Stability of Levitrons

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July 22, 1998

Abstract

The Levitron is a magnetic spinning top which can levitate in the constant field of a repelling base magnet. An explanation for the stability of the Levitron using an adiabatic approximation has been given by M.V. Berry. In experiments the top eventually loses stability at a critical spin rate which can not be predicted by Berry's approach. The present work develops an exact theory of the Levitron with six degrees of freedom which allows for the calculation of the critical spin rate. The main result is a complete classification of possible Levitrons that allow for an interval of stable spin rates. Stability of the relative equilibrium is lost in Hamiltonian Hopf bifurcations if either the spin rate is too large or too small.

1 Introduction

The Levitron is a toy consisting of a magnetic top which can levitate spinning in the air over a ceramic magnetic base. It was invented by Roy Harrigan, for the history of its invention see [6]. A theory for the motion of the top has been developed by M. V. Berry [2]. Berry uses the method of averaging to predict the motion of the center of mass of the top in an averaged force field. He assumes that both the spin of the top and the precession of its axis are fast relative to the motion of its center of mass. A similar approximate treatment has been done in [6]. This paper augments these investigations by studying the exact model of the full twelve dimensional problem. We are able to predict the interval of spin rates for which the top is stable. Linear stability of the relative equilibrium is lost when the top spins too slow or too fast.

In Sec. 2 we treat the kinematics of the Levitron. We take the traditional approach [1,3] of deriving the Lagrangian for the Levitron as the difference of kinetic (Sec. 3) and potential (Sec. 4) energy. The introduction of local coordinates is delayed as long as possible and we treat the symmetries of body and space in a coordinate free approach in Sec. 5. We show that the Levitron has a two dimensional torus symmetry group, hence there are two conserved quantities, the angular momentum in space and the angular momentum in the body. However, the orbits of the symmetry group are not always two-tori, but degenerate into circles for the periodic orbit of the top that we want to analyze in detail: the top with center of mass at rest on the z-axis, and spinning with its axis aligned with the z-axis. Technically speaking we encounter the case of singular reduction. Therefore we introduce local coordinates (Sec. 6) that reduce only one symmetry and are nonsingular in the neighborhood of the periodic orbit. Using Hamilton's equations (Sec. 7) we show that there is a four dimensional invariant set (Sec. 8) in phase space and that it contains the periodic orbit as a relative equilibrium (Sec. 9). The invariant set induces a block decomposition of the linearization (Sec. 10) about our periodic orbit. The 8×8 system can be written in complex form as a 2×2 second order equation. This leads to our principle result: a fourth degree polynomial depending on 3 dimensionless quantities whose roots determine linear stability. The remaining part of the paper deals with the analyses of the parameter dependence of the roots of this polynomial. The stable region in coefficient space is determined and the bifurcations (Sec. 11) corresponding to its boundaries are shown to be Hamiltonian Hopf bifurcations (also called Krein collisions or complex bifurcations). The coefficient space is decomposed into rays which correspond to changing the spin rate and a transverse two dimensional Levitron space (Sec. 12) from which the stability behavior of a Levitron can be determined. The Levitron space captures the essential parameters of any given Levitron; besides the geometry of the top it only depends on the derivatives of the magnetic field of the base on the z-axis. In the final section we calculate the critical spin rates and find good agreement with the experimental values.

2 Kinematics

It is useful to think of two pictures of the top. In the first picture the top is stationary, the center of mass of the top is located at the origin and its symmetry axis is aligned with the vertical axis. We refer to this picture as the body coordinate system. In the second picture the top is moving in a space fixed coordinate system, the laboratory coordinate system. To produce a motion of the top in space, each point in the top is rotated by a rigid rotation R and then translated by adding a vector r. Thus a point Q in the body coordinate system is transformed to a point q in laboratory coordinates. The group of all orientation- and distance-preserving transformations of $\mathbb{R}^3$ is the *configuration space* of the Levitron. It is well known that this space is $\mathbb{R}^3 \times SO(3)$ where $SO(3)$ is the Lie group consisting of all 3×3 orthogonal matrices having determinant equal to 1. A point in $\mathbb{R}^3 \times SO(3)$ is denoted by the pair (r, R).

A *motion* of the top is determined by a curve $\gamma(t) = (r(t), R(t))$ in configuration space. A point Q in the body traces the curve $q(t) = R(t)Q + r(t)$ when viewed in the laboratory coordinate system. The phase space of the Levitron is the (co)tangent bundle of the six dimensional configuration space.

3 Kinetic Energy

The kinetic energy of the top is the sum of the translational kinetic energy due to the motion of its center of mass and the kinetic energy associated with the rotation of the top. In the body coordinate system, the mass density of the top at a point Q is assumed to be an integrable function denoted by $m(Q)$. We assume that the center of mass of the top is at the origin, and its total mass is m. Thus $\int m(Q)QdQ = 0$, and $\int m(Q)dQ = m$.

Now suppose a curve $\gamma(t) = (r(t), R(t))$ determines a motion of the top in space. A point Q in the body is transformed to a moving point $q(t) = R(t)Q + r(t)$. The kinetic energy density associated with this moving point is

$$dT(Q) = \frac{1}{2} m(Q) \langle \dot{q}, \dot{q} \rangle$$

with $\dot{q} = \dot{R}(t)Q + \dot{r}(t)$. The kinetic energy is $T = \int dT(Q)dQ$. The kinetic energy density breaks up into three terms,

$$\begin{aligned} dT_1 &= \frac{1}{2} m(Q) \langle \dot{r}, \dot{r} \rangle, \\ dT_2 &= m(Q) \langle \dot{r}, \dot{R}Q \rangle, \\ dT_3 &= \frac{1}{2} m(Q) \langle \dot{R}Q, \dot{R}Q \rangle. \end{aligned}$$

Integrating the first term gives $T_1 = m\langle \dot{r}, \dot{r} \rangle / 2$, the contribution from the motion of the center of mass. Integrating the second term gives

$$T_2 = \int \langle \dot{r}, \dot{R}Q \rangle m(Q)dQ = \left\langle \dot{r}, \dot{R} \int m(Q)QdQ \right\rangle = 0.$$

The term T_2 is zero because the center of mass of the top in the body coordinate system is at the origin. To integrate the third term we make use of certain properties of $SO(3)$ and its Lie algebra $\mathfrak{so}(3)$. Recall that $SO(3) = \{R : RR^t = I, \det R = 1\}$ where R is a 3×3 matrix, and I denotes the 3×3 identity matrix. As a set, $SO(3)$ may be viewed as a three dimensional submanifold of the 9 dimensional space of all 3×3 matrices. The Lie algebra of $SO(3)$ is associated with the tangent plane to this manifold at the identity matrix I. Tangent vectors arise as the velocity vectors associated with curves in a manifold. Thus given a curve $R(t)$ in $SO(3)$, since $R(t)R^t(t) = I$ we have $\dot{R}(t)R^t(t) + R(t)\dot{R}^t(t) = 0$ by differentiating. It follows that the matrix $\dot{R}(t)R^t(t)$ is skew symmetric. If $R(0) = I$, then $\dot{R}(0)$ is skew symmetric and thus the Lie algebra $\mathfrak{so}(3)$ is the space of all skew symmetric 3×3 matrices.

Definition 1 *The cross product map is a linear isomorphism $g : \mathbb{R}^3 \to \mathfrak{so}(3)$ defined by the formula $g(a) = a_1S_1 + a_2S_2 + a_3S_3$ with*

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrices S_1, S_2, S_3 form a basis for $\mathfrak{so}(3)$.

The map g is called the cross product map because it was constructed to turn the vector cross product into matrix multiplication.

Lemma 1 *The cross product map has the following properties, with $a, b \in \mathbb{R}^3$:*

$$a \times b = g(a)b, \;\; g(a \times b) = [g(a), g(b)], \;\; g(Ra) = Rg(a)R^t$$

These properties may be verified by short computations. The second property is well known, and has a very interesting interpretation. The right side of the equation is the Lie bracket in $\mathfrak{so}(3)$ defined by $[A, B] = AB - BA$. Thus the vector cross product from vector calculus is actually the Lie bracket operation on $\mathfrak{so}(3)$. This is apparently why sometimes the notation $[a, b]$ is used instead of $a \times b$, e.g. in Arnold [1]. In the following we will need the standard identity

$$\langle a \times b, a \times b \rangle = b^t g(a)^t g(a) b = -b^t g(a)^2 b = b^t (I|a|^2 - aa^t) b = |a|^2|b|^2 - \langle a, b \rangle^2.$$

One can easily derive other cross product identities like $\langle a, b \times c \rangle = \langle c, a \times b \rangle$ via the corresponding matrix calculation using g. The computation of T_3 now goes as follows: Inserting $R^t R$ gives

$$T_3 = \frac{1}{2} \int \langle \dot{R} R^t R Q, \dot{R} R^t R Q \rangle m(Q) dQ.$$

Because $\dot{R} R^t \in \mathfrak{so}(3)$ we can define $\omega = g^{-1}(\dot{R} R^t)$; ω is the angular velocity of the spinning top in the laboratory frame. We use below the fact that matrices in $SO(3)$ preserve inner products, $\langle Rx, Ry \rangle = \langle x, y \rangle$. Hence

$$\begin{aligned} T_3 &= \frac{1}{2} \int \langle \omega \times RQ, \omega \times RQ \rangle m(Q) dQ \\ &= \frac{1}{2} \int \{ |\omega|^2 |RQ|^2 - \langle \omega, RQ \rangle^2 \} m(Q) dQ \\ &= \frac{1}{2} \int \{ |\omega|^2 |Q|^2 - \omega^t R Q Q^t R^t \omega \} m(Q) dQ \end{aligned}$$

Now define $\Omega = R^t \omega$, the angular velocity in the body fixed frame. Note that

$$g(\Omega) = g(R^t \omega) = R^t g(\omega) R = R^t \dot{R} R^t R = R^t \dot{R}.$$

Introducing Ω into T_3 gives

$$\begin{aligned} T_3 &= \frac{1}{2} \int \{ \Omega^t |Q|^2 \Omega - \Omega^t Q Q^t \Omega \} m(Q) dQ \\ &= \frac{1}{2} \Omega^t \{ \int (|Q|^2 I - Q Q^t) m(Q) dQ \} \Omega \end{aligned}$$

The 3×3 matrix Θ is the *inertia tensor* given by

$$\Theta = \int (|Q|^2 I - Q Q^t) m(Q) dQ,$$

such that

$$T_3 = \frac{1}{2} \Omega^t \Theta \Omega.$$

Because Θ is a symmetric matrix, the body coordinate system can always be chosen such that the inertia tensor becomes diagonal, $\Theta = \text{diag}(\Theta_1, \Theta_2, \Theta_3)$, which we assume in the following. In our case we additionally have the symmetry of the body, i.e. $\Theta_1 = \Theta_2$. Finally the total kinetic energy of the top is

$$T = \frac{1}{2} m \langle \dot{r}, \dot{r} \rangle + \frac{1}{2} \Omega^t \Theta \Omega, \qquad \text{with } \Omega = g^{-1}(R^t \dot{R}).$$

4 Potential Energy

The potential energy $U(r, R)$ consists of two terms, the gravitational potential energy and the magnetic potential energy. The former is mgz, where m is the total mass of the top, and z is the height of the center of mass of the top above the x-y plane. The magnetic field of the ceramic base at a point $r = (x, y, z)$ in space is specified by the vector $B(r)$. Let e_z denote the direction of the z-axis, $e_z = (0, 0, 1)^t$. Then the vector Re_z is the unit vector pointing in the direction of the axis of the top. A (negative) parameter μ models the strength of the dipole field of the top, and it is assumed that the symmetry axis of the top is aligned with the magnetic axis of the top. Furthermore it is assumed that the position of the dipole is the center of mass. The magnetic moment of the top is described by the vector μRe_z, pointing downward if the top is pointing upward. In this orientation the top is repelled by the base. The magnetic potential energy $-\langle B(r), \mu Re_z \rangle$ is the inner product of the magnetic field of the ceramic base at position r with the dipole vector of the top. The total potential now is

$$U(r, R) = mgz - \langle B(r), \mu Re_z \rangle, \quad \mu < 0.$$

The magnetic field of the base can be written as the gradient of a scalar potential, $B(r) = -\nabla V(r)$, because we have a static magnetic field. The potential $V(r)$ must be a harmonic function, $\Delta V(r) = 0$, so $B(r)$ fulfills Maxwell's equations. Moreover, we assume cylindrical symmetry of the magnetic field, and therefore require that

$$V(r) = V_0(z) + \rho V_1(z) + \rho^2 V_2(z) + \dots,$$

where $\rho^2 = x^2 + y^2$. The function V is harmonic provided that

$$\Delta V(r) = V_0''(z) + \rho V_1''(z) + V_1(z)\Delta\rho + \rho^2 V_2''(z) + V_2(z)\Delta\rho^2 + \dots = 0.$$

Using the formula $\Delta\rho = n^2 \rho^{n-2}$ and setting the terms with equal powers of ρ equal to zero gives $V_j(z) = 0$ for j odd and $V_{2j+2}(z) = -(1/(2j+2)^2) V_{2j}''(z)$ for j even. Introducing the notation

$$\Phi_k(z) = \frac{d^k}{dz^k} V_0(z)$$

we obtain

$$V(r) = \Phi_0(z) - \frac{\rho^2}{4} \Phi_2(z) \pm \dots,$$

and the magnetic field becomes

$$B(r) = -\nabla V(r) = \begin{pmatrix} x\Phi_2(z)/2 + O(\rho^3) \\ y\Phi_2(z)/2 + O(\rho^3) \\ -\Phi_1(z) + \rho^2 \Phi_3(z)/4 + O(\rho^4) \end{pmatrix}.$$

Since we want the magnetic field to repel the top, we require it to be pointing upward on the z-axis, i.e. $\Phi_1(z) < 0$. In order to be able to start the top spinning on a plastic plate close to the base the field can point in the opposite direction close to the base. Combining the gravitational and magnetic contributions we find to quadratic order in x, y

$$U(r, R) = mgz - \mu\left(\frac{1}{2}\Phi_2(z)(xR_{13} + yR_{23}) + (-\Phi_1(z) + \frac{1}{4}(x^2 + y^2)\Phi_3(z))R_{33} + \dots\right).$$

For the field generated by a disk of radius a with a hole of radius w following Jackson [4] we obtain

$$V_0(z) = 2\pi z(\frac{1}{\sqrt{w^2 + z^2}} - \frac{1}{\sqrt{a^2 + z^2}}).$$

The potential of a square slab of side length $2a$ with a hole in the middle has been given by Berry [2]. We refer to this as Berry's second model, see Fig. 4 and Fig. 7.

5 Symmetry and Conservation Laws

The gravitational and magnetic fields in the problem have rotational symmetry around the vertical axis. Further, the top is symmetric with respect to rotation about its axis. These symmetries may be formally expressed as an action of the group $S^1 \times S^1$ on $\mathbb{R}^3 \times SO(3)$. The action is

$$(r, R) \to (r', R') = (R_z(\alpha)r,\ R_z(\alpha)RR_z(\beta))$$

where $R_z(t)$ is the rotation matrix about the z-axis,

$$R_z(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that this action has orbits which are tori parameterized by α and β except when r is on the z-axis and when $Re_z = e_z$, i.e. the top rotates about its symmetry axis on the symmetry axis of space. The symmetry group action on configuration space extends to an action on its tangent bundle via the tangent map, such that

$$(\dot{r}, \dot{R}) \to (\dot{r}', \dot{R}') = (R_z(\alpha)\dot{r}, R_z(\alpha)\dot{R}R_z(\beta)).$$

The behavior of the top is conveniently modeled using Lagrangian mechanics. Hamilton's principle states that the path of motion in phase space is one which is an extremal for the integral of the Lagrangian $\mathcal{L}$. The function $\mathcal{L}$ is the difference between the kinetic and potential energy of the moving top,

$$\mathcal{L} = \frac{m}{2}\langle\dot{r}, \dot{r}\rangle + \frac{1}{2}\Omega^t\Theta\Omega - U(r, R).$$

That the above symmetry is indeed the symmetry group of the Levitron is shown in

Theorem 1 *The Lagrangian $\mathcal{L}$ is constant on orbits of the torus action on $T(\mathbb{R}^3 \times SO(3))$.*

Proof: We must show that $\mathcal{L}(r, R, \dot{r}, \dot{R}) = \mathcal{L}(r', R', \dot{r}', \dot{R}')$, with $(r', R', \dot{r}', \dot{R}')$ given above.

a) The equation $\langle \dot{r}, \dot{r} \rangle = \langle \dot{r}', \dot{r}' \rangle$ holds because $R_z(\alpha)$ is an orthogonal matrix.

b) By symmetry of the magnetic field we have $B(R_z(\alpha)r) = R_z(\alpha)B(r)$. Hence

$$\langle B(r'), R'e_z \rangle = \langle R_z(\alpha)B(r), R_z(\alpha)RR_z(\beta)e_z \rangle = \langle R_z(\alpha)B(r), R_z(\alpha)Re_z \rangle = \langle B(r), Re_z \rangle.$$

c) To compute the new kinetic energy of the top we calculate

$$R'^t \dot{R}' = R_z^t(\beta)R^t R_z^t(\alpha)R_z(\alpha)\dot{R}R_z(\beta) = R_z^t(\beta)R^t\dot{R}R_z(\beta).$$

From the fact that $g(Rv) = Rg(v)R^t$ for $R \in SO(3)$ it follows that $\Omega(R', \dot{R}') = R_z^t(\beta)\Omega(R, \dot{R})$. Because the matrix Θ is diagonal and $\Theta_1 = \Theta_2$ it commutes with $R_z(\beta)$, therefore $\Omega^t(R', \dot{R}')\Theta\Omega(R', \dot{R}') = \Omega^t(R, \dot{R})\Theta\Omega(R, \dot{R})$.

Hence the Lagrangian $\mathcal{L}$ is invariant under the action of the symmetry group. □

By Noether's theorem we know that there exist two conserved quantities for the system of differential equations which govern the motion of the top. Using the method of Lanczos [5] to calculate them we find

$$\begin{aligned} K_z &= (x\dot{y} - y\dot{x})m + \langle \Omega, \Theta R^t e_z \rangle \\ L_3 &= \langle \Omega, \Theta e_z \rangle. \end{aligned}$$

K_z is the angular momentum about the z-axis in space and L_3 is the angular momentum about the symmetry axis in the body. K_z is the sum of contributions from the motion of the center of mass and from the motion of the top.

The fact that the orbit of the symmetry group is just a circle instead of a torus if we start on the z-axis, $r = (0, 0, z)$ and rotate about the body axis, $R(t) = R_z(t)$, causes the two conserved quantities to coincide on this cylinder in configuration space.

6 Local Coordinates

Recall that if $\mathcal{L} : TM \to \mathbb{R}^1$ is a smooth Lagrangian on the tangent bundle of a manifold M, and if $\Psi : \mathbb{R}^n \to M$ is a local coordinate system on M, then $T\Psi : \mathbb{R}^n \times \mathbb{R}^n \to TM$ is a local coordinate system on TM. Lagrange's equations in local coordinates (q, v) with $v = \dot{q}$ are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_k} = \frac{\partial \mathcal{L}}{\partial q_k}.$$

Thus in order to derive equations of motion for the top, local coordinates on $T(\mathbb{R}^3 \times SO(3))$ must be introduced. To fully explore the symmetry it would be appropriate to introduce a coordinate system which has the angles α and β as coordinates. This coordinate system would use the classical Euler angles, see e.g. Goldstein [3], and both angles would be cyclic in the Lagrangian, thereby reducing the 6 degrees of freedom Levitron to a system with 4 nontrivial degrees of freedom and 2 parameters K_z and L_3. However, this coordinate system is singular exactly on the periodic orbit we are most interested in, i.e. the top spinning about its axis which is aligned with the z-axis and with its center of mass at rest on the z-axis.

Since we already remarked that the two conserved quantities are not independent on this orbit, it does not make sense to try to reduce both of them for studying this orbit. We choose to reduce the angle β, i.e. we reduce by the symmetry of the top.

Definition 2 *The following system of local coordinates is constructed to reduce with respect to the symmetry of the top. Define*

$$\mathbb{R}^3 \to SO(3); R(\psi, \vartheta, \varphi) = R_x(\varphi) R_y(\vartheta) R_z(\psi).$$

The matrices R_x, R_y, R_z are rotations about the x, y, and z axes respectively; R_z was given explicitly in the last section. The entries in the third column of the matrix $R(\psi, \vartheta, \varphi)$ are given by

$$Re_z = (\sin\vartheta, \cos\vartheta \sin\varphi, \cos\vartheta\cos\varphi)^t.$$

Therefore the potential energy is independent of ψ. To calculate the kinetic energy T in local coordinates set $\xi = (\psi, \vartheta, \varphi)$. Then from $\Omega = g^{-1}(R^t \dot{R})$ we find

$$\Omega(\xi, \dot{\xi}) = C(\xi)\dot{\xi} \quad \text{with} \quad C(\xi) = \begin{pmatrix} 0 & -\sin\psi & -\cos\vartheta\cos\psi \\ 0 & \cos\psi & -\cos\vartheta\sin\psi \\ -1 & 0 & -\sin\vartheta \end{pmatrix}.$$

Even though $C(\xi)$ does depend on ψ, the kinetic energy of the top $\Omega^t \Theta \Omega / 2$ does not because $\Theta_1 = \Theta_2$.

7 Hamilton's Equations

Suppose that a Lagrangian in local coordinates $(q, v) \in \mathbb{R}^n \times \mathbb{R}^n$ has the form $\mathcal{L}(q, v) = \frac{1}{2} v^t M(q) v - U(q)$, where $M(q)$ is an $n \times n$ invertible symmetric matrix. The classical Legendre transformation is obtained by setting $p_k = (\partial / \partial v_k) L(q, v)$ and then defining a Hamiltonian $\mathcal{H}$ by the formula $\mathcal{H}(q, p) = p^t v - \mathcal{L}(q, v)$ where v is assumed to be an implicitly defined function of q and p. In this case, $p = M(q)v$ and thus

$$\mathcal{H}(q, p) = \frac{1}{2} v^t M(q) v + U(q) = \frac{1}{2} p^t M^{-1}(q) p + U(q).$$

By this transformation the Lagrangian equations of motion are replaced by the Hamiltonian equations

$$\dot{q}_k = \frac{\partial \mathcal{H}(q, p)}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial \mathcal{H}(q, p)}{\partial q_k}.$$

The Lagrangian for the Levitron has the form

$$\mathcal{L}(r, \dot{r}, \xi, \dot{\xi}) = \frac{m}{2} \langle \dot{r}, \dot{r} \rangle + \frac{1}{2} \dot{\xi}^t M(\xi) \dot{\xi} - U(r, \xi)$$

where $M(\xi) = C^t(\xi) \Theta C(\xi)$. When the Legendre transformation is applied, the resulting Hamiltonian is

$$\mathcal{H}(r, \xi, p, \eta) = \frac{1}{2m} \langle p, p \rangle + \frac{1}{2} \eta^t M^{-1}(\xi) \eta + U(r, \xi).$$

We have $p = m\dot{r}, \eta = M(\xi)\dot{\xi}$ with $\eta = (p_\psi, p_\vartheta, p_\varphi)$. The kinetic energy of the top is most efficiently written using the angular momenta in the body fixed frame defined by

$$L = C^{-t}(\xi)\eta = \Theta\Omega(\xi, \dot{\xi}(\eta)).$$

Then the kinetic energy reads $\eta^t M^{-1}\eta/2 = L^t\Theta^{-1}L/2$, and

$$\begin{aligned}\mathcal{H}(r,\xi,p,\eta) &= \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + \frac{1}{2\Theta_1}\left(p_\vartheta^2 + \frac{(p_\varphi - p_\psi \sin\vartheta)^2}{\cos^2\vartheta}\right) \\ &\quad + \frac{p_\psi^2}{2\Theta_3} + U(r,\vartheta,\varphi)\end{aligned}$$

By construction, the Hamiltonian does not depend on ψ, so p_ψ is conserved. Hamilton's equations take the following form, where we use subscripts on U to denote partial derivatives. We set $r = (x, y, z)$ and $p = (p_x, p_y, p_z)$. Then

$$\begin{aligned}\dot{r} &= \frac{p}{m} \\ \dot{p} &= -U_r(r,\vartheta,\varphi) \\ \dot{\psi} &= \frac{p_\psi}{\Theta_3} + \frac{\sin\vartheta}{\Theta_1\cos^2\vartheta}(p_\psi\sin\vartheta - p_\varphi) \\ \dot{\vartheta} &= \frac{p_\vartheta}{\Theta_1} \\ \dot{\varphi} &= \frac{-1}{\Theta_1\cos^2\vartheta}(p_\psi\sin\vartheta - p_\varphi) \\ \dot{p}_\psi &= 0 \\ \dot{p}_\vartheta &= \frac{1}{\Theta_1\cos^3\vartheta}(p_\psi\sin\vartheta - p_\varphi)(p_\varphi\sin\vartheta - p_\psi) - U_\vartheta(r,\vartheta,\varphi) \\ \dot{p}_\varphi &= -U_\varphi(r,\vartheta,\varphi)\end{aligned}$$

Note that these equations are regular around $\vartheta = 0$. The conserved quantities in these coordinates are $L_3 = -p_\psi$ and

$$K_z = xp_y - yp_x + l_z \quad \text{with} \quad l = (l_x, l_y, l_z)^t = RL = RC^{-1}(\xi)\eta.$$

8 Invariant Set

Due to the symmetry of our problem not only is the Lagrangian invariant under a torus action, but additionally in phase space there is a four dimensional invariant set of solutions where the spin axis of the top is vertical and its center of mass is on the z-axis.

Proposition 1 *The set $\mathcal{I}$ given by*

$$\mathcal{I} = \{x = y = 0,\ \vartheta = \varphi = 0,\ p_x = p_y = 0,\ p_\vartheta = p_\varphi = 0\}$$

is an invariant set for Hamilton's equations of the Levitron.

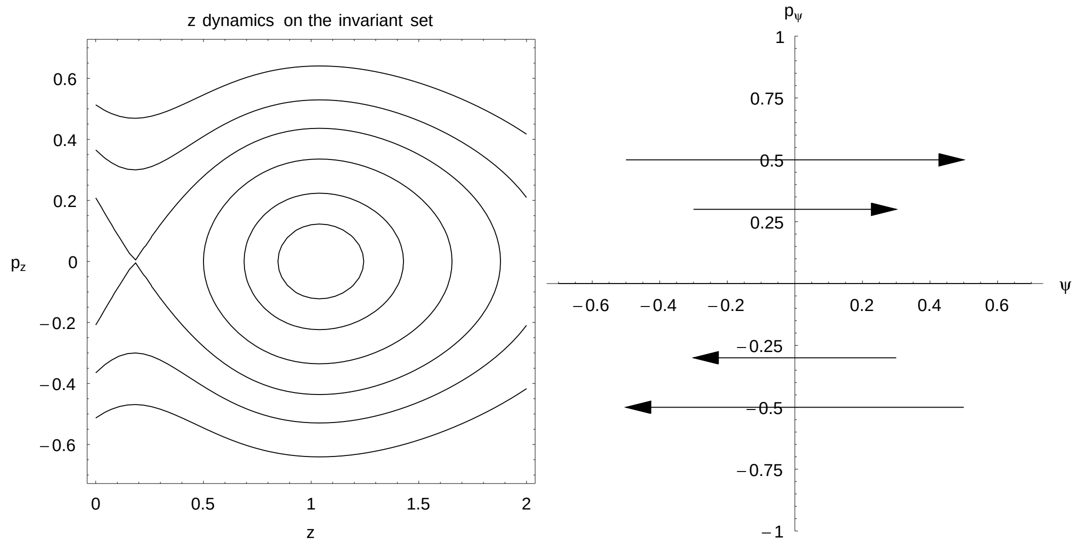

Figure 1: Phase portraits for the Hamiltonian $\mathcal{H}_\mathcal{I}$ on the invariant set $\mathcal{I}$

Proof: One must check that the time derivatives of each of the eight variables used to define $\mathcal{I}$ are equal to zero at each point in the set $\mathcal{I}$. For example, $\dot{x} = p_x/m$, and since $p_x = 0$ on $\mathcal{I}$, it follows that $\dot{x}$ is zero at each point of $\mathcal{I}$. Further, we have $\dot{p}_x = 0$ on $\mathcal{I}$ because the potential energy function U is cylindrically symmetric and consequently $U_x(0, 0, z, 0, 0) = 0$. One inspects the Hamiltonian system of equations above to verify that each derivative is zero. □

The vector field on the invariant set $\mathcal{I}$ is given by the following decoupled equations:

$$\begin{aligned} \dot{z} &= p_z/m, & \dot{p}_z &= -U_z(0, 0, z, 0, 0) \\ \dot{\psi} &= p_\psi/\Theta_3 =: \sigma, & \dot{p}_\psi &= 0, \end{aligned}$$

where we introduced the spin rate σ of the top, which is constant on orbits in the invariant set. The dynamics on $\mathcal{I}$ can be obtained from restricting the Hamiltonian

$$\mathcal{H}|_\mathcal{I} = \mathcal{H}_\mathcal{I}(z, \psi, p_z, p_\psi) = \frac{1}{2m}p_z^2 + \frac{1}{2\Theta_3}p_\psi^2 + mgz + \mu\Phi_1(z).$$

The phase portraits for the integrable system $\mathcal{H}_\mathcal{I}$ with the magnetic potential of a disk shaped base magnet without central hole are shown in Fig. 1.

9 Relative Equilibrium

A relative equilibrium is a periodic orbit of Hamilton's equations that appears as an equilibrium in the reduced system. In the reduced system the symmetry is made explicit by introducing an angle in the local coordinates that becomes cyclic in the Hamiltonian. The corresponding conserved momentum can be treated as a parameter and the systems number of degrees of freedom is effectively reduced by one. In our case we would need to

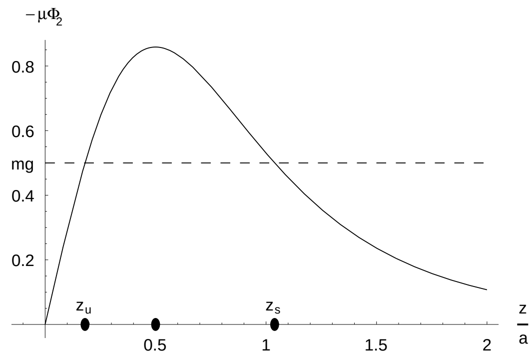

Figure 2: The equilibrium condition $mg = -\mu\Phi_2$

look for an equilibrium of Hamilton's equation ignoring the trivial equations for $\dot{\psi}$ and $\dot{p}_\psi$ equations.

However, since we have already established the existence of an invariant set $\mathcal{I}$ which contains the trivial ψ-dynamics plus the nontrivial z-dynamics, we only need to find an equilibrium in the z-dynamics. By the invariance of $\mathcal{I}$ this automatically gives an equilibrium of the reduced Hamiltonian.

Considering the Hamiltonian $\mathcal{H}_\mathcal{I}$ on $\mathcal{I}$ we see that the equilibrium condition is given by the critical points of the potential $U_\mathcal{I}(z)$ of $\mathcal{H}_\mathcal{I}$, i.e. by

$$U'_\mathcal{I}(z) = mg + \mu\Phi_2(z) = 0.$$

For a typical base magnet without central hole the function $\Phi_2(z)$ starts out at zero, rises to a maximum value and then decreases monotonically to zero as z goes to infinity. For the disk shaped base magnet it is pictured in Fig. 2. Note that the exact formula for $V_0(z)$ is not important, but rather the shape of the function $\Phi_2(z)$.

As long as the constant mg in this equation is less than the maximum value of the function, there are exactly two real solutions $z_u < z_s$. The bottom equilibrium solution is unstable and the higher critical point is stable in the invariant set $\mathcal{I}$, see Fig. 1. The force along the z-axis is positive on the interval (z_u, z_s) and is negative below z_u and above z_s. Thus if the top is placed below the lower equilibrium or above the upper equilibrium gravity predominates and it is pushed downward. If it starts between the equilibria the magnetic repulsion between the base and the top predominates and it is pushed upwards, compare Fig. 1.

We are interested in the equilibrium that is at least stable in the z-dynamics. Vertical stability is obtained if we are at a minimum of $U_\mathcal{I}(z)$, i.e. if

$$U''_\mathcal{I}(z) = \mu\Phi_3(z) > 0.$$

This holds for the upper equilibrium point $z = z_s$. The periodic orbit in full phase space $Z = (x, y, z, \psi, \vartheta, \varphi, p_x, p_y, p_z, p_\psi, p_\vartheta, p_\varphi)$ that corresponds to the higher relative equilibrium is given by

$$Z(t) = (0, 0, z_s, \sigma t, 0, 0, 0, 0, 0, \sigma\Theta_3, 0, 0).$$

The frequency of (linear) oscillation in the z direction around this equilibrium is $(\mu\Phi_3(z_s)/m)^{1/2}$.

10 Linear Stability

At relative equilibrium the center of mass of the top is at rest on the z-axis, and the top is spinning with its axis fixed along the z-axis. The linearized equations of motion near the relative equilibrium must be stable in order that the nonlinear equations have the possibility of being stable. We have already seen that the periodic solution $Z(t)$ is vertically stable inside the invariant set, now we must establish the conditions under which the remaining transverse 4 degrees of freedom (horizontal motions of the center of mass and rotations other than around the z-axis) are also stable. Note that in the general case of a relative equilibrium we would need to deal with a 5 degree of freedom linear system. The existence of the invariant set allows us to reduce to a 4 degree of freedom linear system.

A general Hamiltonian system of equations in $2n$ dimensions has the form $\dot{Z} = JH_Z(Z)$, where J is the symplectic matrix. If $Z_0(t)$ is a solution to this system, then the linearized system of equations associated with this solution is the system $\dot{W} = JH_{ZZ}(Z_0(t))W$ with $W = Z - Z_0 = \Delta Z$. For the Levitron we take the above relative equilibrium as a solution.

We could write the linearized equations as a first order 12 by 12 system. However, since the solution $Z(t)$ is contained in a 4 dimensional invariant set, the linearized equations decouple into a 4 by 4 system which linearizes the equations restricted to this invariant set and an 8 by 8 system of equations in the remaining variables. Because the Hamiltonian does not depend on the variable ψ, the 8 by 8 system of equations has constant coefficients. It has the form $\dot{\zeta} = L\zeta$, where $\zeta = (\Delta x, \Delta y, \Delta\vartheta, \Delta\varphi, \Delta p_x, \Delta p_y, \Delta p_\vartheta, \Delta p_\varphi)$. We have used the algebraic manipulation program Maple to compute the matrix L. It is a surprise that this matrix has so few non-zero entries:

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 1/m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/\Theta_1 & 0 \\ 0 & 0 & p_\psi/\Theta_1 & 0 & 0 & 0 & 0 & 1/\Theta_1 \\ \mu\Phi_3/2 & 0 & \mu\Phi_2/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu\Phi_3/2 & 0 & \mu\Phi_2/2 & 0 & 0 & 0 & 0 \\ \mu\Phi_2/2 & 0 & \mu\Phi_1 - p_\psi^2/\Theta_1 & 0 & 0 & 0 & 0 & -p_\psi/\Theta_1 \\ 0 & \mu\Phi_2/2 & 0 & \mu\Phi_1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Besides the mass m, the moment of inertia Θ_1 and the (negative) dipole strength μ, the linearized equations essentially depend on the angular momentum $p_\psi = \sigma\Theta_3$ and on the properties of the magnetic field encoded in the derivatives Φ_i evaluated at the critical point z_s on the z-axis.

The system of equations $\dot{\zeta} = L\zeta$ can be written as the following much smaller system of second order equations in two complex variables. Define $u = \Delta x + i\Delta y, v = \Delta\vartheta + i\Delta\varphi$. Then the linearized 8×8 equations are equivalent to

$$\begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} = \begin{pmatrix} \frac{\mu\Phi_3}{2m} & \frac{\mu\Phi_2}{2m} \\ \frac{\mu\Phi_2}{2\Theta_1} & \frac{\mu\Phi_1}{\Theta_1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & i\sigma\frac{\Theta_3}{\Theta_1} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}.$$

This equation has the form $\ddot{w} = Aw + B\dot{w}$, with $w = (u, v)^t$ and solutions of this equation have the form $w(t) = e^{\lambda t}w_0$ with w_0 a constant complex column vector. For $w(t)$ to be a solution, the equation $(A + \lambda B - \lambda^2 I)v = 0$ must be satisfied. Therefore λ is a root of the polynomial

$$P(\lambda) = \lambda^4 - i\sigma\varrho\lambda^3 - \mu\left(\frac{\Phi_1}{\Theta_1} + \frac{\Phi_3}{2m}\right)\lambda^2 + i\sigma\varrho\mu\frac{\Phi_3}{2m}\lambda + \mu^2\frac{2\Phi_1\Phi_3 - \Phi_2^2}{4m\Theta_1},$$

where ϱ denotes the ratio of the moments of inertia, $\varrho = \Theta_3/\Theta_1$. The original problem given by the eigenvalue equation of L has λ and $\bar{\lambda}$ as solutions. The multipliers of the periodic orbit $Z(t)$ are then given by $\exp(\lambda T)$ where T is the period $T = 2\pi/\sigma$.

Stability requires that the exponents λ are on the imaginary axis. However, by making the change of variables $\lambda = i\sigma\varrho t$, and dividing by $(\sigma\varrho)^4$, a new polynomial $N(t)$ is obtained. Note that the roots of $N(t)$ are related to the winding numbers ν of the periodic orbit. The ν are defined through the formula $\exp(\lambda T) = \exp(2i\pi\nu)$ and this gives $\nu = t\varrho$. By the above calculations we obtain

Theorem 2 *If the polynomial*

$$N(t) = t^4 - t^3 + (g_1 + g_3)t^2 - g_3 t + g_1 g_3 - g_2^2$$

with dimensionless coefficients given by

$$g_1 = \frac{\mu\Phi_1(z_s)}{\sigma^2\varrho^2\Theta_1}, \quad g_2 = \frac{\mu\Phi_2(z_s)}{2\sigma^2\varrho^2\sqrt{m\Theta_1}}, \quad g_3 = \frac{\mu\Phi_3(z_s)}{2\sigma^2\varrho^2 m},$$

and z_s determined by

$$mg + \mu\Phi_2(z_s) = 0 \quad \textit{and} \quad \mu\Phi_3(z_s) > 0,$$

has four distinct real roots, then the periodic orbit $Z(t)$ of the Levitron is linearly stable.

11 Bifurcations

If the polynomial $N(t)$ has 4 real roots, then this condition is maintained as the coefficients g_i change until a double root of $N(t)$ occurs, i.e. when the discriminant $G(g_1, g_2, g_3)$ of $N(t)$ vanishes. It is given by

$$\begin{aligned} G(g_1, g_2, g_3) &= 4p^3 + 27q^2, \quad \text{where} \\ 3p &= -g_1^2 - 14g_1g_3 - g_3^2 + 12g_2^2 - 3g_3 \\ 27q &= 2g_1^3 + 2g_3^3 + 27g_2^2 - 6(g_1 + g_3)(11g_1g_3 + 3g_3 - 12g_2^2). \end{aligned}$$

The equilibrium condition $\mu\Phi_2 = -mg$ requires $\mu\Phi_2 < 0$, hence $g_2 < 0$. In order to have stability in the vertical motion of the top we need $\mu\Phi_3 > 0$, hence $g_3 > 0$. We already remarked that the magnetic field $B = -\nabla V$ is pointing upward on the z-axis. Therefore $-\Phi_1 > 0$, and hence $g_1 > 0$.

Definition 3 *The coefficient space is the set of all possible g's with $g_2 < 0$ and g_1, $g_3 > 0$. Clearly this space is an octant contained in $\mathbb{R}^3$. Define a subset G of coefficient space to be the set where the polynomial $N(t)$ has 4 real roots.*

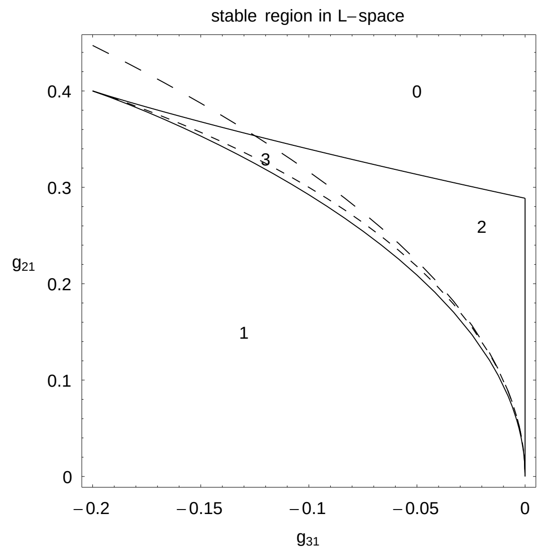

Figure 3: *Bifurcation diagram in Levitron space. The cuspoidal region bounded by $D_2 = 0$ (solid) contains Levitrons which have an interval of stable spin rates. The curve $C_3 = 0$ (long dashed) indicates a zero winding number. The numbers in the connected regions indicate the number of critical spin rates for rays emerging from these points. The corresponding bifurcations are listed in Table 1. Crossing $D_1 = 0$ (dashed) does not change the number of roots. The cusp is located at $(-1/5, 2/5)$, and the intersection with the g_{21}-axis is located at $1/\sqrt{12}$.*

The boundary of the set G is contained in the set $\{G(g_1, g_2, g_3) = 0\} \cup \{g_3 = 0\}$. Although we do not model the effect of weak dissipation on the motion, it is clear that its main effect is a slow decrease of the spin rate. Once the top is set into motion all the parameters except the spin rate σ are constant. As the spin rate changes, a ray through the origin in coefficient space is determined. The intersection of this ray with the set G determines the critical spin rates; for points on the ray inside G the relative equilibrium is stable.

The ray $l(g_1, g_2, g_3)$ with l as a parameter intersects the set G for special values of l determined by the zeroes of the polynomial L,

$$L(l) = G(lg_1, lg_2, lg_3)/l^3 = 0$$

where the trivial zeroes $l^3 = 0$ have been removed, and the remaining polynomial is of degree three. The explicit form of L can easily be calculated with Maple. We just remark that the indefinite factor of the highest power coefficient is

$$C_3(g_1, g_2, g_3) = g_1 g_3 - g_2^2.$$

Hence, one winding number ν is zero if $C_3 = 0$, because then $N(0) = 0$ and correspondingly one multiplier is equal to one.

Each positive critical l given by $L(l) = 0$ corresponds to a critical spin rate that is proportional to $\pm 1/\sqrt{l}$, at which a bifurcation occurs. A given ray usually intersects the set G in a closed interval or not at all. Exceptional rays intersect G in a point. These

$\#\{l>0\}$	bifurcations for increasing σ
0	$CQEE$
1	$CQCQ \to CQEE$
2	$CQEE \to EEEE \to CQEE$
3	$CQCQ \to CQEE \to EEEE \to CQEE$

Table 1: *Transition of multipliers with increasing spin rate σ depending on the number of critical spin rates on a ray, see Fig. 3. EE denotes a pair of elliptic multipliers, CQ a complex quadruple. The stable region $EEEE$ is always entered/left by a Hamiltonian Hopf bifurcation.*

rays are given by the double roots of $L(l)$, i.e. by the discriminant $D(g_1, g_2, g_3)$ of L. The polynomial D is a homogeneous polynomial in the g_i,

$$
\begin{aligned}
D(g_1, g_2, g_3) &= -16 g_2^2 D_1(g_1, g_2, g_3)^2 D_2(g_1, g_2, g_3)^3 \\
D_1(g_1, g_2, g_3) &= g_1 g_3 - g_2^2 - g_3^2 \\
D_2(g_1, g_2, g_3) &= 108 g_2^4 - 9 g_2^2 (g_1^2 + 14 g_1 g_3 + g_3^2) + 8 g_3 (g_1 + g_3)^3
\end{aligned}
$$

Because the factor D_1 appears squared in D, it can not change the sign of the discriminant. Therefore it's vanishing is of minor importance for the bifurcation scenario.

Rays in coefficient space can be coordinatized by the inhomogeneous coordinates $g_{21} = -g_2/g_1$ and $g_{31} = -g_3/g_1$, i.e. dividing C_3, D_1 and D_2 by $g_1^{\deg}$, where deg is the respective degree of the homogeneous polynomial. In the quadrant $g_{21} > 0$, $g_{31} < 0$ the three curves defined by the dehomogenized curves D_1, D_2, and C_3 are shown in Fig. 3. We observe four regions with different sequences of critical spin rates. The vanishing of D_1 does not change these sequences. By calculating the multipliers of the Poincaré map transverse to the periodic orbit for one value in each interval between the critical spin rates for each region we obtain Table 1. All bifurcations are Hamiltonian Hopf bifurcations, also called Krein collisions. At this bifurcation two pairs of elliptic multipliers (real t) collide on the unit circle and move off from the unit circle, forming a complex quadruple of multipliers (complex t).

For $\sigma = 0$ the polynomial $P(\lambda)$ never has 4 imaginary roots because the coefficient of λ^2 is negative. For $\sigma = \infty$ inspection of the matrix L shows that there are nontrivial Jordan blocks and hence instability.

12 Levitron Space

Definition 4 *The set of all possible rays in coefficient space is called the Levitron space or L-space. It is coordinatized by the inhomogeneous coordinates $g_{21} = -g_2/g_1$ and $g_{31} = -g_3/g_1$.*

The minus signs in the definition of g_{21} and g_{31} are meant to cancel the minus sign in Φ_1, such that $\mathrm{sgn}(g_{21}) = \mathrm{sgn}(\Phi_2)$ and $\mathrm{sgn}(g_{31}) = \mathrm{sgn}(\Phi_3)$. For Levitrons inside the

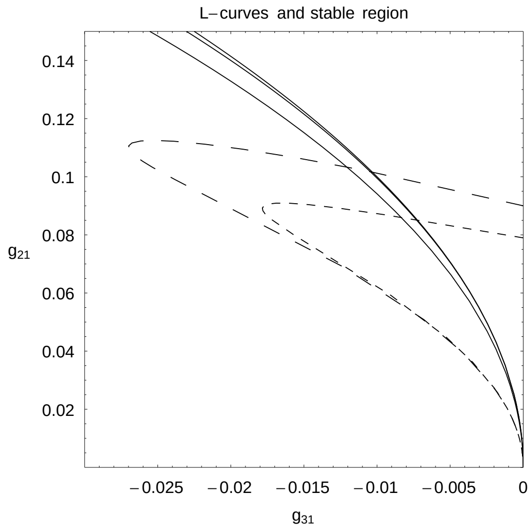

Figure 4: *L-curves in the bifurcation diagram describing two concrete realizations of Levitrons. The curves are parameterized by equilibrium height z_s or, equivalently, mass. Minimal $z_s = z_c$ (maximal mass) corresponds to the point on the g_{21}-axis. As z_s limits to infinity (vanishing mass) the curves approaches the origin. The long dashed L-curve corresponds to Berrys first model, the other one to his second model ($\alpha = 0.0075$). The solid curves are D_2, D_1 and C_3.*

cuspoidal region $D_2 < 0$ there exists a range of σ for which the relative equilibrium of the top is linearly stable. In order to decide whether a given Levitron is capable of stable motion we simply draw the curve

$$(g_{31}(z_s), g_{21}(z_s)) = \left(-\alpha^2 \frac{\Phi_3(z_s)}{2\Phi_1(z_s)}, -\alpha \frac{\Phi_2(z_s)}{2\Phi_1(z_s)}\right)$$

and find out whether it intersects the stable region. The "effective length" α is defined by $\Theta_1 = m\alpha^2$. Such an *L-curve* in L-space describes a particular realization of a Levitron with fixed geometry of the base and of the top. The exact position *on* the L-curve can be adjusted by changing the equilibrium height z_s, which is practically done by changing the mass of the top with the washers. The approximation that putting washers onto the top does not change α is very good. The L-curves for both models of magnetic fields introduced by Berry are shown in Fig. 4. The equilibrium height z_s is the the same as in his adiabatic description.

The parameterization of an L-curve starts with the special z_c for which $\Phi_3(z_c) = 0$. This minimal z_c corresponds to the top with maximal weight at which the vertical equilibrium appears in a tangent bifurcation. A requirement for stability therefore is

$$g_{21}(z_c) = -\alpha \frac{\Phi_2(z_c)}{2\Phi_1(z_c)} < \frac{1}{\sqrt{12}}, \quad \text{where } \Phi_3(z_c) = 0.$$

It seems possible to have L-curves start outside the stable region $D_2 < 0$ and only enter it for $z_s > z_c$, but as we will see $g_{21}(z_c)$ has to be smaller then -0.1 in order to have a sufficiently high upper critical spin rate (see Fig. 6). The two L-curves corresponding to the fields proposed by Berry are shown in Fig. 4. Increasing the height z_s, hence decreasing the mass m, the stability is lost upon leaving the cuspoidal region. In Berry's analyses stability is already lost a little earlier upon crossing of the curve C_3. The two curves are reasonably close to each other, so that Berry's adiabatic stability analyses gives a surprisingly good result. Moreover we have shown that this curve corresponds to a multiplier equal to one, (even independently of σ) so that nonlinear instability due to resonance in the neighborhood of this line is possible.

13 Critical Spin Rate

Our main addition to Berry's analyses is that we can calculate the critical spin rates. For a given Levitron we calculate g_i as given in Theorem 2 and g_{21} and g_{31} as given in Definition 4. This determines a ray $l(g_1, g_2, g_3)$ or equivalently $l(1, -g_{21}, -g_{31})$ and the critical l are now calculated from $L(l) = 0$, i.e. from

$$\begin{aligned}
&-16(g_{21}^2 + g_{31})(g_{31}^2 + 2g_{31} + 1 + 4g_{21}^2)^2 l^3 \\
&+4(15g_{21}^2 g_{31}^3 + 12g_{31}^2 + 22g_{31}^3 + g_{31} + 12g_{21}^4 g_{31} + 31g_{31}^2 g_{21}^2 + g_{21}^2 \\
&\qquad +49g_{21}^2 g_{31} + 12g_{31}^4 + g_{31}^5 + 36g_{21}^4) l^2 \\
&-(8g_{31}^4 + 32g_{31}^3 + 36g_{21}^2 g_{31} + 27g_{21}^4 + 12g_{31}^2 g_{21}^2 + 8g_{31}^2) l + 4g_{31}^3 = 0.
\end{aligned}$$

The two smallest positive roots l_2, l_1 (by definition $l_2 < l_1$) give the critical spin rates σ_1, σ_2 ($\sigma_1 < \sigma_2$). The critical spin rates correspond to the points $(l_i, -l_i g_{21}, -l_i g_{31})$ in coefficient space. Now we take the definition of g_2, substitute $\Theta_1 = m\alpha^2$, and use the equilibrium condition $\Phi_2(z_s) = -mg/\mu$ to eliminate Φ_2; solving for σ^2 gives

$$\sigma^2 = \frac{g}{2\alpha\varrho^2(-g_2)} = \frac{1}{2\varrho(-g_2)}\frac{mg\alpha}{\Theta_3}.$$

The last expression has the form of torque divided by moment of inertia, where the length α measures the effective horizontal radius of the top. If we replace g_2 by the critical value $-l_i g_{21}$, where l_i is found by solving the above cubic equation, we obtain the critical spin rate σ_i as

$$\sigma_i^2 = \frac{g}{2\alpha\varrho^2 g_{21} l_i} = \frac{g}{2\alpha^2\varrho^2 l_i}\frac{-\Phi_1(z_s)}{\Phi_2(z_s)}.$$

It is now obvious that the critical spin rate only depends on the dimensionless quantities g_{21}, g_{31}, and ϱ and on the effective length α. Figure 5 shows contour plots of the minimal and maximal critical spin rates σ_i in the stable region in Levitron space.

From Fig. 5 we see that the lower critical spin rate σ_1 does not change very much on the L-curve. Hence we can safely calculate it at the critical height z_c where $\Phi_3(z_c) = 0$. There l_i is given by a quadratic polynomial

$$L(l; g_{31} = 0)/l = 16(1 + 4g_{21}^2)^2 l^2 + 4(1 + 36g_{21}^2) l - 27g_{21}.$$

Since $g_{21} < 1/\sqrt{12}$ is necessary for stability we expand the solutions of this equation in g_{21} and find

$$l_1 \approx \frac{1}{4}(1 + g_{21}^2), \quad l_2 \approx \frac{27}{4} g_{21}^2,$$

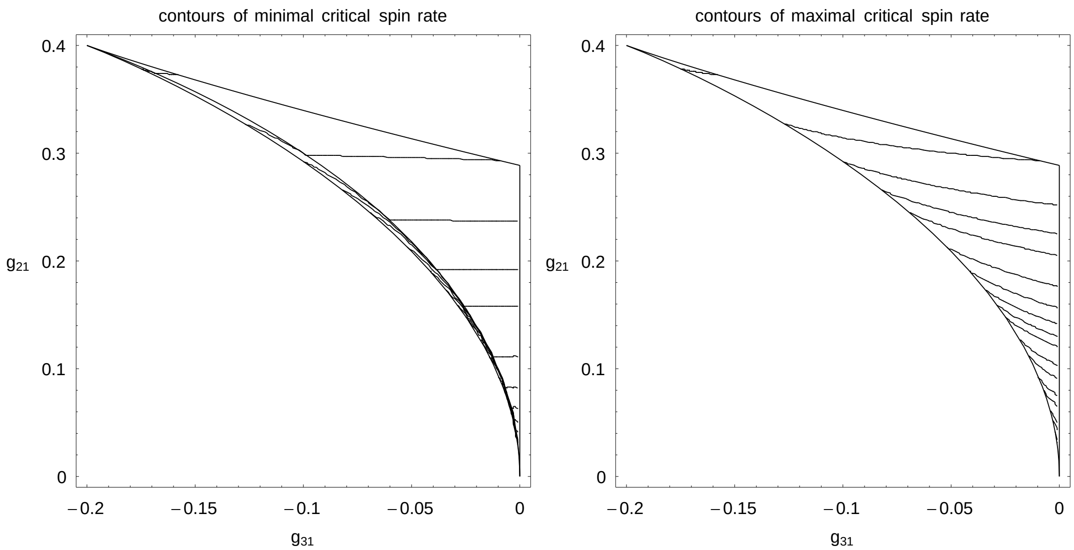

Figure 5: *Minimal and maximal stable spin rates* $\sigma_1/(2\pi)$ *and* $\sigma_2/(2\pi)$ *in* Hz *as a contour plot on the stable region of the bifurcation diagram. For definiteness the constants are* $\sqrt{g/(2\alpha\varrho^2)}/(2\pi) = 2$. *The values of the contour lines in both pictures are from top to bottom* $\{6, 7, 8, 9, 10, 12, 14, 16, 18, 20, 25, 30, 40, 50\}$.

where the approximation for l_1 is very good, while that for l_2 (i.e. the upper critical spin rate) is poor for $g_{21} > 0.15$. For the critical spin rates calculated at the critical height we finally find

$$\sigma_1^2 \approx 2\frac{g}{\alpha\varrho^2 g_{21}(z_c)} = 4\frac{-\Phi_1(z_c)}{\Phi_2(z_c)}\frac{g}{\alpha^2\varrho^2}$$

and

$$\sigma_2^2 \approx \frac{2}{27}\frac{g}{\alpha\varrho^2 g_{21}(z_c)^3} = \frac{16}{27}\left(\frac{-\Phi_1(z_c)}{\Phi_2(z_c)}\right)^3\frac{g}{\alpha^4\varrho^2} \approx \frac{\sigma_1^2}{27g_{21}^2}.$$

The exact dependence of the critical σ for $g_3 = 0$ is shown in Fig. 6.

For a cylinder of radius b (neglecting its height) we have $\Theta_1 = mb^2/4$ and $\Theta_3 = mb^2/2$, such that $\alpha = b/2$ and $\varrho = 2$. For Berry's first model we have $z_c = a/2$, and $-\Phi_1/\Phi_2 = 5a/6$

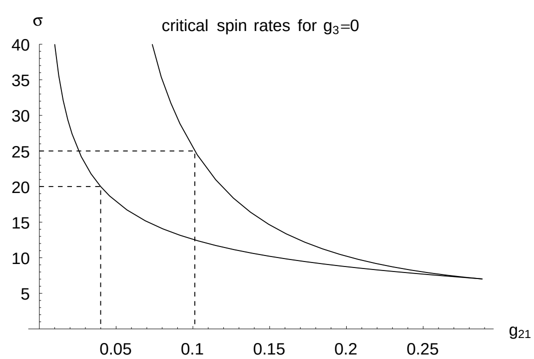

Figure 6: *Critical spin rates for* $g_3 = 0$

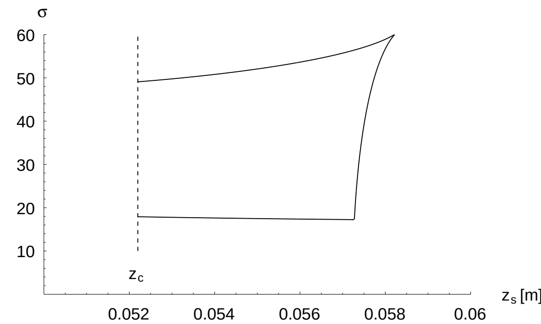

Figure 7: *Critical spin rates for Berry's second model in the dependence on equilibrium height z_s. The small interval of stable z_s translates into an even smaller interval of stable masses via the equilibrium condition. This is why the stability of the Levitron depends so delicately on the mass. Note that this shows the values from the contour plots along the corresponding L-curve.*

and find $\sigma_1^2 = 10ga/(3b^2)$; choosing $b = 0.3\,a$, and $a = 0.05$ as measured from the Levitron gives the lower critical spin rate $\sigma_1/(2\pi) = 13.6$Hz, which is a little too low. Taking the more realistic second model proposed by Berry, we find $\sigma_1/(2\pi) = 17$Hz, which nicely fits the experimentally observed value of 18Hz [6].

Considering the advice that "the top should barely lift of the plate" indicates that in experiment we are probably using it close to the critical height/mass. The complete dependence of the critical spin rate on the mass respectively on the equilibrium height is shown in Fig. 7 for Berry's second model.

14 Discussion

We develop and study an exact model of the Levitron. Two symmetries of the system are fully exploited and lead to the study of a relative equilibrium which is contained inside a four dimensional invariant set in a 12 dimensional phase space. The fact that there is a whole family of degenerate orbits of the symmetry group can be viewed as the origin of the invariant set. Because we encountered a case of singular reduction it is not possible to reduce by both symmetries for the study of this particular orbit. We choose to reduce by the symmetry of the top. It would be slightly more general to reduce by symmetry of space, because then one could even make the top slightly asymmetric. However, since the resulting equations are more complicated, we did not use them. It is interesting to note that eigenvalues λ of the two reductions are not the same, only the multipliers $\exp(2\pi\lambda/\sigma)$ of the orbit are the same. The difference between the two reductions accounts for a twisting of the fibers of the quotient space.

The invariant set induces a block decomposition of the linearization, and the nontrivial 8×8 part can be written as a 2×2 linear complex second order equation. The stability of the motion depends only on 3 dimensionless parameters. Separating the spin rate from the Levitron geometry leads to a Levitron space in which every possible Levitron is represented by a point. The critical spin rates can then be determined from Fig. 5. This information can be used to optimize Levitrons. Since a hand spun top should be initially stable, the upper critical spin rate has to larger than $\approx$ 25Hz. Since the top should be stable for a while, the lower critical spin rate should be smaller than $\approx$ 20Hz. These conditions determine a rather small range of operation for g_{21}, see Fig. 6. The main lesson to be learned is to make g_{21} as large as possible without passing $\sigma_2 \approx$ 25Hz, since this makes σ_1 as small as possible, so that the Levitron spins a long time.

Watching the Levitron shows that often it does not stay on the periodic orbit we described, but rather makes large excursions around it. Part of it we have captured in the invariant set: analytic solutions that describe the spinning top bouncing up and down can be found. These solutions are two-tori within the 6 degree of freedom system, equivalently they are periodic orbits of the reduced 5 degree of freedom system. It would be interesting to study the stability of these two-tori. Another set of two-tori are the non-degenerate equilibria corresponding to motions where the spin axis precesses.

Another concern is the validity of linear stability analysis. Linear stability is the strongest result that can be proved for high dimensional Hamiltonian dynamics since Arnold diffusion can not be ruled out. However, if diffusion occurs, then it happens slowly, and evidently on a time scale which is long compared to the few minutes when the top is spinning in the air.

15 Acknowledgments

Thanks to Jim Howard who got this research going by suggesting we read the paper by M.V. Berry in our dynamics seminar. Thanks to Jim Meiss, and Hector Lomeli for listening to several talks on early versions of this paper and for many valuable suggestions and criticisms. HRD was supported by the DFG under contract number Du 302/1.

16 References

1. V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1978.

2. M.V. Berry, The Levitron: an adiabatic trap for spins, Proc. Royal Society of London, A 452 1207-1220 (1996)

3. H. Goldstein, Classical Mechanics, 2nd edition, Addison-Wesley, Reading, MA, 1980

4. J. Jackson, Classical electrodynamics, 2nd edition, Wiley, New York, 1975

5. C. Lanczos, The Variational Principles of Mechanics, 4th edition, Dover, 1970

6. M.D. Simon, L.O. Heflinger, S.L. Ridgway, Spin stabilized magnetic levitation, Am. J. Phys. 65, 286 (1997)