

# Action variables of the Kovalevskaya top

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## Abstract

An explicit formula for the action variables of the Kovalevskaya top as certain abelian integrals of third kind on the Kovalevskaya curve is found. The linear system of differential equations of Picard-Fuchs type describing the dependence of these variables on the integrals of the Kovalevskaya system is presented in the explicit form. The results are based on the formula for the actions derived by S.P.Novikov and A.P.Veselov within the theory of the algebro-geometric Poisson brackets on the universal bundle of the hyperelliptic Jacobians.

## 1 Introduction

The Kovalevskaya top is one of the most beautiful examples of integrable systems. Kovalevskaya's original paper [K89] is heavily based on deep ideas from the theory of hyperelliptic Riemann surfaces and abelian functions, giving one of the most impressive applications of this theory. From this point of view the Kovalevskaya top is one of the genuine highlights of nineteenth century mathematics.

It is no accident that Kovalevskaya's top became popular again with the modern renaissance of integrable systems, which started in the sixties when the famous paper [GGKM67] about some remarkable properties of the KdV equation appeared. The relation between the KdV theory and the Kovalevskaya top was first mentioned in the review by B. A. Dubrovin, V. B. Matveev, and S. P. Novikov [DMN76] (see the introduction). Later, the Kovalevskaya top was considered as the best example to demonstrate the power of several modern methods (see e.g. M. Adler and P. van Moerbeke [AM82], L. Haine and E. Horozov

[HH87], E. Horozov and P. van Moerbeke [HM89], A. I. Bobenko, A. G. Reyman, M. A. Semenov-Tian-Shansky [BRS89]).

The determination of action variables for the Kovalevskaya top was first discussed by S. P. Novikov and A. P. Veselov [VN84] within the general theory of algebro-geometric Poisson brackets on the universal bundle of hyperelliptic Jacobians. It could be shown that the Kovalevskaya top fits into this theory, and as a result an explicit formula for the action variables was given, see (14) and (18) below. The same formula was rederived later, using separation of variables, by I. Komarov and V. Kuznetsov [KK87]. These results were not known to H. R. Dullin et al. who calculated the action variables for the Kovalevskaya top using a different, numerical method [DJR94, DW94].

In this paper we present an analysis of the action variables based on the Novikov-Veselov result. First we express the action integrals in another, more suitable form as an abelian integral of the third kind on the Kovalevskaya curve. Using this formula we derive the Picard-Fuchs equations for the action variables. We should mention that a similar question was discussed in a paper by J. P. Francoise [F87], but his considerations are wrong: he assumed that the action variables are given by abelian integrals of the first kind, which is not the case.

## 2 Action variables

The classical rigid body is a system with three angular degrees of freedom. Due to rotational symmetry with respect to the direction of gravity, the corresponding angle  $\varphi$  does not appear in the Hamiltonian; its conjugate angular momentum  $l$  is a general constant. Treating  $l$  as a parameter, the system is usually considered to have only two degrees of freedom, the Poisson sphere  $S^2$  ( $\gamma_1, \gamma_2, \gamma_3$ ) acting as a reduced configuration space. The reduced phase space consists of the variables ( $\gamma_1, \gamma_2, \gamma_3$ ) and the three components ( $l_1, l_2, l_3$ ) of the angular momentum in the body-fixed frame of reference. This phase space is equipped with the Poisson structure

$$\{\gamma_i, \gamma_j\} = 0, \quad \{\gamma_i, l_j\} = \epsilon_{ijk} \gamma_k, \quad \{l_i, l_j\} = \epsilon_{ijk} l_k, \quad (1)$$

where  $\epsilon_{ijk}$  is the standard skew-symmetric tensor. This structure has two Casimir functions  $C_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$  and  $C_2 = \gamma_1 l_1 + \gamma_2 l_2 + \gamma_3 l_3$ ; they guarantee the invariance of the Poisson sphere,  $C_1 = 1$ , and the constancy of the angular momentum  $l = C_2$ . This Poisson structure is the reduction of the standard symplectic structure on  $T^*SO(3)$  with respect to rotations around the vertical axis [A78]. The symplectic leaves corresponding to fixed values of  $C_1$  and  $C_2$  are topologically equivalent to the cotangent bundle of the sphere,  $T^*S^2$ . But note that their symplectic structure is known to be different from the standard symplectic structure on  $T^*S^2$  by the magnetic term proportional to  $l$  [DFN90].

The special feature of the Kovalevskaya top is that the three principal moments of inertia are  $(2, 2, 1)$ , and the center of gravity  $(-c, 0, 0)$  lies on the 1-axis. The Hamiltonian is therefore

$$H = \frac{1}{4}l_1^2 + \frac{1}{4}l_2^2 + \frac{1}{2}l_3^2 + c\gamma_1, \quad (2)$$

and the equations of motion, if written in terms of the angular velocities  $(p, q, r) = (\frac{1}{2}l_1, \frac{1}{2}l_2, l_3)$ , are the Euler-Poisson equations

$$\begin{aligned} \dot{\gamma}_1 &= r\gamma_2 - q\gamma_3 & 2\dot{p} &= qr \\ \dot{\gamma}_2 &= p\gamma_3 - r\gamma_1 & 2\dot{q} &= -c\gamma_3 - rp \\ \dot{\gamma}_3 &= q\gamma_1 - p\gamma_2 & \dot{r} &= c\gamma_2. \end{aligned} \tag{3}$$

These equations possess the three usual integrals

$$\begin{aligned} p^2 + q^2 + \frac{1}{2}r^2 + c\gamma_1 &= h, & \text{energy} \\ 2p\gamma_1 + 2q\gamma_2 + r\gamma_3 &= l, & \text{angular momentum} \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1, & \text{Poisson sphere} \end{aligned} \tag{4}$$

and the non-trivial Kovalevskaya integral

$$K := ((p + iq)^2 - c(\gamma_1 + i\gamma_2)) ((p - iq)^2 - c(\gamma_1 - i\gamma_2)) = k^2. \tag{5}$$

Viewed as functions on the symplectic leaves  $C_1 = 1$ ,  $C_2 = l$ , Hamiltonian  $H$  and Kovalevskaya integral  $K$  generate independent commuting flows on the invariant tori  $H = h$ ,  $K = k^2$ .

Integrating the equations of motion, S. Kovalevskaya demonstrated a fascinating analytical skill, the spirit of which seems to be lost nowadays. Complexifying the variables  $(p, q)$  in terms of  $z_1 = p + iq$ ,  $z_2 = p - iq$ , she introduced a pair of variables  $w_1, w_2$ , as the roots of a certain quadratic equation,

$$\begin{aligned} w_{1,2} &= \frac{R(z_1, z_2) \mp \sqrt{R(z_1)R(z_2)}}{(z_1 - z_2)^2} \\ R(z) &= -z^4 + 2hz^2 - 2clz + c^2 - k^2 \\ R(z_1, z_2) &= -z_1^2 z_2^2 + 2hz_1 z_2 - cl(z_1 + z_2) + c^2 - k^2, \end{aligned} \tag{6}$$

and showed that the equations of motion can be written in the form

$$\dot{w}_1 = \frac{\sqrt{P_5(w_1)}}{w_2 - w_1}, \quad \dot{w}_2 = \frac{\sqrt{P_5(w_2)}}{w_1 - w_2}, \tag{7}$$

where

$$P_5(w) = -2P_2(w)P_3(w), \quad P_2(w) = w^2 - k^2, \quad P_3(w) = (w + h)(P_2(w) + c^2) - \frac{1}{2}c^2l^2. \tag{8}$$

Similar equations were known from the classical work of C. G. Jacobi about the geodesics on an ellipsoid. K. Weierstrass and C. Neumann had shown how they may be used to derive explicit formulas for the solutions in terms of theta-functions, so the rest of Kovalevskaya's work was essentially technical.

It is interesting, and more important than it seems at first sight, that S. Kovalevskaya introduced the shifted variables

$$s_i = w_i + H, \quad (9)$$

which of course satisfy similar equations,

$$\dot{s}_1 = \frac{\sqrt{R_5(s_1)}}{s_2 - s_1}, \quad \dot{s}_2 = \frac{\sqrt{R_5(s_2)}}{s_1 - s_2}, \quad (10)$$

where

$$R_5(s) = -2R_2(s)R_3(s), \quad R_2(s) = (s - h)^2 - k^2, \quad R_3(s) = s(R_2(s) + c^2) - \frac{1}{2}c^2l^2. \quad (11)$$

The reasons why she made this shift seem to be unclear today (see e. g. Golubev's book [G53]); somehow the shifted variables were better suited for the theory of elliptic functions from which Kovalevskaya's choice was inspired.

Amazingly, the variables  $s_1, s_2$  were exactly the right ones for Novikov and Veselov [VN84] to apply to the Kovalevskaya top the theory of algebro-geometric Poisson brackets! More precisely, it can be shown by direct calculation that the Poisson bracket of the variables  $s_1$  and  $s_2$  in the Poisson structure (1) is zero

$$\{s_1, s_2\} = 0. \quad (12)$$

This is not true for the unshifted variables  $w_1, w_2$ : one can check that  $\{w_1, w_2\} \neq 0$ . Novikov and Veselov were able to show that in the variables  $s_1$  and  $s_2$ , the symplectic structure  $\omega$  has the form

$$\omega = d\alpha, \quad \text{where} \quad \alpha = Q(h, k, l, s_1) ds_1 + Q(h, k, l, s_2) ds_2 \quad (13)$$

with

$$Q(h, k, l, s) = \frac{1}{2\sqrt{-2s}} \log \left[ \sqrt{-s}R_2(s) - \frac{(2s - l^2)c^2}{4\sqrt{-s}} + \sqrt{-R_2(s)R_3(s)} \right]. \quad (14)$$

To get an intuition for how this result may be obtained, we observe that Kovalevskaya's equations of motion (10) may be generalized to include the flow generated by the Kovalevskaya integral  $K$ , with flow parameter  $\tau$ :

$$\begin{aligned} dt &= \frac{h - s_1}{\sqrt{R_5(s_1)}} ds_1 + \frac{h - s_2}{\sqrt{R_5(s_2)}} ds_2, \\ d\tau &= \frac{-1/2}{\sqrt{R_5(s_1)}} ds_1 + \frac{-1/2}{\sqrt{R_5(s_2)}} ds_2. \end{aligned} \quad (15)$$

For  $\tau = \text{const}$ , we recover the Hamiltonian flow (10); the second equation (15) then describes the trajectory in  $(s_1, s_2)$  coordinates while the first gives the time development. Similarly, for  $t = \text{const}$  we obtain the  $K$ -flow in phase space. Now the periods of the two flows,  $T_{H,i} = \oint_{\gamma_i} dt$  and  $T_{K,i} = \oint_{\gamma_i} d\tau$ , taken along fundamental cycles  $\gamma_1, \gamma_2$  of the Kovalevskaya curve (see Figure 1)

$$\Gamma : y^2 = R_5(x), \quad (16)$$

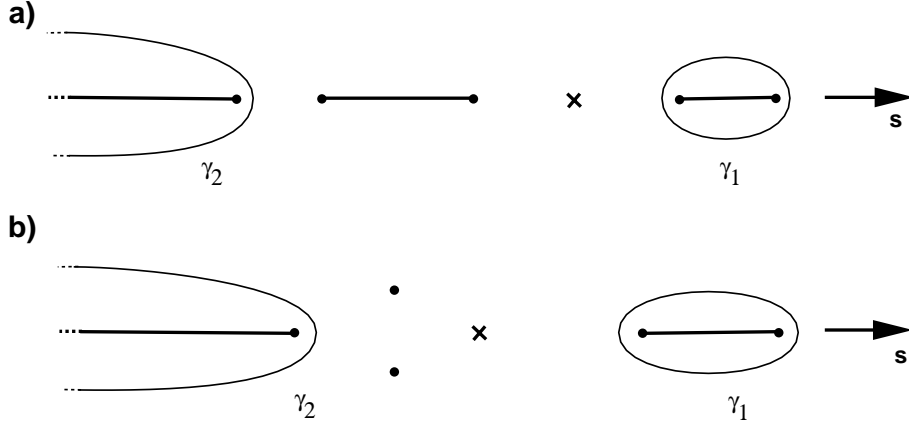


Figure 1: The cycles  $\gamma_1$  and  $\gamma_2$  for the cases a) when  $R_5(s)$  has 5 real roots and b) when  $R_5(s)$  has 3 real roots. The cross marks the position of the pole at  $s = l^2/2$ , which in case a) can be in either of the two gaps; the dots mark the roots of  $R_5$ , connected by intervals in which  $R_5(s) > 0$ .

ought to be expressed as

$$T_{H,i} = 2\pi \frac{\partial I_i}{\partial h} \quad \text{and} \quad T_{K,i} = 2\pi \frac{\partial I_i}{\partial k^2}, \quad (17)$$

where

$$I_i = \frac{1}{2\pi} \oint_{\gamma_i} Q(h, k, l, s) ds \quad (18)$$

are the action variables of the system, and hence  $Q(h, k, l, s_i)$  the canonically conjugate momenta to  $s_i$  ( $i = 1, 2$ ). Comparison with (15) shows that these momenta must fulfill the relations

$$\begin{aligned} \frac{\partial Q}{\partial h} &= \frac{h - s}{\sqrt{R_5(s)}} \\ \frac{\partial Q}{\partial k^2} &= \frac{-1/2}{\sqrt{R_5(s)}}. \end{aligned} \quad (19)$$

The explicit form (14) of  $Q(h, k, l, s)$  may be derived from here by integration.

Notice that in the form (18) the action variables  $I_i$  depend on the branch of the multivalued function  $Q$  on  $\Gamma$ . This implies only changes of  $I_i$  by some constants, but since these constants may be complex there is a problem of choosing real action variables.

To resolve these problems we rewrite first of all the expression for  $Q$  in the following way using the fact that  $(2s - l^2)c^2 = 2R_3(s) - 2sR_2(s)$ :

$$\begin{aligned} Q(h, k, l, s) &= Q_l(s) + \frac{1}{2\sqrt{-2s}} \log \left[ \frac{\sqrt{sR_2(s)} + \sqrt{R_3(s)}}{\sqrt{sR_2(s)} - \sqrt{R_3(s)}} \right] \\ &= Q_l(s) + \frac{1}{\sqrt{-2s}} \log(\Psi + \sqrt{\Psi^2 - 1}) \\ &= Q_l(s) + \frac{1}{\sqrt{2s}} \arccos \Psi, \end{aligned} \quad (20)$$

where

$$Q_l(s) = \frac{1}{2\sqrt{-2s}} \log \frac{(2s-l^2)c^2}{4\sqrt{-s}} \quad \text{and} \quad \Psi = \sqrt{\frac{-2sR_2(s)}{(2s-l^2)c^2}}. \quad (21)$$

Notice that  $Q_l(s)$  does not depend on the constants  $h$  and  $k$ .

Now we can write the 1-form  $Q ds$  as

$$Q ds = \beta + \lambda + d\Phi \quad (22)$$

where

$$\beta = \frac{-4s^3 + (4h + 3l^2)s^2 - 4hl^2s + l^2(h^2 - k^2)}{\sqrt{R_5(s)}(2s - l^2)} ds, \quad (23)$$

$$\lambda = \frac{2s + l^2}{2\sqrt{-2s}(2s - l^2)} ds, \quad (24)$$

and  $d\Phi$  is a total differential. This can be done by means of partial integrations, using

$$\frac{1}{\sqrt{2s}} \arccos \Psi ds = \arccos \Psi d\sqrt{2s} = -\sqrt{2s} d\arccos \Psi + d\Phi_1 = \beta + d\Phi_1, \quad (25)$$

and

$$Q_l ds = \lambda + d\Phi_2, \quad (26)$$

where

$$\Phi_1 = \sqrt{2s} \arccos \Psi, \quad \Phi_2 = -\frac{1}{2} \sqrt{-2s} \log \frac{(2s-l^2)c^2}{4\sqrt{-s}}, \quad \Phi = \Phi_1 + \Phi_2. \quad (27)$$

Taking advantage of the fact that  $\lambda$  does not depend on  $h$  and  $k$ , the integrals (18) with (14) for the actions  $I_1$  and  $I_2$  may now be expressed by the following abelian integrals of the third kind on the curve  $\Gamma$ :

$$I_i = \frac{1}{2\pi} \oint_{\gamma_i} \beta = \frac{1}{2\pi} \oint_{\gamma_i} \frac{-4s^3 + (4h + 3l^2)s^2 - 4hl^2s + l^2(h^2 - k^2)}{\sqrt{R_5(s)}(2s - l^2)} ds \quad (i = 1, 2). \quad (28)$$

Comparison with numerical calculation of action integrals obtained using  $(1/2\pi) \oint p dq$  on  $T^*SO(3)$  [DJR94] shows complete agreement with this formula.

It is interesting that the third action  $I_3 = l$  of the Kovalevskaya top on  $T^*SO(3)$  can also be obtained from the differential (22) because both  $\beta$  and  $\lambda$  have poles at  $s = l^2/2$  with residues  $\pm il/2$ .

Notice that the differential  $\beta + \lambda$  is living not on the Kovalevskaya curve but on its double covering  $\tilde{\Gamma}$  which is the curve of genus 4 given by the system of equations

$$\begin{aligned} y^2 &= R_5(s) \\ z^2 &= -s. \end{aligned} \quad (29)$$

Actually, one can see this curve already in the original Novikov-Veselov formula (14). The fact that one can write down the actions of the Kovalevskaya system (3) as abelian integrals (28), although of third kind, on the Kovalevskaya curve  $\Gamma$  itself, was not clear from the formula (14) and seems to be a new important observation.

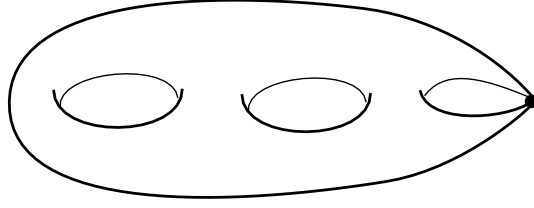


Figure 2: The curve  $\Gamma_{\text{sing}}$

### 3 Picard-Fuchs equation

In this section we derive the Picard-Fuchs equation for the actions  $I_1, I_2$  as functions of  $h$  and  $k$ . Since  $I_1$  and  $I_2$  are periods of the abelian differential  $\beta$  on  $\Gamma$  one can use the standard arguments to derive such equations (see e. g. [BK86]). Let us first explain the general idea underlying this derivation.

First we must overcome the difficulty related to the fact that  $\beta$  has nonzero residues. To do this we use a trick: Consider the following augmented version  $\Gamma_{\text{sing}}$ , of the Kovalevskaya curve  $\Gamma$ , given by the equation

$$\Gamma_{\text{sing}} : y^2 = R_5(x)(2x - l^2)^2. \quad (30)$$

It is a singular algebraic curve of arithmetic genus 3 with a double point at  $x = l^2/2$ . Its nonsingular model coincides with the Kovalevskaya curve  $\Gamma$  and has genus 2. Topologically  $\Gamma_{\text{sing}}$  is a genus 3 surface with one cycle pinched. It is important for us that our differential  $\beta$  can be considered as a second kind differential on  $\Gamma_{\text{sing}}$  since  $\beta$  is regular at the singular point.

The first cohomology group of  $\Gamma_{\text{sing}}$  has dimension 5, therefore by de Rham theory there exist 5 abelian differentials of second kind on  $\Gamma_{\text{sing}}$  which generate the space of all such differentials modulo total derivatives of meromorphic functions on  $\Gamma_{\text{sing}}$ . In particular, we can take the standard basis of hyperelliptic differentials

$$\omega_i = \frac{x^{i-1} dx}{\sqrt{R_5(x)}}, \quad i = 1, 2, 3, 4, \quad (31)$$

plus the additional differential  $\omega_5 = \beta$  which is our 1-form (28). As we have mentioned already, as a differential on  $\Gamma_{\text{sing}}$  the form  $\beta$  is regular at the point  $x = l^2/2$ ; it is de Rham dual to the vanishing cycle on  $\Gamma_{\text{sing}}$ . Now if we take the derivatives

$$\frac{\partial \omega_i}{\partial h}, \quad \frac{\partial \omega_i}{\partial k^2}, \quad i = 1, \dots, 5 \quad (32)$$

they have to be a linear combination of  $\omega_i$  modulo total derivatives:

$$\begin{aligned} \frac{\partial \omega_i}{\partial h} &= \sum_{j=1}^5 A_{ij}(h, k, l) \omega_j + dF_i \\ \frac{\partial \omega_i}{\partial k^2} &= \sum_{j=1}^5 B_{ij}(h, k, l) \omega_j + dG_i, \quad i = 1, \dots, 5, \end{aligned} \quad (33)$$

and therefore the periods

$$\pi_i = \oint_{\gamma} \omega_i, \quad i = 1, \dots, 5, \quad (34)$$

along any cycle  $\gamma$  on  $\Gamma_{\text{sing}}$  should satisfy the Picard-Fuchs equation

$$\begin{aligned} \frac{\partial \pi_i}{\partial h} &= \sum_{j=1}^5 A_{ij}(h, k, l) \pi_j \\ \frac{\partial \pi_i}{\partial k^2} &= \sum_{j=1}^5 B_{ij}(h, k, l) \pi_j. \quad i = 1, \dots, 5. \end{aligned} \quad (35)$$

Actually, we can avoid the consideration of the singular curves because of the following fact. Let  $\pi_5 = \oint \omega_5 = \oint \beta$  be any period of the form  $\beta$ ; then

$$\begin{aligned} \frac{\partial \pi_5}{\partial h} &= h\pi_1 - \pi_2, \\ \frac{\partial \pi_5}{\partial k^2} &= -\frac{1}{2}\pi_1. \end{aligned} \quad (36)$$

This follows from the properties of the form  $Q$ , see (19) above.

Since the abelian differentials  $\omega_1, \dots, \omega_4$  live on the nonsingular Kovalevskaya curve  $\Gamma$ , the rest of the Picard-Fuchs equations can be derived for periods  $\pi_i$  on  $\Gamma$ ,  $i = 1, 2, 3, 4$ . The calculation is quite lengthy, and the result is given in the appendix. Here we only present the algorithm used in an algebraic manipulation program such as Maple.

Consider an abelian differential, polynomially depending on a parameter  $c$ ,

$$\omega = \frac{Q(x, c)}{\sqrt{R(x, c)}} dx, \quad (37)$$

and its derivative with respect to the parameter,

$$\frac{\partial \omega}{\partial c} = \frac{P(x, c)}{\sqrt{R(x, c)}^3} dx, \quad \text{with} \quad P(x, c) = R(x, c) \frac{\partial Q(x, c)}{\partial c} - \frac{1}{2} Q(x, c) \frac{\partial R(x, c)}{\partial c}. \quad (38)$$

The goal is to remove the third power of the square root, and to rewrite the differential in the form

$$\frac{P(x, c)}{\sqrt{R(x, c)}^3} dx = \frac{S(x, c)}{\sqrt{R(x, c)}} dx + dF. \quad (39)$$

This can be achieved by decomposing  $P$  into a combination of  $R$  and its derivative,

$$P(x, c) = A(x, c)R(x, c) + B(x, c) \frac{\partial R(x, c)}{\partial x}, \quad (40)$$

where  $A(x, c)$  and  $B(x, c)$  are polynomials in  $x$  and  $c$ . Then integration by parts gives

$$\begin{aligned} \frac{P(x, c)}{\sqrt{R(x, c)}^3} dx &= \frac{A(x, c)}{\sqrt{R(x, c)}} dx + \frac{B(x, c)}{\sqrt{R(x, c)}^3} \frac{\partial R(x, c)}{\partial x} dx \\ &= \frac{A(x, c)}{\sqrt{R(x, c)}} dx + \frac{2}{\sqrt{R(x, c)}} \frac{\partial B(x, c)}{\partial x} dx - 2d \left( \frac{B(x, c)}{\sqrt{R(x, c)}} \right), \end{aligned} \quad (41)$$



and the result is

$$S(x, c) = A(x, c) + 2 \frac{\partial B(x, c)}{\partial x}. \quad (42)$$

Thus the main task is to find the decomposition (40). We assume that the polynomial  $R$  has no multiple roots for generic  $c$  and its highest coefficient does not depend on  $c$ . If the degree of  $R$  is  $n$ , then for any polynomial  $P(x, c)$  of degree  $\deg P \leq 2n - 2$  there exists a decomposition (40) with some polynomials  $A$  and  $B$  of degree  $n - 2$  and  $n - 1$  respectively. Such a decomposition can be found in the usual way, applying the Euclidian algorithm for two polynomials  $R$  and  $R'$ . A more effective procedure is to write down the polynomials  $A$  and  $B$  in the general form  $A(x, c) = \sum_{i=0}^{n-2} a_i(c)x^i$ ,  $B(x, c) = \sum_{i=0}^{n-1} b_i(c)x^i$ , with unknown coefficients  $a_i(c)$  and  $b_i(c)$ . Then the decomposition (40) is equivalent to a system of  $2n - 1$  linear equations for these coefficients. The determinant of this linear system is proportional to the discriminant of the polynomial  $R$ .

For the Kovalevskaya case we have to solve a system of 9 equations in order to express all partial derivatives of differentials  $\omega_i$  in the form (33).

The discriminant of  $R = R_5(s)$  in this case is  $256 k^2 \delta^2 \Delta$ , where

$$\begin{aligned} \delta &= \text{resultant}\{R_2(s), R_3(s)\} = c^4(h - k - l^2/2)(h + k - l^2/2) \\ \Delta &= \text{discriminant}\{R_3(s)\} \\ &= -\frac{27}{4}c^4l^4 + 8hc^2(h^2 + 9(c^2 - k^2))l^2 - 16(c^2 - k^2)(h^2 + c^2 - k^2)^2 \end{aligned} \quad (43)$$

Each coefficient  $A_{ij}$ ,  $B_{ij}$  is a polynomial in  $k^2$ , with degree at most 5, divided by  $k^2 \delta \Delta$ . Therefore each coefficient of the Picard-Fuchs equation can be written as

$$a + \frac{b}{k^2} + \frac{c}{\delta} + \frac{d_0 + d_1k^2 + d_2k^4}{\Delta}, \quad (44)$$

where  $a, b, c, d_0, d_1, d_2$  are rational functions in  $h$  and  $l$ . Altogether we obtain a set of  $2 \times 16 \times 6$  such functions of  $h$  and  $l$ . The structure of the coefficients reveals that the double roots of  $R_5(s)$  correspond to the singular points of the Picard-Fuchs equation. Consider the appendix for details.

Thus we arrive at our main result: Any action variable  $I$  of the Kovalevskaya system satisfies the following set of Picard-Fuchs equations

$$\begin{aligned} \frac{\partial I}{\partial h} &= h\pi_1 - \pi_2 & \frac{\partial I}{\partial k^2} &= -\frac{1}{2}\pi_1 \\ \frac{\partial \pi_i}{\partial h} &= \sum_{j=1}^4 A_{ij}(h, k, l)\pi_j & \frac{\partial \pi_i}{\partial k^2} &= \sum_{j=1}^4 B_{ij}(h, k, l)\pi_j, \quad i = 1, 2, 3, 4, \end{aligned} \quad (45)$$

where the coefficients  $A_{ij}$  and  $B_{ij}$  are those given in the appendix.

The monodromy group of this system is isomorphic to the modular group  $\text{Sp}(4, \mathbb{Z})$ , acting as automorphisms in the first homology of the corresponding Kovalevskaya curve.

Notice that as a corollary we can deduce that  $I$  satisfies certain linear equations of 5th order,

$$\begin{aligned}\frac{\partial^5 I}{\partial h^5} &= \sum_{i=1}^4 a_i(h, k, l) \frac{\partial^i I}{\partial h^i} \\ \frac{\partial^5 I}{\partial (k^2)^5} &= \sum_{i=1}^4 b_i(h, k, l) \frac{\partial^i I}{\partial (k^2)^i}.\end{aligned}\tag{46}$$

Since there are only derivatives of  $I$  in these equations, there is a constant solution which is  $I = l$ . The remaining four solutions are periods of the 1-form  $\beta$  on the Kovalevskaya curve. The actions  $I_1$  and  $I_2$  correspond to two special real solutions of these equations.

## 4 Conclusion

The determination of action variables for an integrable mechanical system is an important part of its analysis. The explicit representation of a Hamiltonian in terms actions,  $H = H(\{I_i\})$ , tends to change at bifurcations of the momentum mapping. Nevertheless, within domains of regularity, the geometry of surfaces  $H(\{I_i\}) = h$  contains all relevant information about physical periods and resonances. It is a highly convenient starting point for studies of stability against non-integrable perturbations, and for semi-classical quantization [B78].

Strangely enough, many well-known integrable systems of classical mechanics have not been evaluated to this point, until very recently. Meanwhile, a few cases have been worked out in detail, including graphical rendering of the energy surfaces. As examples, we mention the work [RWKK95, WR96] on ellipsoidal billiards, and [RDWW96, WWD97] on quantization. For the Euler and Lagrange cases of rigid body dynamics, the action representation of energy surfaces was first given in [R90]. The Kovalevskaya case turned out to be much more difficult. The explicit integral (18) with  $Q$  in the form (14) was given as early as 1984 by [VN84], but this was not used in the numerical study [DJR94]. The non-algebraic structure of this integral causes computational difficulties.

The present paper offers two sets of formulas which overcome these difficulties. One is the abelian integral (28). Its explicit evaluation requires integration, for each set  $(h, k, l)$  of integration constants, along paths  $\gamma_i$  on the Kovalevskaya curve, as shown in Fig. 1. This is a lot of work, but it can be done. The alternative is to use the set (45) of Picard-Fuchs equations to obtain, for given  $l$ , the actions  $I_{1,2}$  along paths in the  $(h, k)$  plane. Integration at fixed energy  $h$  thus produces an entire line on the energy surface. However, the set of coefficients  $A_{ij}$  and  $B_{ij}$  is so cumbersome that it is by no means obvious which of the two methods gives faster access to the desired results. This will be investigated in future work.

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## References

- [AM82] M. Adler, P. van Moerbeke. *Kowalewski's asymptotic method, Kac-Moody Lie algebras and regularization*. Comm. Math. Phys. **83** (1982), 83-106.
- [A78] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer, New York 1978.
- [B78] M. V. Berry. "Regular and irregular motion", in: *Topics in Nonlinear Dynamics*. AIP Conf. Proc. **46**(1978), 16-120.
- [BRS89] A. I. Bobenko, A. G. Reyman, and M. A. Semenov-Tian-Shansky. *The Kowalewski top 99 years later: A Lax pair, generalizations and explicit solutions*. Commun. Math. Phys. **122**(1989), 321-354.
- [BK86] E. Brieskorn, H. Knörrer. *Plane Algebraic Curves*. Birkhäuser, Basel 1986
- [DFN90] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov. *Modern Geometry – Methods and Applications. Part 3: Introduction to Homology Theory*. Springer, Berlin 1990.
- [DMN76] B. A. Dubrovin, V. B. Matveev, S. P. Novikov. *Nonlinear equations of KdV type, finite-gap linear operators and abelian varieties*. Russ. Math. Surv. **31/1** (1976), 59-146.
- [DJR94] H. R. Dullin, M. Juhnke, and P. H. Richter. *Action integrals and energy surfaces of the Kovalevskaya top*. Bifurcation and Chaos **4/6**(1994), 1535-1562.
- [DW94] H. R. Dullin and A. Wittek. *Efficient calculation of actions*. J. Phys. A **27**(1994), 7461-7474.
- [F87] J. P. Francoise. *Monodromy and the Kovalevskaya top*. Astérisque **150-151**(1987), 87-108.
- [GGKM67] C. S. Gardner, J. Green, M. Kruskal, R. Miura. *Method for solving the Korteweg-de Vries equation*. Phys. Rev. Lett. **19**(1967), 1095-1097.
- [G53] V. V. Golubev. *Lectures on Integration of the Equations of Motion of a Rigid Body about a Fixed Point*. State Publ. House of Theoretical Technical Literature, Moscow 1953.
- [HH87] L. Haine and E. Horozov. *A Lax pair for Kowalewski's top*. Physica **29D**(1987), 173-180.
- [HM89] E. Horozov and P. van Moerbeke. *The full geometry of Kowalewski's top and (1,2)-Abelian surfaces*. Commun. Pure Appl. Math. **42**(1989), 357-407.

- [K89] S. Kowalewski. *Sur le problème de la rotation d'un corps solide d'un point fixe*. Acta Math. **12**(1889), 177-232.
- [KK87] I. V. Komarov, V. B. Kuznetsov. *Quasiclassical quantization of the Kovalevski top*. Theor. and Math. Physics **73** (1987), 335-343.
- [R90] P. H. Richter. *Die Theorie des Kreisels in Bildern*. Report Nr. 226 Inst. f. Dynamische Systeme, Bremen 1990.
- [RWKK95] P. H. Richter, A. Wittek, M. P. Kharlamov, A. P. Kharlamov. *Action integrals for ellipsoidal billiards*. Z. Naturforsch. **50a**(1995), 693-710.
- [RDWW96] P. H. Richter, H. R. Dullin, H. Waalkens, J. Wiersig. *Spherical pendulum, actions, and spin*. J. Phys. Chem. **100/49**(1996), 19124-19135.
- [VN84] A. P. Veselov and S. P. Novikov. *Poisson brackets and complex tori*. Proc. Steklov Inst. of Math. **165**(1985), 53-65.
- [WWD97] H. Waalkens, J. Wiersig, H. R. Dullin. *Elliptic quantum billiard*. Ann. Phys. **260**(1997), 50-90.
- [WR96] J. Wiersig, P. H. Richter. *Energy surfaces of ellipsoidal billiards*. Z. Naturforsch. **51a**(1996), 219-241.

## Appendix

### A Coefficients $A_{ij}$

In addition to the abbreviations  $\delta$  and  $\Delta$  introduced in (43), the following expressions use

$$\mu = k^2 - 1 \quad \text{and} \quad \nu = l^4 k^2 - 1.$$

The 16 coefficients  $A_{ij}$  in (45) are

$$\begin{aligned}
A_{0,0} = & -\frac{1}{8} \frac{l^2}{\mu} + \frac{1}{16} \frac{(-l^8 - 4(4k^2 + 3)l^4)h}{\nu\delta} + \frac{1}{32} \frac{l^{10} + 10l^6 - 8(4k^2 + 1)l^2}{\nu\delta} \\
& + \frac{1}{4} \frac{(l^8 + (16k^6 - 20k^4 - 7k^2 + 10)l^4)h^3}{\mu\nu\Delta} + \frac{1}{8} \frac{((34k^2 - 1)l^6 + 4(-11 + 2k^2 + 8k^4)l^2)h^2}{\nu\Delta} \\
& + \frac{1}{16} \frac{(-9l^8 - 4(-11k^2 + 23 + 16k^6 - 52k^4)l^4)h}{\nu\Delta} \\
& + \frac{1}{32} \frac{-27l^{10} + (88k^6 - 228k^4 + 426k^2 - 259)l^6 - 16(8k^4 - 14k^2 + 9)(k^2 - 1)^2 l^2}{\mu\nu\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{0,1} = & \frac{1}{8} \frac{(5l^6 + 4(4k^2 + 7)l^2)h}{\nu\delta} + \frac{1}{16} \frac{-5l^8 + 2(16k^2 - 9)l^4 + 32k^2 + 8}{\nu\delta} \\
& + \frac{1}{2} \frac{(5l^6 - (16k^4 + 12k^2 - 23)l^2)h^3}{\nu\Delta} + \frac{1}{4} \frac{(-(112k^4 - 5 - 58k^2)l^4 - 8(k^2 - 1)(4k^2 - 5))h^2}{\nu\Delta} \\
& + \frac{1}{8} \frac{(-3(68k^2 - 15)l^6 + 4(-71k^2 - 36k^4 + 94 + 16k^6)l^2)h}{\nu\Delta} \\
& + \frac{1}{16} \frac{-135l^8 + 2(-283 + 326k^2 + 96k^6 - 292k^4)l^4 + 32(4k^2 - 5)(k^2 - 1)^2}{\nu\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{0,2} = & \frac{1}{4} \frac{(-16 - 7l^4)h}{\nu\delta} + \frac{1}{8} \frac{7l^6 - 2(16k^2 - 1)l^2}{\nu\delta} + \frac{(7(k^2 - 1)l^4 + 16k^2 - 16)h^3}{\nu\Delta} \\
& + \frac{1}{2} \frac{l^2(16k^2 + 7)(2k^2 - 1)h^2}{\nu\Delta} + \frac{1}{4} \frac{((100k^4 + 32k^2 - 63)l^4 - 64(k^2 - 1)^2)h}{\nu\Delta} \\
& + \frac{1}{8} \frac{9(21 + 2k^2)l^6 - 4(k^2 - 1)(32k^4 - 66k^2 + 37)l^2}{\nu\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{0,3} = & \frac{3}{2} \frac{l^2 h}{\nu\delta} + \frac{3}{4} \frac{2 - l^4}{\nu\delta} + \frac{(-6k^2 + 6)l^2 h^3}{\nu\Delta} + 3 \frac{(-l^4 k^2 + 2 - 2k^2)h^2}{\nu\Delta} + \frac{1}{2} \frac{(12k^4 - 48k^2 + 27)l^2 h}{\nu\Delta} \\
& + \frac{3}{4} \frac{-3(9 + 4k^4 - 10k^2)l^4 + 8(k^2 - 1)^2}{\nu\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{1,0} = & -\frac{1}{8} \frac{l^2 h}{\mu} + \frac{1}{16} \frac{-(k^2 - 2)l^4 + 8(k^2 - 1)^2}{\mu^2} + \frac{1}{32} \frac{(-l^{10} - 4(4k^2 + 3)l^6)h}{\nu\delta} \\
& + \frac{1}{64} \frac{l^{12} + 10l^8 - 8(4k^2 + 1)l^4}{\nu\delta} \\
& + \frac{1}{8} \frac{(-l^{10} + (16k^6 - 43k^4 + 37k^2 - 9)l^6 + 4(4k^2 - 5)(k^2 - 1)^2 l^2)h^3}{\nu\mu^2\Delta} \\
& + \frac{1}{16} \frac{(-l^8 + 2(-5 + 4k^2 + 16k^4)l^4)h^2}{\nu\Delta} \\
& + \frac{1}{32} \frac{(9l^{10} + (64k^6 + 60k^4 - 216k^2 + 83)l^6 - 16(4k^2 - 5)(k^2 - 1)^2 l^2)h}{\nu\mu\Delta} \\
& + \frac{1}{64} \frac{27l^{12} - (669k^2 - 232 - 534k^4 + 124k^6)l^8 - 8(16k^6 - 44k^4 + 35k^2 - 16)(k^2 - 1)^2 l^4}{\nu\mu^2\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{1,1} = & \frac{5}{8} \frac{l^2}{\mu} + \frac{1}{16} \frac{(5l^8 + 4(4k^2 + 7)l^4)h}{\nu\delta} + \frac{1}{32} \frac{-5l^{10} + 2(16k^2 - 9)l^6 + 8(4k^2 + 1)l^2}{\nu\delta} \\
& + \frac{1}{4} \frac{(-5l^8 - (-35k^2 + 18 - 4k^4 + 16k^6)l^4)h^3}{\mu\nu\Delta} + \frac{1}{8} \frac{(-58k^2 - 5)l^6 - 4(8k^4 + 10k^2 - 19)l^2)h^2}{\nu\Delta} \\
& + \frac{1}{16} \frac{(45l^8 + 4(-71k^2 + 43 + 16k^6 - 36k^4)l^4)h}{\nu\Delta} \\
& + \frac{1}{32} \frac{135l^{10} - (248k^6 - 756k^4 + 1074k^2 - 431)l^6 + 16(4k^2 - 1)(2k^2 - 1)(k^2 - 1)^2 l^2}{\mu\nu\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{1,2} = & \frac{1}{8} \frac{(-7l^6 - 16l^2)h}{\nu\delta} + \frac{1}{16} \frac{7l^8 - 2(16k^2 - 1)l^4}{\nu\delta} + \frac{1}{2} \frac{(-7l^6 + (16k^2 - 9)l^2)h^3}{\nu\Delta} \\
& + \frac{1}{4} \frac{l^4(16k^2 + 7)(2k^2 - 1)h^2}{\nu\Delta} + \frac{1}{8} \frac{(3(44k^2 - 21)l^6 - 4(k^2 - 1)(16k^2 - 41)l^2)h}{\nu\Delta} \\
& + \frac{1}{16} \frac{189l^8 - 2(64k^6 - 196k^4 - 83 + 206k^2)l^4}{\nu\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{1,3} = & \frac{3}{4} \frac{l^4 h}{\nu\delta} + \frac{1}{8} \frac{-3l^6 + 6l^2}{\nu\delta} + \frac{(-3k^2 + 3)l^4 h^3}{\nu\Delta} + \frac{1}{2} \frac{(-6k^2 + 3)l^2 h^2}{\nu\Delta} + \frac{1}{4} \frac{(12k^4 - 48k^2 + 27)l^4 h}{\nu\Delta} \\
& + \frac{1}{8} \frac{27(-3 + 2k^2)l^6 + 12(k^2 - 1)(2k^2 - 5)l^2}{\nu\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{2,0} = & -\frac{1}{8} \frac{l^2 h^2}{\mu} + \frac{1}{16} \frac{(-k^2 + 2)l^4 h}{\mu^2} + \frac{1}{32} \frac{-(3 + k^4 - 3k^2)l^6 - 4(k^2 - 2)(k^2 - 1)^2 l^2}{\mu^3} \\
& + \frac{1}{64} \frac{(-l^{12} - 4(4k^2 + 3)l^8)h}{\nu\delta} + \frac{1}{128} \frac{l^{14} + 10l^{10} - 8(4k^2 + 1)l^6}{\nu\delta} \\
& + \frac{1}{16} \frac{(l^{12} - (-40k^4 - 8 + 15k^6 + 34k^2)l^8 + 4(4k^4 - 5k^2 + 2)(k^2 - 1)^2 l^4)h^3}{\nu\mu^3\Delta} \\
& + \frac{1}{32} \frac{(l^{10} + (32k^6 - 24k^4 - 18k^2 + 9)l^6)h^2}{\nu\mu\Delta} \\
& + \frac{1}{64} \frac{(-9l^{12} + (236k^6 - 564k^4 + 411k^2 - 74)l^8 - 16(4k^4 - 13k^2 + 12)(k^2 - 1)^2 l^4)h}{\mu^2\nu\Delta} \\
& + \frac{1}{128} \frac{-27l^{14} + (394k^6 - 1047k^4 + 885k^2 - 205)l^{10} - 4(32k^8 - 120k^6 + 158k^4 - 71k^2 + 28)(k^2 - 1)^2 l^6}{\nu\mu^3\Delta}
\end{aligned}$$

$$\begin{aligned}
A_{2,1} = & \frac{5}{8} \frac{l^2 h}{\mu} + \frac{1}{16} \frac{5(k^2 - 2)l^4 + 24(k^2 - 1)^2}{\mu^2} + \frac{1}{32} \frac{(5l^{10} + 4(4k^2 + 7)l^6)h}{\nu \delta} \\
& + \frac{1}{64} \frac{-5l^{12} + 2(16k^2 - 9)l^8 + 8(4k^2 + 1)l^4}{\nu \delta} \\
& + \frac{1}{8} \frac{(5l^{10} - (73k^2 - 87k^4 - 13 + 32k^6)l^6 - 4(4k^2 - 5)(k^2 - 1)^2 l^2)h^3}{\nu \mu^2 \Delta} \\
& + \frac{1}{16} \frac{(5l^8 - 2(16k^4 + 20k^2 - 9)l^4)h^2}{\nu \Delta} \\
& + \frac{1}{32} \frac{(-45l^{10} - (-536k^2 + 364k^4 + 127)l^6 + 16(4k^2 - 5)(k^2 - 1)^2 l^2)h}{\nu \mu \Delta} \\
& + \frac{1}{64} \frac{-135l^{12} + (332k^6 - 1230k^4 + 1329k^2 - 296)l^8 + 8(16k^6 - 28k^4 - 17k^2 + 20)(k^2 - 1)^2 l^4}{\nu \mu^2 \Delta}
\end{aligned}$$

$$\begin{aligned}
A_{2,2} = & -\frac{7}{8} \frac{l^2}{\mu} + \frac{1}{16} \frac{(-7l^8 - 16l^4)h}{\nu \delta} + \frac{1}{32} \frac{7l^{10} - 2(16k^2 - 1)l^6}{\nu \delta} + \frac{1}{4} \frac{(7l^8 + (16k^4 - 25k^2 + 2)l^4)h^3}{\mu \nu \Delta} \\
& + \frac{1}{8} \frac{((30k^2 - 7)l^6 + 32(k^2 - 1)l^2)h^2}{\nu \Delta} + \frac{1}{16} \frac{(-63l^8 - 4(16k^4 - 57k^2 + 8)l^4)h}{\nu \Delta} \\
& + \frac{1}{32} \frac{-189l^{10} + (136k^6 - 420k^4 + 450k^2 + 23)l^6 - 128(k^2 - 1)^3 l^2}{\mu \nu \Delta}
\end{aligned}$$

$$\begin{aligned}
A_{2,3} = & \frac{3}{8} \frac{l^6 h}{\nu \delta} + \frac{1}{16} \frac{-3l^8 + 6l^4}{\nu \delta} + \frac{1}{2} \frac{(3l^6 - 3l^2)h^3}{\nu \Delta} + \frac{1}{4} \frac{(-6k^2 + 3)l^4 h^2}{\nu \Delta} \\
& + \frac{1}{8} \frac{(-9(4k^2 - 3)l^6 + 12(k^2 - 1)l^2)h}{\nu \Delta} + \frac{1}{16} \frac{-81l^8 + 6(4k^4 - 14k^2 + 19)l^4}{\nu \Delta}
\end{aligned}$$

$$\begin{aligned}
A_{3,0} = & -\frac{1}{8} \frac{l^2 h^3}{\mu} + \frac{1}{16} \frac{(-k^2 + 2)l^4 h^2}{\mu^2} + \frac{1}{32} \frac{(-3 - 3k^2 + k^4)l^6 + 4(k^2 - 1)^2 k^2 l^2)h}{\mu^3} \\
& + \frac{1}{64} \frac{-(k^2 - 2)(k^4 - 2k^2 + 2)l^8 - 2(k^2 + 1)(2k^2 - 1)(k^2 - 1)^2 l^4}{\mu^4} + \frac{1}{128} \frac{(-l^{14} - 4(4k^2 + 3)l^{10})h}{\nu \delta} \\
& + \frac{1}{256} \frac{l^{16} + 10l^{12} - 8(4k^2 + 1)l^8}{\nu \delta} \\
& + \frac{1}{32} \frac{(-l^{14} + (14k^6 - 37k^4 + 31k^2 - 7)l^{10} + (16k^6 - 36k^4 + 13k^2 + 3)(k^2 - 1)^2 l^6)h^3}{\nu \mu^4 \Delta} \\
& + \frac{1}{64} \frac{(-l^{12} + (48k^6 - 106k^4 + 67k^2 - 8)l^8 + 8(4k^2 - 5)(k^2 - 1)^2 l^4)h^2}{\mu^2 \nu \Delta} \\
& + \frac{1}{128} \frac{(9l^{14} - (285k^2 + 128k^6 - 65 - 339k^4)l^{10} - 4(16k^6 - 68k^4 + 41k^2 + 2)(k^2 - 1)^2 l^6)h}{\mu^3 \nu \Delta} + \\
& \frac{1}{256} (27l^{16} - (804k^2 - 178 - 966k^4 + 367k^6)l^{12} \\
& + 2(32k^8 - 124k^6 + 154k^4 + 79k^2 - 87)(k^2 - 1)^2 l^8 - 32(4k^2 - 5)(k^2 - 1)^5 l^4) / (\nu \mu^4 \Delta)
\end{aligned}$$

$$\begin{aligned}
A_{3,1} = & \frac{5}{8} \frac{l^2 h^2}{\mu} + \frac{1}{16} \frac{(5k^2 - 10)l^4 h}{\mu^2} + \frac{1}{32} \frac{5(3 - 3k^2 + k^4)l^6 - 4(3k^2 - 2)(k^2 - 1)^2 l^2}{\mu^3} \\
& + \frac{1}{64} \frac{(5l^{12} + 4(4k^2 + 7)l^8)h}{\nu \delta} + \frac{1}{128} \frac{-5l^{14} + 2(16k^2 - 9)l^{10} + 8(4k^2 + 1)l^6}{\nu \delta} \\
& + \frac{1}{16} \frac{(-5l^{12} + (27k^6 - 72k^4 + 58k^2 - 8)l^8 - 4(4k^4 - k^2 - 2)(k^2 - 1)^2 l^4)h^3}{\nu \mu^3 \Delta} \\
& + \frac{1}{32} \frac{(-5l^{10} - (-58k^2 + 8k^4 + 32k^6 + 13)l^6)h^2}{\nu \mu \Delta} \\
& + \frac{1}{64} \frac{(45l^{12} - (-996k^4 - 82 + 412k^6 + 711k^2)l^8 + 16(4k^4 - 9k^2 + 8)(k^2 - 1)^2 l^4)h}{\mu^2 \nu \Delta} \\
& + \frac{1}{128} \frac{135l^{14} - (674k^6 - 1779k^4 + 1401k^2 - 161)l^{10} + 4(32k^8 - 88k^6 + 22k^4 + 157k^2 - 96)(k^2 - 1)^2 l^6}{\nu \mu^3 \Delta}
\end{aligned}$$

$$\begin{aligned}
A_{3,2} = & -\frac{7}{8} \frac{l^2 h}{\mu} + \frac{1}{16} \frac{-7(k^2 - 2)l^4 + 40(k^2 - 1)^2}{\mu^2} + \frac{1}{32} \frac{(-7l^{10} - 16l^6)h}{\nu \delta} + \frac{1}{64} \frac{7l^{12} - 2(16k^2 - 1)l^8}{\nu \delta} \\
& + \frac{1}{8} \frac{(-7l^{10} + (16k^6 - 41k^4 + 27k^2 + 5)l^6)h^3}{\nu \mu^2 \Delta} + \frac{1}{16} \frac{(-7l^8 + 2(16k^2 - 1)l^4)h^2}{\nu \Delta} \\
& + \frac{1}{32} \frac{(63l^{10} - (260k^2 - 292k^4 + 64k^6 + 31)l^6)h}{\nu \mu \Delta} \\
& + \frac{1}{64} \frac{189l^{12} - (-462k^4 + 212 + 148k^6 + 291k^2)l^8 - 8(16k^2 - 33)(k^2 - 1)^3 l^4}{\nu \mu^2 \Delta}
\end{aligned}$$

$$\begin{aligned}
A_{3,3} = & \frac{3}{8} \frac{l^2}{\mu} + \frac{3}{16} \frac{l^8 h}{\nu \delta} + \frac{1}{32} \frac{-3l^{10} + 6l^6}{\nu \delta} + \frac{1}{4} \frac{(-3l^8 - 3(k^2 - 2)l^4)h^3}{\mu \nu \Delta} + \frac{1}{8} \frac{(-6k^2 + 3)l^6 h^2}{\nu \Delta} \\
& + \frac{1}{16} \frac{(27l^8 + 12(k^2 - 4)l^4)h}{\nu \Delta} + \frac{1}{32} \frac{81l^{10} + 3(2k^2 - 5)(4k^4 - 8k^2 + 13)l^6}{\mu \nu \Delta}
\end{aligned}$$

## B Coefficients $B_{ij}$

In addition to the abbreviations  $\delta$  and  $\Delta$  introduced in (43), the following expressions use

$$\xi = 2h - l^2 \quad \text{and} \quad \eta = l^4 + 2 - 2hl^2.$$

The 16 coefficients  $B_{ij}$  in (45) are



$$\begin{aligned}
B_{0,0} = & -\frac{1}{2} \frac{(h^2+1)h}{\xi k^2} + \frac{1}{8} \frac{-l^6 + 8hl^4 - 4(4h^2+1)l^2}{\eta \xi \delta} + \frac{1}{2} \frac{(-3l^4 - 2h(2h^2+1)l^2 + 4(h-1)(h+1))k^4}{\eta \Delta} \\
& + \frac{1}{4} \frac{(3(8h^2+5)l^4 + 8h(2h^4+h^2+3)l^2 - 16(h-1)(h+1)(h^2+1))k^2}{\eta \Delta} \\
& + \frac{1}{8} \frac{63hl^6 + (28h^4 - 94h^2 - 27)l^4 - 8h(2h^6 + h^4 + 4h^2 - 1)l^2 + 16(h-1)(h+1)(h^2+1)^2}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{0,1} = & \frac{1}{2} \frac{5h^2+1}{\xi k^2} + \frac{1}{4} \frac{4-3l^4+8hl^2+16h^2}{\xi \eta \delta} + \frac{(5(2h^2+1)l^2-2h)k^4}{\eta \Delta} \\
& + \frac{1}{2} \frac{(-10hl^4 - (40h^4+68h^2+35)l^2 + 4(6h^2+1)h)k^2}{\eta \Delta} \\
& + \frac{1}{8} \frac{-63l^6 - 8h(27h^2-8)l^4 + 4(20h^6+26h^4+91h^2+16)l^2 - 80(h^2+1)h^3}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{0,2} = & -\frac{7}{2} \frac{h}{\xi k^2} + \frac{1}{2} \frac{-16h+l^2}{\xi \eta \delta} + 2 \frac{(-1-7hl^2)k^4}{\eta \Delta} + \frac{1}{2} \frac{(-3l^4+2h(28h^2+53)l^2-24h^2+8)k^2}{\eta \Delta} \\
& + \frac{1}{4} \frac{(166h^2+15)l^4 - 4h(14h^4+21h^2+69)l^2 + 8(7h^2-1)(h^2+1)}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{0,3} = & \frac{3}{2} \frac{1}{\xi k^2} + 3 \frac{1}{\xi \eta \delta} + 6 \frac{l^2 k^4}{\eta \Delta} + 3 \frac{(-(4h^2+7)l^2+2h)k^2}{\eta \Delta} \\
& + \frac{3}{2} \frac{-12hl^4 + (4h^4+6h^2+19)l^2 - 4(h^2+1)h}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{1,0} = & -\frac{1}{2} \frac{(h^2+1)h^2}{\xi k^2} + \frac{1}{16} \frac{-l^8+8hl^6-4(4h^2+1)l^4}{\eta \xi \delta} \\
& + \frac{1}{2} \frac{(-6hl^4 - (-1-6h^2+4h^4)l^2 + 4(h-1)(h+1)h)k^4}{\eta \Delta} \\
& + \frac{1}{8} \frac{(9l^6+4(14h^2+13)hl^4+4(8h^6-20h^2-4h^4+1)l^2-32(h-1)(h+1)(h^2+1)h)k^2}{\eta \Delta} \\
& + \frac{1}{16} (3(44h^2-9)l^6 + 2(32h^4-124h^2-67)hl^4 - 8(4h^8+h^4+2h^6-27h^2+2)l^2 \\
& + 32(h-1)(h+1)(h^2+1)^2h)/(\eta \Delta)
\end{aligned}$$

$$\begin{aligned}
B_{1,1} = & \frac{1}{2} \frac{(5h^2+1)h}{\xi k^2} + \frac{1}{8} \frac{-3l^6+8hl^4+4(4h^2+1)l^2}{\eta \xi \delta} + \frac{1}{2} \frac{(7l^4+2(10h^2-3)hl^2-20h^2+4)k^4}{\eta \Delta} \\
& + \frac{1}{4} \frac{(-56h^2+43)l^4-40(2h^2-1)(h^2+1)hl^2+16(5h^2-1)(h^2+1)k^2}{\eta \Delta} \\
& + \frac{1}{8} \frac{-75hl^6-(-246h^2-103+236h^4)l^4+8(10h^6+13h^4+28h^2-13)hl^2-16(5h^2-1)(h^2+1)^2}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{1,2} = & -\frac{7}{2} \frac{h^2}{\xi k^2} + \frac{1}{4} \frac{l^4-16hl^2}{\xi \eta \delta} + \frac{(-1+14h^2)l^2+14h)k^4}{\eta \Delta} \\
& + \frac{1}{2} \frac{(22hl^4+(56h^4+44h^2+7)l^2-56(h^2+1)h)k^2}{\eta \Delta} \\
& + \frac{1}{8} \frac{9l^6+8(45h^2-26)hl^4-4(42h^4+28h^6+99h^2+5)l^2+112(h^2+1)^2h}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{1,3} = & \frac{3}{2} \frac{h}{\xi k^2} + \frac{3}{2} \frac{l^2}{\xi \eta \delta} + 6 \frac{(-1+hl^2)k^4}{\eta \Delta} + \frac{3}{2} \frac{(-3l^4-2(4h^2+3)hl^2+8h^2+8)k^2}{\eta \Delta} \\
& + \frac{3}{4} \frac{-(26h^2-15)l^4+4(2h^4+3h^2+7)hl^2-8(h^2+1)^2}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{2,0} = & \frac{1}{4} - \frac{1}{2} \frac{(h^2+1)h^3}{\xi k^2} + \frac{1}{32} \frac{-l^{10}+8hl^8-4(4h^2+1)l^6}{\eta \xi \delta} \\
& + \frac{1}{4} \frac{(-10h^2-1)l^4-4(h^2-2)(2h^2+1)hl^2+8(h-1)(h+1)h^2)k^4}{\eta \Delta} + \frac{1}{8} ((15hl^6 \\
& + (2h+1)(2h-1)(12h^2+7)l^4+8(4h^6-2h^4-11h^2-3)hl^2 \\
& - 32(h-1)(h+1)(h^2+1)h^2)k^2)/(\eta \Delta) + \frac{1}{32} (-27l^8+6(40h^2-17)hl^6 \\
& + 8(-61h^4-18h^2+18h^6-2)l^4-32(2h^8+h^6+h^4-15h^2-1)hl^2 \\
& + 64(h-1)(h+1)(h^2+1)^2h^2)/(\eta \Delta)
\end{aligned}$$

$$\begin{aligned}
B_{2,1} = & \frac{1}{2} \frac{(5h^2+1)h^2}{\xi k^2} + \frac{1}{16} \frac{-3l^8+8hl^6+4(4h^2+1)l^4}{\eta \xi \delta} \\
& + \frac{1}{2} \frac{(6hl^4+(20h^4-14h^2-5)l^2-4(5h^2-1)h)k^4}{\eta \Delta} \\
& + \frac{1}{8} \frac{(-21l^6-4(22h^2+5)hl^4-4(12h^4+40h^6-52h^2-7)l^2+32(5h^2-1)(h^2+1)h)k^2}{\eta \Delta} + \\
& \frac{1}{16} (-68h^2-87)l^6-2(256h^4-252h^2-47)hl^4+8(20h^8+53h^4-51h^2+26h^6-2)l^2 \\
& - 32(5h^2-1)(h^2+1)^2h)/(\eta \Delta)
\end{aligned}$$

$$\begin{aligned}
B_{2,2} = & -\frac{7}{2} \frac{h^3}{\xi k^2} + \frac{1}{8} \frac{l^6 - 16 h l^4}{\xi \eta \delta} + \frac{1}{2} \frac{(-l^4 - 14(2h^2 - 1) h l^2 + 28 h^2) k^4}{\eta \Delta} \\
& + \frac{1}{4} \frac{((32 h^2 + 7) l^4 + 4(28 h^4 + 14 h^2 - 25) h l^2 - 112 (h^2 + 1) h^2) k^2}{\eta \Delta} \\
& + \frac{1}{8} \frac{-51 h l^6 + (388 h^4 - 230 h^2 - 19) l^4 - 8(14 h^6 + 21 h^4 + 45 h^2 - 18) h l^2 + 112 (h^2 + 1)^2 h^2}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{2,3} = & \frac{3}{2} \frac{h^2}{\xi k^2} + \frac{3}{4} \frac{l^4}{\xi \eta \delta} + 3 \frac{((2h^2 - 1) l^2 - 2h) k^4}{\eta \Delta} + \frac{3}{2} \frac{(-2 h l^4 - (8 h^4 + 4 h^2 - 7) l^2 + 8 (h^2 + 1) h) k^2}{\eta \Delta} \\
& + \frac{3}{8} \frac{9 l^6 - 8(7 h^2 - 4) h l^4 + 4(6 h^4 + 4 h^6 + 13 h^2 - 5) l^2 - 16 (h^2 + 1)^2 h}{\eta \Delta}
\end{aligned}$$

$$\begin{aligned}
B_{3,0} = & \frac{1}{4} h - \frac{1}{16} l^2 - \frac{1}{2} \frac{(h^2 + 1) h^4}{\xi k^2} + \frac{1}{64} \frac{-l^{12} + 8 h l^{10} - 4(4 h^2 + 1) l^8}{\eta \xi \delta} \\
& + \frac{1}{8} \frac{(3 l^6 - 4(2 h - 1)(2 h + 1) h l^4 - 4(-1 - 3 h^2 + 4 h^6 - 6 h^4) l^2 + 16 (h - 1)(h + 1) h^3) k^4}{\eta \Delta} + \\
& \frac{1}{16} ((3(2 h^2 - 5) l^6 + 2(40 h^4 + 12 h^2 - 19) h l^4 + 16(4 h^8 - 2 h^6 - 10 h^4 - 1 - 3 h^2) l^2 \\
& - 64 (h - 1)(h + 1)(h^2 + 1) h^3) k^2) / (\eta \Delta) + \frac{1}{32} (-90 h l^8 + (212 h^4 - 8 h^2 + 27) l^6 \\
& + 8(20 h^6 - 60 h^4 - 14 h^2 - 3) h l^4 - 16(2 h^8 + 4 h^{10} + 3 h^6 - 29 h^4 - 3 h^2 - 1) l^2 \\
& + 64 (h - 1)(h + 1)(h^2 + 1)^2 h^3) / (\eta \Delta)
\end{aligned}$$

$$\begin{aligned}
B_{3,1} = & \frac{3}{4} + \frac{1}{2} \frac{(5 h^2 + 1) h^3}{\xi k^2} + \frac{1}{32} \frac{-3 l^{10} + 8 h l^8 + 4(4 h^2 + 1) l^6}{\eta \xi \delta} \\
& + \frac{1}{4} \frac{((2 h^2 - 5) l^4 + 4(10 h^4 - 7 h^2 - 2) h l^2 - 8(5 h^2 - 1) h^2) k^4}{\eta \Delta} + \frac{1}{8} ((-11 h l^6 \\
& - (-48 h^2 + 48 h^4 - 35) l^4 - 8(h - 1)(h + 1)(20 h^4 + 26 h^2 + 3) h l^2 + 32(5 h^2 - 1)(h^2 + 1) h^2) \\
& k^2) / (\eta \Delta) + \frac{1}{32} (63 l^8 + 10(8 h^2 + 11) h l^6 - 8(-113 h^4 + 22 h^2 + 138 h^6 + 8) l^4 \\
& + 32(10 h^8 + 13 h^6 + 29 h^4 - 23 h^2 - 1) h l^2 - 64(5 h^2 - 1)(h^2 + 1)^2 h^2) / (\eta \Delta)
\end{aligned}$$

$$\begin{aligned}
B_{3,2} = & -\frac{7}{2} \frac{h^4}{\xi k^2} + \frac{1}{16} \frac{l^8 - 16 h l^6}{\eta \xi \delta} + \frac{1}{2} \frac{(6 h l^4 - (-1 - 14 h^2 + 28 h^4) l^2 + 28 h^3) k^4}{\eta \Delta} \\
& + \frac{1}{8} \frac{(3 l^6 + 4(2 h^2 - 23) h l^4 + 8(14 h^4 + 28 h^6 - 22 h^2 - 1) l^2 - 224(h^2 + 1) h^3) k^2}{\eta \Delta} + \frac{1}{16} ( \\
& -(268 h^2 + 15) l^6 + 2(416 h^4 - 188 h^2 + 119) h l^4 - 8(42 h^6 + 28 h^8 + 97 h^4 - 1 - 30 h^2) l^2 \\
& + 224(h^2 + 1)^2 h^3) / (\eta \Delta)
\end{aligned}$$

$$\begin{aligned}
B_{3,3} = & \frac{3}{2} \frac{h^3}{\xi k^2} + \frac{3}{8} \frac{l^6}{\xi \eta \delta} + \frac{3}{2} \frac{(-l^4 + 2(2 h^2 - 1) h l^2 - 4 h^2) k^4}{\eta \Delta} \\
& + \frac{3}{4} \frac{(7 l^4 - 4(4 h^4 + 2 h^2 - 3) h l^2 + 16(h^2 + 1) h^2) k^2}{\eta \Delta} \\
& + \frac{3}{8} \frac{21 h l^6 - (-26 h^2 + 60 h^4 + 19) l^4 + 8(2 h^6 + 3 h^4 + 7 h^2 - 2) h l^2 - 16(h^2 + 1)^2 h^2}{\eta \Delta}
\end{aligned}$$