3.7) takes the form

$$A_N\, S_1(u)\Big(1+\frac{Q_{12}+\dots+Q_{1N}}{2u-1}\Big).$$

Apply this operator to a basis vector

$$v_{ij} = e_{a_j}\otimes e_{a_1}\otimes\cdots\otimes \widehat{e}_{a_i}\otimes\cdots\otimes e_{a_N}, \qquad i,j\in\{1,\dots,N\},$$

where $(a_1,\dots,a_N)=(-n,\dots,n)$. The coefficient at $A_N\, v_{ii}$ will be equal to $s_{a_ia_j}(u)$ if $a_j=-a_i$, and equal to the expression

$$\frac{2u}{2u-1} s_{a_ia_j}(u) \mp \frac{1}{2u-1}\theta_{a_ia_j} s_{-a_j,-a_i}(u)$$

if $a_j\neq -a_i$. In both cases the coefficient coincides with

$$\frac{2u+1}{2u\pm 1}\, s^t_{a_ia_j}(-u)$$

due to the symmetry relation (3.1). □

As before, we set $(a_1,\dots,a_N)=(-n,\dots,n)$.

Proposition 3.4. *For any* $i,j\in\{1,\dots,N\}$ *we have the relation*

$$\widehat{s}^{\,t}_{a_ia_j}(u) = (-1)^{i+j}\cdot \alpha_{N-1}(u)\cdot s^{\,a_1\cdots\widehat{a}_j\cdots a_N}_{\,a_1\cdots\widehat{a}_i\cdots a_N}(-u+N-2).$$

Proof. The relations (2.9) and (2.19) with $M=N-2$ imply that

$$A_N\,\langle S_1,\dots,S_{N-1}\rangle = A_N\, \widehat{S}_N(u)\,(R^t_{N-1,N})^{-1}\cdots(R^t_{1,N})^{-1}. \tag{3.8}$$

Using the notation (2.20) and the definition of the matrix $S^*(u)$ we can write $\widehat{S}(u) = \alpha_n(u)\, S^*(u_N^\circ)$ so that (3.8) becomes

$$A_N \, \langle S_1, \dots, S_{N-1} \rangle = \alpha_n(u)\, A_N\, S_N^*(u_N^\circ)\, R_{N-1,N}^\circ \cdots R_{1N}^\circ. \tag{3.9}$$

However, $S(u) \mapsto S^*(u)$ defines a homomorphism $\mathrm{Y}(\mathfrak{g}_N) \to \mathrm{Y}(\mathfrak{g}_N)$. Therefore, writing $A_N = \operatorname{sgn} \sigma \cdot A_N\, P_\sigma$, where $\sigma = (1N)(2, N-1)\cdots$, and applying Lemma 3.3 we can simplify the right hand side of (3.9) as

$$\alpha_n(u)\, A_N\, S_N^*(u_N^\circ)\, R_{N-1,N}^\circ \cdots R_{1N}^\circ = \alpha_n(u)\, \frac{2u_N^\circ + 1}{2u_N^\circ \pm 1}\, A_N \;\, {S^*}^t_N(-u_N^\circ).$$

Hence, we come to the identity

$$A_N \, \langle S_1, \dots, S_{N-1} \rangle = \alpha_{n-1}(u)\, A_N \;\, {S^*}^t_N(-u_N^\circ).$$

Applying both sides to the basis vector $e_{a_1} \otimes \cdots \otimes \widehat{e}_{a_i} \otimes \cdots \otimes e_{a_N} \otimes e_{a_j}$ we get

$$\alpha_{n-1}(u)\, s^{*\,t}_{a_i a_j}(u - N/2 + 1) = (-1)^{i+j}\, s^{\, a_1 \cdots \widehat{a}_j \cdots a_N}_{\, a_1 \cdots \widehat{a}_i \cdots a_N}(u).$$

The argument is completed by using the definition of $S^*(u)$. $\square$

Suppose now that $m = n - 1$. As before, we identify the subalgebra of $\mathrm{Y}(\mathfrak{g}_N)$ generated by the elements $s_{ab}^{(r)}$ for $a, b \in \{-n, n\}$ with the twisted Yangian $\mathrm{Y}(\mathfrak{g}_2)$. The restriction of the homomorphism (3.4) to the subalgebra $\mathrm{Y}(\mathfrak{g}_2)$ defines the homomorphism

$$\phi : \mathrm{Y}(\mathfrak{g}_2) \to \mathrm{Y}(\mathfrak{g}_N).$$

The defining relations of the twisted Yangian imply that the map

$$s_{ab}(u) \mapsto \operatorname{sgn} a \cdot \operatorname{sgn} b \cdot s_{ab}(u), \qquad a, b \in \{-n, n\}$$

defines an automorphism $\psi : \mathrm{Y}(\mathfrak{g}_2) \to \mathrm{Y}(\mathfrak{g}_2)$.

Corollary 3.5. *The homomorphism ϕ coincides with the homomorphism* (3.5) *in the symplectic case, while in the orthogonal case the homomorphism* (3.5) *coincides with the composition $\phi \circ \psi$.*

Proof. This is immediate from Proposition 3.4 and the skew-symmetry of the Sklyanin minors. $\square$

The symmetry relation (3.1) allows one to obtain the following expansion of the auxiliary minors (see [8, Proposition 4.4]): if $-b_1 \in \{a_1, \dots, a_{k-1}, c\}$ and $c \notin \{-b_2, \dots, -b_{k-1}\}$ then

$$\check{s}^{\, a_1 \cdots a_{k-1}, c}_{\, b_1 \cdots b_{k-1}, c}(u) = \frac{2u+1}{2u \pm 1} \sum_{i=1}^{k-1} (-1)^{i-1}\, s^t_{a_i b_1}(-u)\, s^{\, a_1 \cdots \widehat{a}_i \cdots a_{k-1}}_{\, b_2 \cdots b_{k-1}}(u-1). \tag{3.10}$$

Using this relation together with (2.13) and (2.14) one can derive explicit formulas for the Sklyanin determinant and some Sklyanin minors. The formulas use a special map

$$\omega_N : \mathfrak{S}_N \to \mathfrak{S}_N, \qquad p \mapsto p' \tag{3.11}$$

from the symmetric group $\mathfrak{S}_N$ into itself which is defined by the following inductive procedure. Given a set of positive integers $c_1 < \cdots < c_N$ we regard $\mathfrak{S}_N$ as the group of their permutations. If $N = 2$ we define ω_2 as the map $\mathfrak{S}_2 \to \mathfrak{S}_2$ whose image is the identity permutation. For $N > 2$ define a map from the set of ordered pairs (c_k, c_l) with $k \neq l$ into itself by the rule

$$\begin{aligned}
(c_k, c_l) &\mapsto (c_l, c_k), && k, l < N,\\
(c_k, c_N) &\mapsto (c_{N-1}, c_k), && k < N-1,\\
(c_N, c_k) &\mapsto (c_k, c_{N-1}), && k < N-1,\\
(c_{N-1}, c_N) &\mapsto (c_{N-1}, c_{N-2}), &&\\
(c_N, c_{N-1}) &\mapsto (c_{N-1}, c_{N-2}). &&
\end{aligned} \tag{3.12}$$

Let $p = (p_1, \dots, p_N)$ be a permutation of the indices $c_1, \dots, c_N$. Its image under the map ω_N is the permutation $p' = (p'_1, \dots, p'_{N-1}, c_N)$, where the pair (p'_1, p'_{N-1}) is the image of the ordered pair (p_1, p_N) under the map (3.12). Then the pair (p'_2, p'_{N-2}) is found as the image of (p_2, p_{N-1}) under the map (3.12) which is defined on the set of ordered pairs of elements obtained from $(c_1, \dots, c_N)$ by deleting p_1 and p_N. The procedure is completed in the same manner by determining consequently the pairs (p'_i, p'_{N-i}).

Now suppose that M is a positive integer and $M = 2m$ or $M = 2m+1$. For the proof of the following formula for the Sklyanin minor in the twisted Yangian $\mathrm{Y}(\mathfrak{g}_N)$ see [8].

Proposition 3.6. *Suppose that $a_1, \dots, a_M, b_M$ are arbitrary indices from the set $\{-n, \dots, n\}$. We have*

$$s^{\,-a_1 \cdots -a_M}_{\,a_1 \cdots a_{M-1}, b_M}(u) = \alpha_m(u) \sum_{p \in \mathfrak{S}_M} \operatorname{sgn} pp' \cdot s^{\,t}_{-a_{p(1)}, a_{p'(1)}}(-u) \cdots s^{\,t}_{-a_{p(m)}, a_{p'(m)}}(-u+m-1)$$
$$\times\, s_{-a_{p(m+1)}, a_{p'(m+1)}}(u-m) \cdots s_{-a_{p(M)}, b_{p'(M)}}(u-M+1),$$

where the $s^t_{ij}(u)$ denote the entries of the matrix $S^t(u)$. □

Remark 3.7. Although this determinant-like formula does not apply to arbitrary Sklyanin minors $s^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(u)$, it provides explicit formulas for the Sklyanin determinant $\operatorname{sdet} S(u)$ and the matrix elements $\widehat{s}_{ij}(u)$ of the Sklyanin comatrix; see Proposition 3.4.

For the use in Section 5 we shall prove the following simple property of the map (3.11).

Lemma 3.8. *The map $\mathfrak{S}_N \to \mathfrak{S}_N$ defined by $p \mapsto p\,(p')^{-1}$is bijective.*

Proof. Suppose that p and q are two elements of $\mathfrak{S}_N$ such that $p\,(p')^{-1} = q\,(q')^{-1}$. It suffices to show that $p = q$. By definition of the map ω_N we have $p'_N = q'_N = N$ which implies that $p_N = q_N$. Then, due to the formulas (3.12), we have $p'_1 = q'_1$. Hence, $p_1 = q_1$. Now, since the pairs (p_1, p_N) and (q_1, q_N) coincide, so do their images under the map (3.12). In particular, $p'_{N-1} = q'_{N-1}$. This implies that $p_{N-1} = q_{N-1}$ and the proof is completed by repeating this argument for the pairs (p_{i+1}, p_{N-i}) and (q_{i+1}, q_{N-i}) with $i = 1, 2, \dots$. □

4 Skew representations

As before, we suppose that $N = 2n$ or $N = 2n + 1$ so that

$$\mathfrak{g}_N = \mathfrak{o}_{2n+1}, \quad \mathfrak{sp}_{2n}, \quad \text{or} \quad \mathfrak{o}_{2n}. \tag{4.1}$$

The finite-dimensional irreducible representations of $\mathfrak{g}_N$ are in a one-to-one correspondence with n-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ where the numbers λ_i satisfy the conditions

$$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \dots, n-1,$$

and

$$\begin{aligned} -2\,\lambda_1 &\in \mathbb{Z}_+ && \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n+1}, \\ -\lambda_1 &\in \mathbb{Z}_+ && \text{for} \quad \mathfrak{g}_N = \mathfrak{sp}_{2n}, \\ -\lambda_1 - \lambda_2 &\in \mathbb{Z}_+ && \text{for} \quad \mathfrak{g}_N = \mathfrak{o}_{2n}. \end{aligned}$$

Such an n-tuple λ is called the *highest weight*[1] of the corresponding representation which we shall denote by $V(\lambda)$. It contains a unique, up to a constant factor, nonzero vector ξ (the *highest vector*) such that

$$\begin{aligned} F_{ii}\,\xi &= \lambda_i\,\xi && \text{for} \quad i = 1, \dots, n, \\ F_{ij}\,\xi &= 0 && \text{for} \quad -n \leqslant i < j \leqslant n. \end{aligned}$$

Let M be a nonnegative integer such that $N - M$ is even and positive. So, $M = 2m$ or $M = 2m+1$ for some $m < n$. We shall identify the Lie algebra $\mathfrak{g}_M$ with the subalgebra

[1] In a more common notation, the highest weight is the n-tuple $(-\lambda_n, \dots, -\lambda_1)$. In particular, in the symplectic case this n-tuple is a partition.

of $\mathfrak{g}_N$ spanned by the elements F_{ij} with the indices satisfying $-m \leqslant i, j \leqslant m$. Denote by $V(\lambda)^+$ the subspace of $\mathfrak{g}_M$-highest vectors in $V(\lambda)$:

$$V(\lambda)^+ = \{\eta \in V(\lambda) \mid F_{ij}\, \eta = 0, \qquad -m \leqslant i < j \leqslant m\}.$$

Given a $\mathfrak{g}_M$-weight $\mu = (\mu_1, \dots, \mu_m)$ we denote by $V(\lambda)^+_\mu$ the corresponding weight subspace in $V(\lambda)^+$:

$$V(\lambda)^+_\mu = \{\eta \in V(\lambda)^+ \mid F_{ii}\, \eta = \mu_i\, \eta, \qquad i = 1, \dots, m\}.$$

We have a natural vector space isomorphism $V(\lambda)^+_\mu \cong \mathrm{Hom}_{\mathfrak{g}_M}(V(\mu), V(\lambda))$.

For any $i, j \in \{-n, \dots, n\}$ introduce the series in u^{-1} with coefficients in the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$ by

$$f_{ij}(u) = \delta_{ij} + F_{ij}\Big(u \pm \frac{1}{2}\Big)^{-1}.$$

The mapping

$$\pi : s_{ij}(u) \mapsto f_{ij}(u) \tag{4.2}$$

defines a surjective homomorphism $\mathrm{Y}(\mathfrak{g}_N) \to \mathrm{U}(\mathfrak{g}_N)$ called the *evaluation homomorphism*; see [18] and [13, Proposition 3.11]. Let $F(u)$ denote the $N \times N$ matrix whose ij-th entry is the series $f_{ij}(u)$. We may introduce the Sklyanin minors $f^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(u)$ of this matrix as the images of the corresponding minors of the matrix $S(u)$ with respect to the evaluation homomorphism,

$$\pi : s^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(u) \mapsto f^{\,a_1 \cdots a_k}_{\,b_1 \cdots b_k}(u).$$

By Theorem 3.1, we have a homomorphism $\mathrm{Y}(\mathfrak{g}_{N-M}) \to \mathrm{U}(\mathfrak{g}_N)$ given by

$$\rho : s_{ab}(u) \mapsto \alpha_{-m}(u)\, f^{\,-m \cdots m,\, a}_{\,-m \cdots m,\, b}\,(u + M/2). \tag{4.3}$$

Due to Corollary 2.2, the image of this homomorphism is contained in the centralizer $\mathrm{U}(\mathfrak{g}_N)^{\mathfrak{g}_M}$ of the subalgebra $\mathfrak{g}_M$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$. On the other hand, the vector space $V(\lambda)^+_\mu$ is obviously a representation of $\mathrm{U}(\mathfrak{g}_N)^{\mathfrak{g}_M}$. Thus, $V(\lambda)^+_\mu$ becomes equipped with the $\mathrm{Y}(\mathfrak{g}_{N-M})$-module structure defined via the homomorphism ρ. We call this module the *skew representation* of $\mathrm{Y}(\mathfrak{g}_{N-M})$. In the particular case $M = 0$ (with even N) the skew representation is just the *evaluation module* $V(\lambda)$ over $\mathrm{Y}(\mathfrak{g}_N)$ defined via the evaluation homomorphism (4.2).

The universal enveloping algebra $\mathrm{U}(\mathfrak{g}_{N-M})$ can be identified with a subalgebra of the twisted Yangian $\mathrm{Y}(\mathfrak{g}_{N-M})$ via the embedding $F_{ab} \mapsto s^{(1)}_{ab}$, see [13, Proposition 3.12]. The elements F_{ab} are stable under the composition of this embedding with the homomorphism ρ. Indeed, if $a \neq b$ then this is verified directly from the definition of the Sklyanin minors with the use of (2.22). If $a = b$ then we may assume without loss of generality that $m = n - 1$ and $a = \pm n$. So, the claim follows from Corollary 3.5 and (3.3). In other words, the restriction of the $\mathrm{Y}(\mathfrak{g}_{N-M})$-module $V(\lambda)^+_\mu$ to the subalgebra $\mathrm{U}(\mathfrak{g}_{N-M})$ coincides with its natural action defined by the $\mathrm{U}(\mathfrak{g}_N)$-action on $V(\lambda)$.

We shall now concentrate on the symplectic case $\mathfrak{g}_N = \mathfrak{sp}_N$ where $N = 2n$. Our next goal is to prove that the skew representations of $\mathrm{Y}(\mathfrak{sp}_{N-M})$ are irreducible. First we show the following.

Proposition 4.1. *The centralizer* $\mathrm{U}(\mathfrak{sp}_N)^{\mathfrak{sp}_M}$ *is generated by the image of the homomorphism* ρ *and the center of* $\mathrm{U}(\mathfrak{sp}_N)$*.*

Proof. For a different homomorphism $\rho' : \mathrm{Y}(\mathfrak{sp}_{N-M}) \to \mathrm{U}(\mathfrak{sp}_N)^{\mathfrak{sp}_M}$ given by

$$s_{ab}(u) \mapsto \alpha_n(u)\, \widehat{f}_{ab}(-u + n - 1), \qquad a, b \in \mathcal{A}, \tag{4.4}$$

where $\widehat{f}_{ab}(u)$ is the image of $\widehat{s}_{ab}(u)$ under the evaluation homomorphism (4.2), this statement was proved in [14, Section 4]. Denote by U' the subalgebra of $\mathrm{U}(\mathfrak{sp}_N)^{\mathfrak{sp}_M}$ generated by the center of $\mathrm{U}(\mathfrak{sp}_N)$ and the coefficients of the series $f^{\,-m\cdots m,\, a}_{\,-m\cdots m,\, b}(u)$ with $a, b \in \mathcal{A}$. It is sufficient to prove that all the coefficients of the series $\widehat{f}_{ab}(u)$ belong to U'. However, the center of $\mathrm{U}(\mathfrak{sp}_N)$ is generated by the coefficients of the Sklyanin determinant $\operatorname{sdet} F(u)$, i.e., the image of $\operatorname{sdet} S(u)$ under the evaluation homomorphism (4.2); see [8, Theorem 5.2]. Then by Theorem 3.1 the coefficients of the series $\operatorname{sdet} F_{\mathcal{B}\mathcal{B}}(u)$ belong to U'. Due to Proposition 2.4, the coefficients of the series $\widehat{f}_{ab}(u)$ also belong to U'. □

Corollary 4.2. *The skew representation* $V(\lambda)^+_\mu$ *of the twisted Yangian* $\mathrm{Y}(\mathfrak{sp}_{N-M})$ *is irreducible.*

Proof. Since the representation $V(\lambda)^+_\mu$ of $\mathrm{U}(\mathfrak{sp}_N)^{\mathfrak{sp}_M}$ is irreducible [1, Section 9.1], the statement follows from Proposition 4.1 as the central elements of $\mathrm{U}(\mathfrak{sp}_N)$ act on $V(\lambda)^+_\mu$ by scalar operators. □

Remark 4.3. In general, the corresponding statement for the orthogonal twisted Yangian $\mathrm{Y}(\mathfrak{o}_{N-M})$ is false; see [12] for the particular case $N - M = 2$. If N is even, then the $\mathrm{Y}(\mathfrak{o}_2)$-module $V(\lambda)^+_\mu$ is still irreducible. If N is odd, then for general parameters λ and μ the $\mathrm{Y}(\mathfrak{o}_2)$-module $V(\lambda)^+_\mu$ is isomorphic to the direct sum of two irreducibles. It looks plausible that, in general, the $\mathrm{Y}(\mathfrak{o}_{N-M})$-module $V(\lambda)^+_\mu$ is completely reducible. It would be interesting to obtain its irreducible decomposition.

Now recall the classification results for representations of the twisted Yangian $\mathrm{Y}(\mathfrak{sp}_N)$; see [9]. If V is a finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{sp}_N)$ then V contains a unique, up to a scalar factor, vector $\xi \neq 0$ such that

$$\begin{aligned} s_{ii}(u)\,\xi &= \mu_i(u)\,\xi \qquad &&\text{for} \quad i = 1, \dots, n, \\ s_{ij}(u)\,\xi &= 0 \qquad &&\text{for} \quad -n \leqslant i < j \leqslant n, \end{aligned}$$

where each $\mu_i(u)$ is a formal series in u^{-1} with coefficients in $\mathbb{C}$. Moreover, there exist monic polynomials $P_1(u), \dots, P_n(u)$ in u with $P_1(u) = P_1(-u+1)$ such that

$$\frac{\mu_{i-1}(u)}{\mu_i(u)} = \frac{P_i(u+1)}{P_i(u)}, \qquad i = 2, \dots, n \tag{4.5}$$

and

$$\frac{\mu_1(-u)}{\mu_1(u)} = \frac{P_1(u+1)}{P_1(u)}. \tag{4.6}$$

The n-tuple $\mu(u) = (\mu_1(u), \dots, \mu_n(u))$ is called the *highest weight* and the $P_i(u)$ are called the *Drinfeld polynomials* of the representation V. Furthermore, given an n-tuple $(P_1(u), \dots, P_n(u))$ of monic polynomials with $P_1(u) = P_1(-u+1)$ there exists a finite-dimensional irreducible representation V of $\mathrm{Y}(\mathfrak{sp}_N)$ having this n-tuple as the family of its Drinfeld polynomials. The isomorphism class of such representation V is determined uniquely, up to the twisting by an automorphism of $\mathrm{Y}(\mathfrak{sp}_N)$ of the form

$$s_{ij}(u) \mapsto g(u)\, s_{ij}(u),$$

where $g(u)$ is a series in u^{-2} with constant term 1.

Since the Sklyanin determinant $\mathrm{sdet}\, S(u)$ is central in $\mathrm{Y}(\mathfrak{sp}_N)$, it acts on V by scalar multiplication. The scalar can be calculated with the use of Proposition 3.6; see also [8]. We have

$$\mathrm{sdet}\, S(u)|_V = \alpha_n(u) \prod_{i=1}^{n} \mu_i(-u+i-1)\, \mu_i(u-N+i). \tag{4.7}$$

Proposition 4.4. *With the above notation, for any $k = 1, \dots, n$ in the representation V we have*

$$s^{\,-k+1\cdots k}_{\,-k+1\cdots k}(u)\,\xi = \mu_k(u-2k+2)\, s^{\,-k+1\cdots k-1}_{\,-k+1\cdots k-1}(u)\,\xi.$$

Proof. By (2.13) we can write

$$s^{\,-k+1\cdots k}_{\,-k+1\cdots k}(u) = \sum_{c=-n}^{n} \check{s}^{\,-k+1\cdots k}_{\,-k+1\cdots k-1,c}(u)\ s_{ck}(u-2k+2).$$

Apply both sides to the highest vector ξ. We have $s_{ck}(u-2k+2)\,\xi = 0$ if $c < k$. On the other hand, if $c > k$ then $\check{s}\,{}^{-k+1\cdots k}_{-k+1\cdots k-1,c}(u) = 0$ by (2.14). Finally, if $c = k$ then (2.15) gives

$$\check{s}\,{}^{-k+1\cdots k}_{-k+1\cdots k-1,c}(u) = s\,{}^{-k+1\cdots k-1}_{-k+1\cdots k-1}(u)$$

completing the proof. $\square$

Our aim now is to identify the representation $V(\lambda)^+_\mu$ of $\mathrm{Y}(\mathfrak{sp}_{N-M})$ by calculating its highest weight and Drinfeld polynomials. Note that for the evaluation module $V(\lambda)$ over $\mathrm{Y}(\mathfrak{sp}_N)$ these can be immediately found from (4.2). In particular, the i-th component of the highest weight is given by

$$\frac{u+\lambda_i-1/2}{u-1/2}, \qquad i = 1,\dots,n. \tag{4.8}$$

In the case $M > 0$ we employ the basis in the $\mathfrak{sp}_N$-module $V(\lambda)$ constructed in [10]. This basis is parameterized by the *patterns* Λ which are arrays of non-positive integers of the form

$$\begin{array}{ccccccccc}
 & \lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\
\lambda'_{n1} & & \lambda'_{n2} & & \cdots & & \lambda'_{nn} & \\
 & \lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & \\
\lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & & & \\
 & \cdots & & \cdots & & & & \\
 & \lambda_{11} & & & & & & \\
\lambda'_{11} & & & & & & &
\end{array}$$

where $\lambda = (\lambda_{n1},\dots,\lambda_{nn})$ is the top row of Λ and the following *betweenness conditions* hold

$$0 \geqslant \lambda'_{k1} \geqslant \lambda_{k1} \geqslant \lambda'_{k2} \geqslant \lambda_{k2} \geqslant \cdots \geqslant \lambda'_{k,k-1} \geqslant \lambda_{k,k-1} \geqslant \lambda'_{kk} \geqslant \lambda_{kk}$$

for $k = 1,\dots,n$, and

$$0 \geqslant \lambda'_{k1} \geqslant \lambda_{k-1,1} \geqslant \lambda'_{k2} \geqslant \lambda_{k-1,2} \geqslant \cdots \geqslant \lambda'_{k,k-1} \geqslant \lambda_{k-1,k-1} \geqslant \lambda'_{kk}$$

for $k = 2,\dots,n$. The representation $V(\lambda)$ admits a basis ζ_Λ parameterized by all patterns Λ. The formulas for the matrix elements of a family of generators of the Lie algebra $\mathfrak{sp}_N$ in this basis can be explicitly written down; see [10]. We shall only need these formulas for the generators F_{kk}. We have

$$F_{kk}\,\zeta_\Lambda = \left(2\sum_{i=1}^{k}\lambda'_{ki} - \sum_{i=1}^{k}\lambda_{ki} - \sum_{i=1}^{k-1}\lambda_{k-1,i}\right)\zeta_\Lambda, \qquad k = 1,\dots,n. \tag{4.9}$$

Given a highest weight $\mu = (\mu_1, \dots, \mu_m)$ for $\mathfrak{sp}_M$, a basis of the space $V(\lambda)^+_\mu$ is formed by those vectors ζ_Λ for which the row $(\lambda_{m1}, \dots, \lambda_{mm})$ of Λ coincides with μ and

$$\lambda_{ki} = \lambda'_{ki} = \mu_i, \qquad 1 \leqslant i \leqslant k \leqslant m.$$

Omitting the same triangle part below the m-th row of all such patterns Λ we get trapezium-like patterns (still denoted by Λ) with the top row λ and the bottom row μ, as illustrated:

$$\begin{array}{ccccccccc}
 & \lambda_1 & & \lambda_2 & & \cdots & & & \lambda_n \\
\lambda'_{n1} & & \lambda'_{n2} & & \cdots & & & \lambda'_{nn} & \\
 & \cdots & & \cdots & & & & & \\
\lambda'_{m+1,1} & & \lambda'_{m+1,2} & & \cdots & & \lambda'_{m+1,m+1} & & \\
 & \mu_1 & & \mu_2 & \cdots & \mu_m & & &
\end{array}$$

Due to the betweenness conditions, the space $V(\lambda)^+_\mu$ is nonzero if and only if

$$\mu_i \geqslant \lambda_{i+n-m}, \qquad i = 1, \dots, m,$$

and

$$\lambda_i \geqslant \mu_{i+n-m}, \qquad i = 1, \dots, n,$$

assuming $\mu_i = -\infty$ for $i > m$. We also set $\mu_i = 0$ for $i \leqslant 0$. In what follows we suppose that $V(\lambda)^+_\mu$ is nonzero.

Denote by $\mathfrak{h}$ the diagonal Cartan subalgebra of $\mathfrak{sp}_{N-M}$ spanned by the basis vectors F_{kk} with $k = m+1, \dots, n$. Let the $\varepsilon_k \in \mathfrak{h}^*$ with $k = m+1, \dots, n$ be the dual basis vectors of $\mathfrak{h}^*$. Consider the root system for $\mathfrak{sp}_{N-M}$ with respect to $\mathfrak{h}$ where the basis vectors F_{ij} with $i < j$ are positive root vectors so that the positive roots are $-2\varepsilon_k$ with $k = m+1, \dots, n$ and $\pm\,\varepsilon_i - \varepsilon_j$ with $m+1 \leqslant i < j \leqslant n$. By (4.9), the weight $w(\Lambda) = (w_{m+1}, \dots, w_n)$ of a trapezium pattern Λ with respect to $\mathfrak{h}$ is given by

$$w_k = 2\sum_{i=1}^{k} \lambda'_{ki} - \sum_{i=1}^{k} \lambda_{ki} - \sum_{i=1}^{k-1} \lambda_{k-1,i}, \qquad k = m+1, \dots, n.$$

We shall need the standard partial ordering on the set of weights of an $\mathfrak{sp}_{N-M}$-module V. We shall write $w \preccurlyeq w'$ for two weights w and w' of V if $w' - w$ is a linear combination of positive roots with nonnegative integral coefficients. Since the vectors ζ_Λ corresponding to the trapezium patterns Λ form a basis of the $\mathfrak{sp}_{N-M}$-module $V(\lambda)^+_\mu$, the set of weights of this module is comprised by the weights $w(\Lambda)$ for all possible patterns Λ.

Given three integers i, j, k we shall denote by $\mathrm{mid}\{i, j, k\}$ that of the three which is between the two others. If one of the indices, say, k is the symbol $-\infty$ then $\mathrm{mid}\{i, j, k\}$ is understood as $\min\{i, j\}$. Consider the trapezium array Λ_0 whose entries are determined by

$$\lambda_{ki} = \mathrm{mid}\{\lambda_i, \mu_{i+k-m}, \mu_{i+m-k}\} \tag{4.10}$$

and

$$\lambda'_{ki} = \mathrm{mid}\{\lambda_i, \mu_{i+k-m-1}, \mu_{i+m-k}\}$$

for all possible values of i and k. One easily verifies that Λ_0 is a pattern.

Proposition 4.5. *The $\mathfrak{sp}_{N-M}$-module $V(\lambda)^+_\mu$ has a unique maximal weight. This weight coincides with $w(\Lambda_0)$.*

Proof. Suppose that $w(\Lambda)$ is a maximal weight of $V(\lambda)^+_\mu$ for some pattern Λ. Then the entries of Λ should satisfy

$$\lambda'_{ki} = \max\{\lambda_{ki}, \lambda_{k-1,i}\} \tag{4.11}$$

for all $k = m+1, \dots, n-1$ and $i = 1, \dots, k$, where we assume $\lambda_{ki} = -\infty$ for $i > k$. Indeed, if the equality is not attained for some k and i then by the betweenness conditions we have $\lambda'_{ki} > \max\{\lambda_{ki}, \lambda_{k-1,i}\}$. Therefore, decreasing the entry λ'_{ki} by 1 we get a pattern of greater weight than $w(\Lambda)$ which contradicts the maximality of $w(\Lambda)$. Thus, the weight $w(\Lambda) = (w_{m+1}, \dots, w_n)$ can now be written as

$$w_k = \sum_{i=1}^{k-1} |\lambda_{ki} - \lambda_{k-1,i}| + \lambda_{kk}, \qquad k = m+1, \dots, n.$$

We shall argue by induction on $n - m$ to show that the λ_{ki} must be given by (4.10). This will imply the statement since the entries of Λ_0 do satisfy (4.11) which is easily seen. In the case $n - m = 1$ there is nothing to prove. Suppose that $n - m = 2$. Omitting the primed entries, we can depict Λ as

$$\begin{array}{cccccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_{n-2} & \lambda_{n-1} & \lambda_n \\
\rho_1 & \rho_2 & \cdots & \rho_{n-2} & \rho_{n-1} & \\
\mu_1 & \mu_2 & \cdots & \mu_{n-2} & &
\end{array}$$

where we have put $\rho_i = \lambda_{n-1,i}$. The weight $w(\Lambda) = (w_{n-1}, w_n)$ is given by

$$\begin{aligned}
w_{n-1} &= \sum_{i=1}^{n-2} |\rho_i - \mu_i| + \rho_{n-1}, \\
w_n &= \sum_{i=1}^{n-1} |\lambda_i - \rho_i| + \lambda_n.
\end{aligned} \tag{4.12}$$

Suppose that (4.10) is violated for some i so that $\rho_i \neq \mathrm{mid}\{\lambda_i, \mu_{i+1}, \mu_{i-1}\}$. Observe that by the betweenness conditions we have

$$\rho_{i+1} \leqslant \mathrm{mid}\{\lambda_i, \mu_{i+1}, \mu_{i-1}\} \leqslant \rho_{i-1}.$$

Hence, if $\rho_i < \mathrm{mid}\{\lambda_i, \mu_{i+1}, \mu_{i-1}\}$ (respectively, $\rho_i > \mathrm{mid}\{\lambda_i, \mu_{i+1}, \mu_{i-1}\}$) then we can increase (respectively, decrease) the value of ρ_i by 1 without violating the betweenness conditions and thus to get another pattern Λ'. Due to the formulas (4.12), we have $w(\Lambda) \prec w(\Lambda')$ which contradicts the maximality of $w(\Lambda)$. This proves the statement in the case under consideration.

Suppose now that $n - m > 2$. Let us set $\rho_i = \lambda_{n-1,i}$ as above, and consider the set of patterns having $\rho = (\rho_1, \dots, \rho_{n-1})$ as the top row and μ as the bottom row. Then, by the maximality of $w(\Lambda)$, the trapezium subpattern of Λ with the top row ρ and bottom row μ will clearly be of a maximal weight amongst all patterns of this set. By the induction hypothesis, we must have

$$\lambda_{ki} = \mathrm{mid}\{\rho_i, \mu_{i+k-m}, \mu_{i+m-k}\} \tag{4.13}$$

for all $k = m+1, \dots, n-2$ and $i = 1, \dots, k$. Similarly, the subpattern of Λ having the top row λ and the bottom row $\sigma = (\sigma_1, \dots, \sigma_{n-2})$ is of a maximal weight amongst all patterns of this form, where we have put $\sigma_i = \lambda_{n-2,i}$. By the statement for the case $n - m = 2$, we must have

$$\rho_i = \mathrm{mid}\{\lambda_i, \sigma_{i+1}, \sigma_{i-1}\}, \qquad i = 1, \dots, n-1.$$

Combining this with the relations $\sigma_i = \mathrm{mid}\{\rho_i, \mu_{i+n-m-2}, \mu_{i+m-n+2}\}$ implied by (4.13), we get the desired relation for the row ρ of Λ,

$$\rho_i = \mathrm{mid}\{\lambda_i, \mu_{i+n-m-1}, \mu_{i+m-n+1}\}, \qquad i = 1, \dots, n-1.$$

Indeed, this is easily verified by looking at all possible combinations of the three values of each of σ_{i-1} and σ_{i+1}. Finally, substituting these values of ρ_i into (4.13) we conclude that (4.10) holds for all possible k and i. Thus, Λ coincides with Λ_0. $\square$

Corollary 4.6. *The vector* ζ_{Λ_0} *is the highest vector of the* $\mathrm{Y}(\mathfrak{sp}_{N-M})$*-module* $V(\lambda)^+_\mu$.

Proof. It suffices to demonstrate that ζ_{Λ_0} is annihilated by all generators $s_{ab}(u)$ of $\mathrm{Y}(\mathfrak{sp}_{N-M})$ with $a < b$, since by [9, Remark 4.4] the vector ζ_{Λ_0} will then have to be an eigenvector for all $s_{aa}(u)$ with $a = m+1, \dots, n$.

Recall that $s_{ab}(u)$ acts on $V(\lambda)^+_\mu$ as a Sklyanin minor which is a series in u^{-1} with coefficients in the universal enveloping algebra $\mathrm{U}(\mathfrak{sp}_N)$; see (4.3). By (2.28), the weight of all these coefficients coincides with the weight of the element F_{ab} with respect to the adjoint action of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sp}_{N-M}$. Therefore, if $a < b$ then the vector $s_{ab}(u)\,\zeta_{\Lambda_0}$ has the weight $w(\Lambda_0) + \alpha$ for a positive root α. Now the claim follows from Proposition 4.5. □

Corollary 4.6 implies that the highest weight $\mu(u) = (\mu_{m+1}(u), \dots, \mu_n(u))$ of the $\mathrm{Y}(\mathfrak{sp}_{N-M})$-module $V(\lambda)^+_\mu$ is determined by the relations

$$s_{aa}(u)\,\zeta_{\Lambda_0} = \mu_a(u)\,\zeta_{\Lambda_0}, \qquad a = m+1, \dots, n.$$

In order to calculate $\mu(u)$ we shall use the results of [10], where the particular case $m = n-1$ was considered. In [10] the vector space $V(\lambda)^+_\mu$ was endowed with the $\mathrm{Y}(\mathfrak{sp}_2)$-module structure defined by the composition of an automorphism of the type (3.2) and the homomorphism (4.4). Therefore, using Corollary 3.5, we can reformulate the result for the $\mathrm{Y}(\mathfrak{sp}_2)$-module structure on $V(\lambda)^+_\mu$ defined by the homomorphism (4.3) to obtain the following.

Proposition 4.7. *For $m = n-1$ in the $\mathrm{Y}(\mathfrak{sp}_2)$-module $V(\lambda)^+_\mu$ we have*

$$\alpha_{-n+1}(u)\, f\,{}^{-n+1\,\cdots\, n}_{-n+1\,\cdots\, n}(u+n-1)\,\zeta_{\Lambda_0} = \prod_{i=2}^{n} \frac{u - \min\{\lambda_{i-1}, \mu_{i-1}\} + i - 1/2}{u + i - 1/2} \cdot \prod_{i=1}^{n} \frac{u + \max\{\lambda_i, \mu_i\} - i + 1/2}{u - i + 1/2}\,\zeta_{\Lambda_0}.$$

In the case of arbitrary $m < n$ introduce the following notation:

$$\nu(u) = \prod_{i=1}^{m} \frac{(u + \mu_i - i + 1/2)\,(u - \mu_i + i + 1/2)}{(u - i + 1/2)\,(u + i + 1/2)}. \tag{4.14}$$

Theorem 4.8. *The highest weight $\mu(u) = (\mu_{m+1}(u), \dots, \mu_n(u))$ of the $\mathrm{Y}(\mathfrak{sp}_{N-M})$-module $V(\lambda)^+_\mu$ is given by the formulas*

$$\begin{aligned}\mu_k(u) = \nu(u) \cdot & \prod_{\substack{i=1\\ \lambda_i < \mu_{i+k-m-1}}}^{k-1} \frac{u - \max\{\lambda_i, \mu_{i+k-m}\} + k - m + i - 1/2}{u - \mu_{i+k-m-1} + k - m + i - 1/2}\\ \times & \prod_{\substack{i=1\\ \lambda_i > \mu_{i+m-k+1}}}^{k-1} \frac{u + \min\{\lambda_i, \mu_{i+m-k}\} + k - m - i - 1/2}{u + \mu_{i+m-k+1} + k - m - i - 1/2}\\ & \times \frac{u + \min\{\lambda_k, \mu_m\} - m - 1/2}{u - m - 1/2},\end{aligned}$$

where $k = m+1, \dots, n$.

Proof. Let us denote by $\varphi_{ab}(u)$ the image of the series $s_{ab}(u)$ under the homomorphism ρ defined in (4.3), that is,

$$\varphi_{ab}(u) = \alpha_{-m}(u)\, f^{\,-m\cdots m,\, a}_{\,-m\cdots m,\, b}\,(u+M/2). \tag{4.15}$$

Combine these series into the $(N-M)\times(N-M)$ matrix $\Phi(u)$. We shall use the usual notation for the Sklyanin comatrix and the Sklyanin minors of $\Phi(u)$. Applying Proposition 2.4, we obtain the following expression for the nn-th entry of the Sklyanin comatrix $\widehat{\Phi}(u)$,

$$\widehat{\varphi}_{nn}(u) = \alpha_{-m}(u)\, \widehat{f}_{nn}(u+M/2)\cdot f_{\mathcal{B}}(u-1)\cdots f_{\mathcal{B}}(u-N+M+2), \tag{4.16}$$

where we have put $f_{\mathcal{B}}(u) = \alpha_{-m}(u)\,\mathrm{sdet}\, F_{\mathcal{B}\mathcal{B}}(u+M/2)$. On the other hand, by Proposition 3.4, we have

$$\widehat{\varphi}_{nn}(u) = \alpha_{N-M-1}(u)\, \varphi^{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n}_{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n}\,(-u+N-M-2)$$

and

$$\widehat{f}_{nn}(u) = \alpha_{N-1}(u)\, f^{\,-n+1\,\cdots\, n}_{\,-n+1\,\cdots\, n}\,(-u+N-2).$$

Therefore, by (4.16),

$$\begin{aligned}\alpha_{N-M-1}(u)\, \varphi^{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n}_{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n}\,(-u+N-M-2) = \\ \alpha_{N-1}(u+M/2)\, f^{\,-n+1\,\cdots\, n}_{\,-n+1\,\cdots\, n}\,(-u+N-M/2-2) \\ \times\, \alpha_{-m}(u)\, f_{\mathcal{B}}(u-1)\cdots f_{\mathcal{B}}(u-N+M+2).\end{aligned} \tag{4.17}$$

Similarly, applying Theorem 3.1, we derive the identity

$$\begin{aligned}\alpha_{N-M}(u)\, \varphi^{\,-n\,\cdots\,-m-1,\, m+1\cdots\, n}_{\,-n\,\cdots\,-m-1,\, m+1\cdots\, n}\,(-u+N-M-1) = \\ \alpha_{N}(u+M/2)\, f^{\,-n\,\cdots\, n}_{\,-n\,\cdots\, n}\,(-u+N-M/2-1) \\ \times\, \alpha_{-m}(u)\, f_{\mathcal{B}}(u-1)\cdots f_{\mathcal{B}}(u-N+M+1).\end{aligned}$$

Replacing here n by $n-1$ and u by $u-1$ we get

$$\begin{aligned}\alpha_{N-M-2}(u-1)\, \varphi^{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n-1}_{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n-1}\,(-u+N-M-2) = \\ \alpha_{N-2}(u+M/2-1)\, f^{\,-n+1\,\cdots\, n-1}_{\,-n+1\,\cdots\, n-1}\,(-u+N-M/2-2) \\ \times\, \alpha_{-m}(u-1)\, f_{\mathcal{B}}(u-2)\cdots f_{\mathcal{B}}(u-N+M+2).\end{aligned} \tag{4.18}$$

Due to Proposition 4.4, in $V(\lambda)^+_\mu$ we have

$$\varphi^{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n}_{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n}\,(v)\,\zeta_{\Lambda_0} = \mu_n(v-N+M+2)\, \varphi^{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n-1}_{\,-n+1\,\cdots\,-m-1,\, m+1\cdots\, n-1}\,(v)\,\zeta_{\Lambda_0}.$$

Hence, comparing (4.17) and (4.18) we come to the relation

$$f_{\mathcal{B}}(u-1)\, f^{\,-n+1\cdots n}_{\,-n+1\cdots n}(-u+2n-m-2)\,\zeta_{\Lambda_0} = \mu_n(-u)\, f^{\,-n+1\cdots n-1}_{\,-n+1\cdots n-1}(-u+2n-m-2)\,\zeta_{\Lambda_0}.$$

Therefore, for each $k = m+1,\dots,n$ the corresponding component of the highest weight of the $\mathrm{Y}(\mathfrak{sp}_{N-M})$-module $V(\lambda)^+_\mu$ can be found from the relation

$$f_{\mathcal{B}}(-u-1)\, f^{\,-k+1\cdots k}_{\,-k+1\cdots k}(u+2k-m-2)\,\zeta_{\Lambda_0} \\ = \mu_k(u)\, f^{\,-k+1\cdots k-1}_{\,-k+1\cdots k-1}(u+2k-m-2)\,\zeta_{\Lambda_0}. \tag{4.19}$$

Observe that each of the Sklyanin minors $f^{\,-k+1\cdots k}_{\,-k+1\cdots k}(v)$ and $f^{\,-k+1\cdots k-1}_{\,-k+1\cdots k-1}(v)$ commutes with all elements of the subalgebra $\mathfrak{sp}_{2k-2}$ by Corollary 2.2. Therefore, since the basis $\{\zeta_\Lambda\}$ of $V(\lambda)$ is consistent with the embeddings $\mathfrak{sp}_{2k-2}\subset\mathfrak{sp}_{2k}$ [10], the vector ζ_{Λ_0} is an eigenvector for each of these minors. The corresponding eigenvalue for the first minor can be calculated from Proposition 4.7, which gives

$$\alpha_{-k+1}(u)\, f^{\,-k+1\cdots k}_{\,-k+1\cdots k}(u+k-1)\,\zeta_{\Lambda_0} \\ = \prod_{i=2}^{k} \frac{u-\min\{\lambda_{k,i-1},\lambda_{k-1,i-1}\}+i-1/2}{u+i-1/2}\cdot \prod_{i=1}^{k} \frac{u+\max\{\lambda_{ki},\lambda_{k-1,i}\}-i+1/2}{u-i+1/2}\,\zeta_{\Lambda_0}, \tag{4.20}$$

where the λ_{ki} are the entries of Λ_0. The second minor coincides with the image of the Sklyanin determinant for $\mathrm{Y}(\mathfrak{sp}_{2k-2})$ under the evaluation homomorphism (4.2). Hence the corresponding eigenvalue can be found from (4.7) and (4.8) which gives

$$\alpha_{-k+1}(u)\, f^{\,-k+1\cdots k-1}_{\,-k+1\cdots k-1}(u+k-1)\,\zeta_{\Lambda_0} \\ = \prod_{i=2}^{k} \frac{u-\lambda_{k-1,i-1}+i-1/2}{u+i-1/2}\cdot \prod_{i=1}^{k-1} \frac{u+\lambda_{k-1,i}-i+1/2}{u-i+1/2}\,\zeta_{\Lambda_0};$$

see also [8]. Note that since $f_{\mathcal{B}}(u) = \alpha_{-m}(u)\, f^{\,-m\cdots m}_{\,-m\cdots m}(u+m)$, the last formula with $k=m+1$ also applies for the calculation of $f_{\mathcal{B}}(u)\,\zeta_{\Lambda_0}$. Hence,

$$f_{\mathcal{B}}(-u-1)\,\zeta_{\Lambda_0} = \nu(u)\,\zeta_{\Lambda_0} \tag{4.21}$$

with $\nu(u)$ defined in (4.14). Furthermore, the formulas (4.10) for the entries of Λ_0 imply that

$$\max\{\lambda_{ki},\lambda_{k-1,i}\} = \mathrm{mid}\{\lambda_i,\mu_{i+k-m-1},\mu_{i+m-k}\} \tag{4.22}$$

and

$$\min\{\lambda_{ki},\lambda_{k-1,i}\} = \mathrm{mid}\{\lambda_i,\mu_{i+k-m},\mu_{i+m-k+1}\}. \tag{4.23}$$

Using (4.19) we obtain the following expression for $\mu_k(u)$:

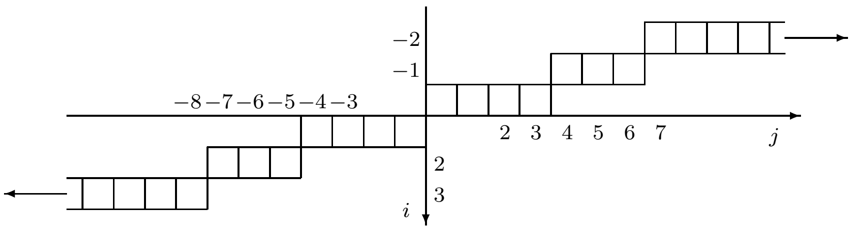
$$\mu_k(u) = \nu(u) \prod_{i=2}^{k} \frac{u - \mathrm{mid}\{\lambda_{i-1}, \mu_{i+k-m-1}, \mu_{i+m-k}\} + k - m + i - 3/2}{u - \mathrm{mid}\{\lambda_{i-1}, \mu_{i+k-m-2}, \mu_{i+m-k}\} + k - m + i - 3/2}$$
$$\times \prod_{i=1}^{k-1} \frac{u + \mathrm{mid}\{\lambda_i, \mu_{i+k-m-1}, \mu_{i+m-k}\} + k - m - i - 1/2}{u + \mathrm{mid}\{\lambda_i, \mu_{i+k-m-1}, \mu_{i+m-k+1}\} + k - m - i - 1/2}$$
$$\times \frac{u + \min\{\lambda_k, \mu_m\} - m - 1/2}{u - m - 1/2}.$$

Finally, replace the index i in the first product by $i+1$ and note that if $\lambda_i \geqslant \mu_{i+k-m-1}$ then the corresponding factor equals 1. Similarly, if $\lambda_i \leqslant \mu_{i+m-k+1}$ then the corresponding factor in the second product equals 1. This brings the expression for $\mu_k(u)$ to the required form. $\square$

We can now compute the Drinfeld polynomials for the $\mathrm{Y}(\mathfrak{sp}_{N-M})$-module $V(\lambda)^+_\mu$ by using Theorem 4.8. Given any $\mathfrak{sp}_{2n}$-highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$, set $\lambda_{-i} = -\lambda_i$ for $i = 1, \dots, n$. We also assume that $\lambda_0 = 0$ while $\lambda_k = -\infty$ and $\lambda_{-k} = +\infty$ for $k > n$. Introduce the *diagram* $\Gamma(\lambda)$ as a certain infinite set of unit squares (cells) on the plane whose centers have integer coordinates. The coordinates (i, j) of a cell are interpreted as the row and column number so that i increases from top to bottom and j increases from left to the right. With these assumptions[2],

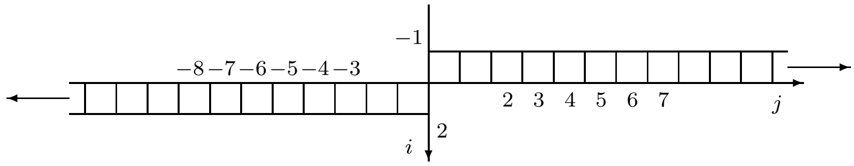
$$\Gamma(\lambda) = \{(i,j) \in \mathbb{Z}^2 \mid -n \leqslant i \leqslant n+1, \quad \lambda_i \leqslant j < \lambda_{i-1}\}.$$

The diagram has a central symmetry, as illustrated below for $\lambda = (-4, -7)$ and $n = 2$:



Note that in the case $n = 0$ the definition of the diagram formally makes sense with λ considered to be empty. Thus, $\Gamma(\emptyset)$ consists of two infinite rows of cells:

[2]This definition of $\Gamma(\lambda)$ corresponds to the one outlined in the Introduction for the partition $(\lambda_{-n}, \dots, \lambda_{-1})$.



By the *content* of a cell $\alpha = (i, j)$ with coordinates i and j we shall mean the number $c(\alpha) = j - i$. For any nonnegative integer p we shall denote by $\Gamma(\lambda)^{(p)}$ the image of $\Gamma(\lambda)$ with respect to the shift operator $(i, j) \mapsto (i - p, j)$. In other words, $\Gamma(\lambda)^{(p)}$ is obtained from the diagram $\Gamma(\lambda)$ by lifting each cell p units up.

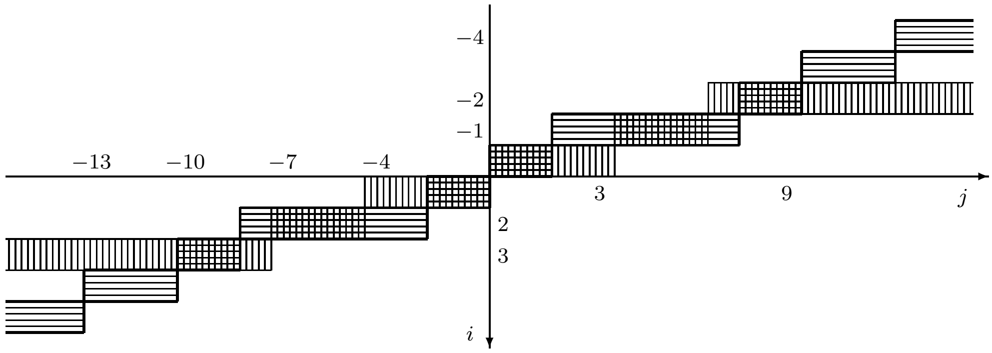
We let $P_1(u), \dots, P_{n-m}(u)$ denote the Drinfeld polynomials corresponding to the $\mathrm{Y}(\mathfrak{sp}_{N-M})$-module $V(\lambda)^+_\mu$.

Theorem 4.9. *For each $k = 1, \dots, n - m$ the Drinfeld polynomial $P_k(u)$ is given by*

$$P_k(u) = \prod_\alpha (u + c(\alpha) + 1/2),$$

where α runs over the cells of the intersection $\Gamma(\mu) \cap \Gamma(\lambda)^{(k-1)}$.

Example 4.10. Let $\lambda = (-2, -8, -10, -13)$ and $\mu = (-4, -7)$ so that $n = 4$ and $m = 2$. (In a more standard notation, λ and μ can be thought of as the partitions $(13, 10, 8, 2)$ and $(7, 4)$, respectively). The polynomial $P_1(u)$ is calculated from the figure:

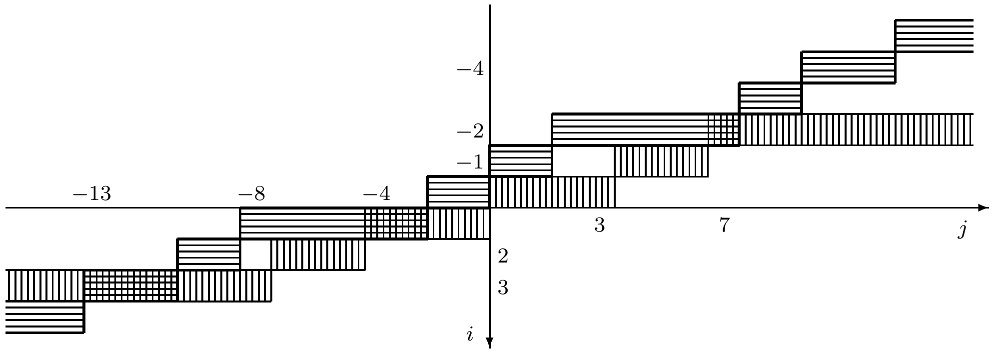


The horizontal and vertical shadings indicate the diagrams $\Gamma(\lambda)$ and $\Gamma(\mu)$, respectively. The cells belonging to the intersection $\Gamma(\mu) \cap \Gamma(\lambda)$ have the coordinates $(3, -10)$, $(3, -9)$, $(2, -7)$, $(2, -6)$, $(2, -5)$, $(1, -2)$, $(1, -1)$, $(0, 0)$, $(0, 1)$, $(-1, 4)$, $(-1, 5)$, $(-1, 6)$, $(-2, 8)$, $(-2, 9)$. Hence,

$$\begin{aligned} P_1(u) = {} & (u - 25/2)(u - 23/2)(u - 17/2)(u - 15/2)(u - 13/2)(u - 5/2)(u - 3/2) \\ & (u + 1/2)(u + 3/2)(u + 11/2)(u + 13/2)(u + 15/2)(u + 21/2)(u + 23/2). \end{aligned}$$

Note that the property $P_1(u) = P_1(-u+1)$ is implied by the central symmetry of the set $\Gamma(\mu) \cap \Gamma(\lambda)$.

The polynomial $P_2(u)$ is calculated from the figure:



The cells which belong to the intersection $\Gamma(\mu) \cap \Gamma(\lambda)^{(1)}$ have the coordinates $(3,-13)$, $(3,-12)$, $(3,-11)$, $(1,-4)$, $(1,-3)$, $(-2,7)$. Hence,

$$P_2(u) = (u - 31/2)(u - 29/2)(u - 27/2)(u - 9/2)(u - 7/2)(u + 19/2).$$

Example 4.11. In the case $m = 0$ the vector space $V(\lambda)^+_\mu$ can be identified with $V(\lambda)$ and the corresponding $\mathrm{Y}(\mathfrak{sp}_{2n})$-module coincides with the evaluation module defined by the evaluation homomorphism (4.2). Applying Theorem 4.9 to the diagrams $\Gamma(\lambda)$ and $\Gamma(\emptyset)$ we obtain

$$\begin{aligned} P_1(u) = (u + \lambda_1 - 1/2)(u + \lambda_1 + 1/2) \cdots (u - 3/2) \\ \times (u + 1/2)(u + 3/2) \cdots (u - \lambda_1 - 1/2) \end{aligned}$$

and

$$P_k(u) = (u + \lambda_k - 1/2)(u + \lambda_k + 1/2) \cdots (u + \lambda_{k-1} - 3/2), \qquad k = 2, \dots, n.$$

On the other hand, the highest weight of this module is given by (4.8). Due to (4.5) and (4.6) this obviously agrees with the above calculation of the $P_k(u)$.

Proof of Theorem 4.9. We shall derive the statement from Theorem 4.8 and the definition of the Drinfeld polynomials (4.5) and (4.6). In order to calculate $P_1(u)$ observe that the component $\mu_{m+1}(u)$ of the highest weight can also be found from the formula

$$\varphi_{m+1,m+1}(u)\, \zeta_{\Lambda_0} = \mu_{m+1}(u)\, \zeta_{\Lambda_0};$$

see (4.15). Hence, applying (4.20) with $k = m+1$ and using (4.22) and (4.23) we get

$$\mu_{m+1}(u) = \prod_{i=1}^{m+1} \frac{u - \operatorname{mid}\{\lambda_{i-1}, \mu_{i-1}, \mu_i\} + i - 1/2}{u + i - 1/2} \times \prod_{i=1}^{m+1} \frac{u + \operatorname{mid}\{\lambda_i, \mu_{i-1}, \mu_i\} - i + 1/2}{u - i + 1/2}.$$

Therefore, by (4.6),

$$\frac{P_1(u+1)}{P_1(u)} = \prod_{i=1}^{m+1} \frac{u + \operatorname{mid}\{\lambda_{i-1}, \mu_{i-1}, \mu_i\} - i + 1/2}{u + \operatorname{mid}\{\lambda_i, \mu_{i-1}, \mu_i\} - i + 1/2} \times \prod_{i=1}^{m+1} \frac{u - \operatorname{mid}\{\lambda_i, \mu_{i-1}, \mu_i\} + i - 1/2}{u - \operatorname{mid}\{\lambda_{i-1}, \mu_{i-1}, \mu_i\} + i - 1/2}.$$

This gives $P_1(u) = Q(u)\, Q(-u+1)\, (-1)^{\deg Q}$, where

$$Q(u) = \prod_{i=1}^{m+1} (u + \beta_i)(u + \beta_i + 1) \cdots (u + \alpha_i - 1) \tag{4.24}$$

with

$$\alpha_i = \operatorname{mid}\{\lambda_{i-1}, \mu_{i-1}, \mu_i\} - i + 1/2 \qquad \text{and} \qquad \beta_i = \operatorname{mid}\{\lambda_i, \mu_{i-1}, \mu_i\} - i + 1/2.$$

Thus, the factor corresponding to the index i in the product in (4.24) can be interpreted as the product $\prod_\alpha (u + c(\alpha) + 1/2)$, where α runs over the cells of the intersection of the diagrams $\Gamma(\lambda) \cap \Gamma(\mu)$ whose first coordinate is i. Taking into account the central symmetry of $\Gamma(\lambda) \cap \Gamma(\mu)$, we derive the desired formula for $P_1(u)$.

Now using (4.5) and replacing the index k by $r = k - m$ in Theorem 4.8 we derive the following expression for the Drinfeld polynomial $P_{r+1}(u)$ with $1 \leqslant r < n - m$:

$$\begin{aligned} \frac{P_{r+1}(u+1)}{P_{r+1}(u)} = & \prod_{\substack{i=1 \\ \lambda_i < \mu_{i+r}}}^{r+m-2} \frac{u - \max\{\lambda_{i+1}, \mu_{i+r+1}\} + r + i + 1/2}{u - \max\{\lambda_i, \mu_{i+r+1}\} + r + i + 1/2} \\ & \times \prod_{\substack{i=1 \\ \lambda_i < \mu_{i+r-1} \leqslant \lambda_{i-1}}}^{r+m-1} \frac{u - \max\{\lambda_i, \mu_{i+r}\} + r + i - 1/2}{u - \mu_{i+r-1} + r + i - 1/2} \\ & \times \prod_{\substack{i=2 \\ \lambda_i > \mu_{i-r}}}^{r+m+1} \frac{u + \min\{\lambda_{i-1}, \mu_{i-r-1}\} + r - i + 1/2}{u + \min\{\lambda_i, \mu_{i-r-1}\} + r - i + 1/2} \\ & \times \prod_{\substack{i=1 \\ \lambda_{i+1} \leqslant \mu_{i-r+1} < \lambda_i}}^{r+m-1} \frac{u + \min\{\lambda_i, \mu_{i-r}\} + r - i - 1/2}{u + \mu_{i-r+1} + r - i + 1/2}. \end{aligned} \tag{4.25}$$

Note that the expression has the form

$$\frac{P_{r+1}(u+1)}{P_{r+1}(u)}=\prod_j\frac{u+\alpha_j}{u+\beta_j}$$

for some parameters α_j and β_j with $\alpha_j-\beta_j\in\mathbb{Z}_+$. This implies that $P_{r+1}(u)$ is given by

$$P_{r+1}(u)=\prod_j(u+\beta_j)(u+\beta_j+1)\cdots(u+\alpha_j-1). \tag{4.26}$$

It is straightforward to verify that this product coincides with $\prod_\alpha(u+c(\alpha)+1/2)$, where α runs over the cells of the intersection $\Gamma(\mu)\cap\Gamma(\lambda)^{(r-1)}$. Indeed, changing the product index i respectively by $i-r$ and by $i-r+1$ in the third and fourth products in (4.25) we bring these products to the form

$$\prod_{\substack{i=1\\ \lambda_{i+r}>\mu_i}}^{m+1}\frac{u+\min\{\lambda_{i+r-1},\mu_{i-1}\}-i+1/2}{u+\min\{\lambda_{i+r-1},\mu_{i-1}\}-i+1/2}$$
$$\times\prod_{\substack{i=1\\ \lambda_{i+r}\leqslant\mu_i<\lambda_{i+r-1}}}^{m}\frac{u+\min\{\lambda_{i+r-1},\mu_{i-1}\}-i+1/2}{u+\mu_i-i+1/2}. \tag{4.27}$$

Therefore, the factor corresponding to the index i contributes into (4.26) the product $\prod_\alpha(u+c(\alpha)+1/2)$, where α runs over the cells of the intersection $\Gamma(\mu)\cap\Gamma(\lambda)^{(r-1)}$ whose first coordinate is i. Note that writing

$$-\max\{\lambda_i,\mu_{i+r+1}\}=\min\{\lambda_{-i},\mu_{-i-r-1}\}$$

we can similarly bring the first and second products in (4.25) to the form (4.27), where the product is taken over negative indices. This contributes $\prod_\alpha(u+c(\alpha)+1/2)$ into (4.26), where α runs over the cells of the intersection $\Gamma(\mu)\cap\Gamma(\lambda)^{(r-1)}$ whose first coordinate is non-positive. ◻

5 Centralizer construction

We now return to our general situation so that $\mathfrak{g}_N$ denotes either the orthogonal or symplectic Lie algebra; see (4.1). We start by recalling the construction of the *Olshanski algebra* A_M; see [18, 14]. Fix a nonnegative integer M such that $N-M$ is even. So, if $N=2n$ or $N=2n+1$ then $M=2m$ or $M=2m+1$, respectively, for some $m\leqslant n$. Denote by $\mathfrak{g}'_{N-M}$ the subalgebra of $\mathfrak{g}_N$ spanned by the elements F_{ij} subject to the condition $m+1\leq|i|,|j|\leq n$. Let $\mathrm{A}_M(N)$ denote the centralizer of

$\mathfrak{g}'_{N-M}$ in the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$. Let $\mathrm{U}(\mathfrak{g}_N)^0$ denote the centralizer of F_{nn} in $\mathrm{U}(\mathfrak{g}_N)$ and let $\mathrm{I}(N)$ be the left ideal in $\mathrm{U}(\mathfrak{g}_N)$ generated by the elements F_{in}, $i=-n,\dots,n$. Then $\mathrm{I}(N)^0=\mathrm{I}(N)\cap\mathrm{U}(\mathfrak{g}_N)^0$ is a two-sided ideal in $\mathrm{U}(\mathfrak{g}_N)^0$ which coincides with the intersection $\mathrm{J}(N)\cap\mathrm{U}(\mathfrak{g}_N)^0$, where $\mathrm{J}(N)$ is the right ideal in $\mathrm{U}(\mathfrak{g}_N)$ generated by the elements F_{ni}, $i=-n,\dots,n$. One has a vector space decomposition

$$\mathrm{U}(\mathfrak{g}_N)^0=\mathrm{I}(N)^0\oplus\mathrm{U}(\mathfrak{g}_{N-2}).$$

Therefore the projection of $\mathrm{U}(\mathfrak{g}_N)^0$ onto $\mathrm{U}(\mathfrak{g}_{N-2})$ with the kernel $\mathrm{I}(N)^0$ is an algebra homomorphism. Its restriction to the subalgebra $\mathrm{A}_M(N)$ defines a filtration-preserving homomorphism

$$\pi_N:\mathrm{A}_M(N)\to\mathrm{A}_M(N-2) \tag{5.1}$$

so that one can define the algebra A_M as the projective limit with respect to this sequence of homomorphisms in the category of filtered algebras; see [14] for more details.

Taking the composition of the homomorphism $\mathrm{Y}(\mathfrak{g}_M)\to\mathrm{Y}(\mathfrak{g}_N)$ defined in Proposition 3.2 and the evaluation homomorphism (4.2) we obtain another homomorphism $\psi_N:\mathrm{Y}(\mathfrak{g}_M)\to\mathrm{U}(\mathfrak{g}_N)$ which takes $s_{ij}(u)$ to the series

$$\alpha_{m-n}(u)\cdot f^{-n\cdots -m-1,\,i,\,m+1\cdots n}_{-n\cdots -m-1,\,j,\,m+1\cdots n}(u+n-m),\qquad -m\leqslant i,j\leqslant m, \tag{5.2}$$

where we have used the notation of Section 4 for the images of the Sklyanin minors with respect to (4.2). Observe now that by Corollary 2.2 the image of ψ_N is contained in the centralizer $\mathrm{A}_M(N)$, so that $\psi_N:\mathrm{Y}(\mathfrak{g}_M)\to\mathrm{A}_M(N)$.

Proposition 5.1. *The sequence of homomorphisms* $(\psi_N|\ N=M+2k, k=0,1,\dots)$ *defines a homomorphism*

$$\psi:\mathrm{Y}(\mathfrak{g}_M)\to\mathrm{A}_M.$$

Proof. We have to verify that the homomorphisms ψ_N are compatible with the sequence of homomorphisms (5.1), that is, the following diagram is commutative:

$$\begin{array}{ccccccccc}
\mathrm{Y}(\mathfrak{g}_M) & = & \mathrm{Y}(\mathfrak{g}_M) & = & \cdots & = & \mathrm{Y}(\mathfrak{g}_M) & = & \cdots\\
{\scriptstyle\psi_M}\downarrow & & {\scriptstyle\psi_{M+2}}\downarrow & & & & {\scriptstyle\psi_N}\downarrow & & \\
\mathrm{A}_M(M) & \underset{\pi_{M+2}}{\longleftarrow} & \mathrm{A}_M(M+2) & \longleftarrow & \cdots & \underset{\pi_N}{\longleftarrow} & \mathrm{A}_M(N) & \longleftarrow & \cdots .
\end{array}$$

Let us calculate the image of the series $\psi_N(s_{ij}(u))$ under the homomorphism π_N. Applying (2.13) we obtain

$$f^{-n\cdots -m-1,\,i,\,m+1\cdots n}_{-n\cdots -m-1,\,j,\,m+1\cdots n}(u+n-m)=\sum_c \check f^{-n\cdots -m-1,\,i,\,m+1\cdots n}_{-n\cdots -m-1,\,j,\,m+1\cdots n-1,c}(u+n-m)\,f_{cn}(u-n+m),$$

where by $\check{f}^{\,a_1\cdots a_k}_{\,b_1\cdots b_{k-1},c}(u)$ we denote the image of the auxiliary minor $\check{s}^{\,a_1\cdots a_k}_{\,b_1\cdots b_{k-1},c}(u)$ under the evaluation homomorphism (4.2). Now observe that $f_{cn}(u-n+m)$ belongs to the left ideal $\mathrm{I}(N)$ unless $c=n$. In this case $f_{nn}(u-n+m)\equiv 1 \mod \mathrm{I}(N)$. Furthermore, by (3.10),

$$\check{f}^{-n\cdots-m-1,\,i,\,m+1\cdots n}_{-n\cdots-m-1,\,j,\,m+1\cdots n}(v)$$
$$=\frac{2v+1}{2v\pm 1}\sum_{i=1}^{2n-2m}(-1)^{i-1}\,f^{\,t}_{a_i,-n}(-v)\,f^{\;a_1\cdots\widehat{a}_i\cdots a_{2n-2m}}_{\;-n+1\cdots-m-1,\,j,\,m+1\cdots n-1}(v-1),$$

where we have set $(a_1,\dots,a_{2n-2m})=(-n\,\cdots\,-m-1,\,i,\,m+1\cdots n-1)$ and $v=u+n-m$. Now $f^{\,t}_{a_i,-n}(-v)=\theta_{-n,a_i}\,f_{n,-a_i}(-v)$ belongs to the right ideal $\mathrm{J}(N)$ unless $a_i=-n$, that is, $i=1$. In this case $f_{nn}(-v)\equiv 1 \mod \mathrm{J}(N)$. Note that

$$\alpha_{m-n}(u)\,\frac{2v+1}{2v\pm 1}=\alpha_{m-n+1}(u).$$

Hence, π_N takes $\psi_N(s_{ij}(u))$ to the series

$$\alpha_{m-n+1}(u)\cdot f^{-n+1\cdots-m-1,\,i,\,m+1\cdots n-1}_{-n+1\cdots-m-1,\,j,\,m+1\cdots n-1}(u+n-m-1)$$

which coincides with $\psi_{N-2}(s_{ij}(u))$. Thus, the sequence of coefficients at each power of u^{-1} in $\psi_N(s_{ij}(u))$ with $N=M+2k$, $k=0,1,\dots$ defines an element of A_M. $\square$

Corollary 2.2 implies that all the coefficients of the series

$$\alpha_{m-n}(u)\cdot f^{-n\cdots-m-1,\,m+1\cdots n}_{-n\cdots-m-1,\,m+1\cdots n}(u+n-m)=1+c^{(n)}_1u^{-1}+c^{(n)}_2u^{-2}+\cdots \tag{5.3}$$

belong to the center of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}'_{N-M})$. Hence, for any i the coefficient $c^{(n)}_i$ is an element of the centralizer $\mathrm{A}_M(N)$. The argument of the proof of Proposition 5.1 shows that the image of the series (5.3) under the homomorphism π_N is

$$\alpha_{m-n+1}(u)\cdot f^{-n+1\cdots-m-1,\,m+1\cdots n-1}_{-n+1\cdots-m-1,\,m+1\cdots n-1}(u+n-m-1).$$

Therefore, for each i the sequence $c_i=(c^{(n)}_i\mid n\geqslant m+1)$ determines an element of the projective limit algebra A_M. Denote by C_M the subalgebra of A_M generated by all c_i with $i\geq 1$. The subalgebra C_M is studied in detail in [14, Section 3] where it was identified with the algebra of *virtual Laplace operators*. Up to an obvious change of notation, the series (5.3) coincides with the Sklyanin minor $f_{\mathcal{B}}(u)$ introduced in the proof of Theorem 4.8. So its Harish-Chandra image can be found from (4.21) which shows that the elements c_{2i} with even indices are algebraically independent and generate the algebra C_M; cf. [14, Section 3].

Theorem 5.2. *The homomorphism $\psi : \mathrm{Y}(\mathfrak{g}_M) \hookrightarrow \mathrm{A}_M$ is injective. Moreover, one has an isomorphism*

$$\mathrm{A}_M = \mathrm{C}_M \otimes \mathrm{Y}(\mathfrak{g}_M),$$

where $\mathrm{Y}(\mathfrak{g}_M)$ is identified with its image under the embedding ψ.

Proof. Our argument is similar to the proof of [14, Theorem 4.17]. Consider the canonical filtration of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_N)$. The corresponding graded algebra $\mathrm{gr}\,\mathrm{U}(\mathfrak{g}_N)$ is isomorphic to the symmetric algebra $\mathrm{S}(\mathfrak{g}_N)$ of the space $\mathfrak{g}_N$. Elements of $\mathrm{S}(\mathfrak{g}_N)$ can be naturally identified with polynomials in matrix elements of an $N \times N$ matrix $x = (x_{ij})_{i,j=-n}^{n}$ such that $x^t = -x$. Denote by $\mathrm{P}_M(N)$ the subalgebra of the elements of $\mathrm{S}(\mathfrak{g}_N)$ which are invariant under the adjoint action of the Lie algebra $\mathfrak{g}'_{N-M}$. The algebra A_M possesses a natural filtration induced by the canonical filtrations on the centralizers $\mathrm{A}_M(N)$. The corresponding graded algebra $\mathrm{gr}\,\mathrm{A}_M$ is naturally isomorphic to the projective limit P_M of the commutative algebras $\mathrm{P}_M(N)$ with respect to homomorphisms $\mathrm{P}_M(N) \to \mathrm{P}_M(N-2)$ analogous to (5.1); see [14, Section 4]. The images in $\mathrm{P}_M(N)$ of the coefficients the series (5.2) and (5.3) can be found from the explicit formulas for the Sklyanin minors; see Proposition 3.6. Indeed, apply the proposition to the Sklyanin minor $s\,{}^{-n\cdots -m-1,\, m+1\cdots n}_{-n\cdots -m-1,\, m+1\cdots n}(u+n-m)$ then replace each series $s_{ij}(u)$ by its image $f_{ij}(u)$ under the evaluation homomorphism (4.2). Observe that the image of $s^t_{ij}(-u)$ coincides with $f_{ij}(u \mp 1)$. Since we are only interested in the highest degree component of the coefficient at each power of u^{-1}, we may replace each expression of type $f_{ij}(u+c)$ by $\delta_{ij} + F_{ij}u^{-1}$. Hence, denoting the elements of the set $\mathcal{A} = \{-n, \dots, -m-1, m+1, \dots, n\}$ by $a_1, \dots, a_{N-M}$ we can write, modulo lower degree terms at each power of u,

$$\begin{aligned} f\,{}^{-a_1\cdots -a_{N-M}}_{a_1\cdots a_{N-M}}(u+n-m) &\equiv \alpha_{n-m}(u+n-m) \\ &\times \sum_{p\in\mathfrak{S}_M} \mathrm{sgn}\, pp' \cdot (1+Fu^{-1})_{-a_{p(1)},a_{p'(1)}} \cdots (1+Fu^{-1})_{-a_{p(N-M)},a_{p'(N-M)}}, \end{aligned}$$

where F denotes the $(N-M)\times(N-M)$ matrix whose rows and columns are enumerated by the elements of the set $\mathcal{A}$ and whose ij-th entry is F_{ij}. Since these matrix elements commute modulo lower degree terms, taking into account the relation

$$\alpha_{m-n}(u)\,\alpha_{n-m}(u+n-m) = 1,$$

we can conclude from Lemma 3.8 that the image of the series (5.3) in $\mathrm{P}_M(N)$ is the determinant $\det(1+xu^{-1})_{\mathcal{A}\mathcal{A}}$. The same argument shows that the image of the series (5.2) in $\mathrm{P}_M(N)$ is $\det(1+xu^{-1})_{\mathcal{A}_i\mathcal{A}_j}$, where

$$\mathcal{A}_i = \{-n, \dots, -m-1, i, m+1, \dots, n\}.$$

Our next step is to show that every element ϕ of the algebra $\mathrm{P}_M(N)$ such that $\deg\phi < n - m$ can be represented as a polynomial in the coefficients of the series $\det(1+xu^{-1})_{\mathcal{A}\mathcal{A}}$ and $\det(1+xu^{-1})_{\mathcal{A}_i\mathcal{A}_j}$. However, it was proved in [14, Section 4.9] that ϕ can be presented as a polynomial in the elements $\mathrm{tr}\,(x_{\mathcal{A}\mathcal{A}})^k$ and $\Lambda^{(k)}_{ij}$ with $-m \leqslant i, j \leqslant m$ and $k \geqslant 1$, where $\Lambda^{(k)}_{ij} = \sum x_{ic_1}x_{c_1c_2}\cdots x_{c_{k-1}j}$, summed over the indices $c_r \in \mathcal{A}$. On the other hand, each element $\mathrm{tr}\,(x_{\mathcal{A}\mathcal{A}})^k$ is a polynomial in the coefficients of the series $\det(1 + xu^{-1})_{\mathcal{A}\mathcal{A}}$. This follows from the fact that the coefficients of the characteristic polynomial $\det(u+x)_{\mathcal{A}\mathcal{A}}$ form a complete set of invariants of the matrix $x_{\mathcal{A}\mathcal{A}}$. (Explicit expressions for the elements $\mathrm{tr}\,(x_{\mathcal{A}\mathcal{A}})^k$ in terms of the coefficients of the characteristic polynomial can be derived e.g. from the Liouville formula; cf. [13, Remark 5.8]). The claim now follows from the well known identity

$$\det(1 + xu^{-1})_{\mathcal{A}_i\mathcal{A}_j} = \det(1 + xu^{-1})_{\mathcal{A}\mathcal{A}} \cdot \Big(\delta_{ij} + \sum_{k=1}^{\infty}(-1)^{k-1}\Lambda^{(k)}_{ij}u^{-k}\Big); \qquad (5.4)$$

see e.g. [5, 6]. Indeed, the identity implies that the elements $\Lambda^{(k)}_{ij}$ are polynomials in the coefficients of the series $\det(1 + xu^{-1})_{\mathcal{A}_i\mathcal{A}_j}$ and $\det(1 + xu^{-1})_{\mathcal{A}\mathcal{A}}$. This allows us to conclude that the algebra A_M is generated by the subalgebra C_M and the image of the homomorphism ψ.

Observe that since the matrix x satisfies $x^t = -x$ we have the relations

$$\det(1 + xu^{-1})_{\mathcal{A}\mathcal{A}} = \det(1 - xu^{-1})_{\mathcal{A}\mathcal{A}}$$

and

$$\Lambda^{(k)}_{ij} = (-1)^k\,\theta_{ij}\,\Lambda^{(k)}_{-j,-i}.$$

Hence, we can write

$$\det(1 + xu^{-1})_{\mathcal{A}\mathcal{A}} = 1 + \sum_{r=1}^{\infty}\Lambda^{(2r)}\,u^{-2r}$$

for some polynomials $\Lambda^{(2r)}$ in the matrix elements of x. For the elements $\Lambda^{(k)}_{ij}$ we shall impose the following restrictions on i, j, k:

$$i + j < 0 \quad \text{for } k \text{ odd}, \qquad i + j \le 0 \quad \text{for } k \text{ even}$$

in the orthogonal case, and

$$i + j < 0 \quad \text{for } k \text{ even}, \qquad i + j \le 0 \quad \text{for } k \text{ odd}$$

in the symplectic case. Fix a positive integer K and assume that the index k satisfies $1 \leqslant k \leqslant K$. It follows from the argument of [14, Section 4.10] that there exists a large enough value of N such that the polynomials $\Lambda^{(k)}$ for even k, and $\Lambda_{ij}^{(k)}$ with the above restrictions on i, j, k are algebraically independent. Due to the identity (5.4), the same statement will hold if each polynomial $\Lambda_{ij}^{(k)}$ is replaced by the coefficient at u^{-k} of the series $\det(1 + xu^{-1})_{\mathcal{A}_i\mathcal{A}_j}$. This proves the injectivity of ψ and the tensor product decomposition for the algebra A_M. $\square$

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