A Two-Parameter Study of the Extent of Chaos in a Billiard System

Holger R. Dullin, Peter H. Richter, Andreas Wittek

Institut für Theoretische Physik and
Institut für Dynamische Systeme
University of Bremen

Postfach 33 04 40, D-28334 Bremen, Germany

E-mail: prichter@physik.uni-bremen.de

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Abstract

The billiard system of Benettin and Strelcyn [2] is generalized to a two-parameter family of different shapes. Its boundaries are composed of circular segments. The family includes the integrable limit of a circular boundary, convex boundaries of various shapes with mixed dynamics, stadiums, and a variety of non-convex boundaries, partially with ergodic behaviour.

The extent of chaos has been measured in two ways: (i) in terms of phase space volume occupied by the main chaotic band, and (ii) in terms of the Lyapunov exponent of that same region. The results are represented as a kind of phase diagram of chaos.

We observe complex regularities, related to the bifurcation scheme of the most prominent resonances. A detailed stability analysis of these resonances up to period six explains most of these features. The phenomenon of *breathing chaos* [13] – that is, the non-monotonicity of the amount of chaos as a function of the parameters – observed earlier in a one-parameter study of the gravitational wedge billiard, is part of the picture, giving support to the conjecture that this is a fairly common global scenario.

Key words: Billiard; chaos; Lyapunov exponent; periodic orbits; stability; Poincaré map

1 Introduction

We present the results of a numerical study of a two-parameter family of two-dimensional billiards. The family is a natural generalization of the system introduced by Benettin and Strelcyn [2]. In addition to *ovals* it contains billiard shapes reminiscent of *peanuts*, *violins*, *moons*, *flippers*, *convex* and *concave drops*. For all possible values of parameters we determine two measures of chaoticity of the system's Poincaré map: the relative area A of the main chaotic region, as a static measure, and its Lyapunov exponent λ as a dynamic characterization. The area A is determined by pixel counting on a sufficiently fine grid, the Lyapunov exponent is evaluated per iteration step of the Poincaré map. (It is *not* taken per unit of time because that would make it dependent on the scaling of length in the various parameter regions.)

The investigation produces a surprising complexity in the parameter dependence of A and λ. In about 40% of parameter space the billiard is ergodic as is proved in a classical theorem by Bunimovich [5]. This is in agreement with our numerical experiments. The rest exhibits mixed behavior, except for a few integrable limiting cases. We attempt to explain the most conspicuous features of the parameter dependence on the basis of the stability properties of periodic orbits with periods up to 6. The analysis is elementary but fairly complicated in detail. We think that our results contain valuable knowledge about the "typical" behavior of billiards as simple prototypes of nonintegrable Hamiltonian systems [3], [12]. The focus is on global rather than local scenarios in the development of chaos. Of course there are numerous instances of period doubling or tangent bifurcations (and rather less KAM scenarios, as the system contains discontinuities not tolerated by the assumptions of the KAM theorem), but the more interesting features concern the "breathing of chaos" discussed previously for the gravitational wedge billiard [13]. It refers to more or less regular oscillations in the amount and strength of chaos as parameters are varied. The comprehensive view on the parameter plane of a two-parameter family reveals a number of different ways in which such behavior can be observed.

The organization of the paper is as follows. In Sec. 2 we present the different billiard shapes. Sec. 3 introduces the Poincaré map and gives one example for each type of billiard. Sec. 4 contains the global results for relative area A and Lyapunov exponent λ of the main chaotic part of the energy surface. Sec. 5 provides information about stability properties of periodic orbits which is used in the final Sec. 6 to explain the major features of our numerical findings on the global characteristics of the extent of chaos.

2 A generalized Benettin-Strelcyn family of billiards

Benettin and Strelcyn [2] introduced an *oval* billiard system whose boundary is composed of four circular arcs joining with common tangents at the corners of a square. There is a one-parameter family of such systems, as the direction of the tangent at the corners may change. The family includes the integrable case of a circular billiard, and the ergodic case of a *stadium*. The original paper as well as a later analysis by Hénon and Wisdom [11] restricted the family to convex shapes, but an extension to *peanut* shapes is straightforward. Hayli and Dumont [9] considered a generalization to a four-parameter family where the points of discontinuous curvature are free to move along a circle instead of being fixed to the corners of a square. However, rather than exploring the entire parameter space, the authors selected just three one-parameter families. Their aim was to identify differences between billiards with and without reflection symmetries.

In the following we describe a two-parameter family whose dynamics will be analyzed comprehensively. The generalization with respect to Benettin and Strelcyn consists of having the four circular arcs meet at the corners of a rectangle instead of a square. The two reflection symmetries of the original system are thereby preserved, but a second parameter is introduced in order to describe the shape of the rectangle.

Consider Fig. 1 for notation and a typical example. The basic rectangle has corners

$$\mathrm{P}_1 \;=\; a\,(\cos\rho, -\sin\rho)\ , \tag{1}$$

and $\mathrm{P}_{2,3,4}$ obtained by reflection at the x- and y-axes, where a is a length to be normalized later, eq. (6), and ρ determines the aspect ratio of the rectangle. The centers of the four circular arcs $\overset{\frown}{\mathrm{P}_\mathrm{i}\mathrm{P}}_{i+1}$ are

$$\mathrm{C}_{1,3} \;=\; a\,(\pm c_x, 0)\ , \qquad \mathrm{C}_{2,4} \;=\; a\,(0, \pm c_y) \tag{2}$$

with

$$c_x \;=\; \frac{\sin\epsilon}{\sin\gamma}\ , \qquad c_y \;=\; -\,\frac{\sin\epsilon}{\cos\gamma}\ . \tag{3}$$

The angle γ or, alternatively, $\epsilon = \gamma - \rho$, determines the direction of the tangents to the billiard boundary in the corner points P_i. The circular arcs with centers C_1 and C_3 have radius $a \cdot |r_x|$, those around C_2 and C_4 have radius $a \cdot |r_y|$, where

$$r_x \;=\; \frac{\sin\rho}{\sin\gamma}\ , \qquad r_y \;=\; \frac{\cos\rho}{\cos\gamma}\ . \tag{4}$$

Fig. 1 *about here*

Fig. 2 *about here*

Oval billiards of the type shown in Fig. 1 are obtained for the parameter range

$$0 < \rho < \pi/2 \ , \qquad \rho \le \gamma \le \pi/2 \ . \tag{5}$$

For $\gamma = \rho < \pi/2$, or $\epsilon = 0$, they are *circles*, for $\gamma = \pi/2$ they are *stadiums* [6] (with straight line segments P_2P_3 and P_4P_1). The original Benettin-Strelcyn family of billiards [2] is the line $\rho = \pi/4$ where the rectangle becomes a square. For $\rho = 0$, or $\epsilon = \gamma$, the rectangle degenerates into a horizontal line, and the billiard assumes the *lemon* shape discussed by Heller [10]. We find it convenient to normalize the horizontal width of these billiards to a total of 2 which requires to choose a according to

$$a \ = \ 1/(c_x + r_x) \ . \tag{6}$$

The triangle (5) in parameter space is not yet a complete family of shapes. The construction of Fig. 1 can be done with any pair (γ, ρ) from the 2-torus $\gamma, \rho \in (-\pi, \pi)$. Moreover, note that instead of the circular arcs shown in Fig. 1, we might as well have taken the complementary arcs around each center. This would produce the self-intersecting curve of Fig. 2 which cuts five billiards out of the plane, with three different shapes: a *lemon* formed by the two circular arcs between T and $\overline{\mathrm{T}}$; a *flipper* made of five circular pieces, $\mathrm{SP_2T\overline{T}P_1S}$, and its mirror image under reflection Y; a *moon* also made of five pieces, $\mathrm{S\overline{S}P_3TP_2S}$, and its mirror image under reflection X.

The set of billiards obtained by allowing γ, ϵ to vary on the full 2-torus and taking the two choices of arcs, can be reduced with respect to the various symmetries of the system. For example, the operation $(\gamma, \epsilon) \mapsto (\frac{\pi}{2} - \gamma, -\epsilon)$ reflects a given billiard at the line $x = y$. We may thus ignore the parameter range $\epsilon < 0$. Starting from *ovals* of the type shown in Fig. 1, and extending the parameter range in such a way that the billiards change continuously, with every possible shape represented just once, we obtain the scheme of Fig. 3. In Fig refFig:Shape2 thus the right *flipper* is chosen. The billiards consist of a single piece or break up into two parts which are then mirror images of each other with respect to the y-axis. The normalization in case of a single piece is chosen to have a horizontal diameter of 2. When the billiard has two components, each one of them has horizontal width 1.

There are three kinds of billiards that come as a single piece, possessing reflection symmetry with respect to both x- and y-axis:

1. *Ovals:* $0 < \gamma \le \pi/2, \quad 0 \le \epsilon \le \gamma$.
   They are the shapes of Fig. 1 and have been discussed above. There are three different limiting cases: *circles* for $\epsilon = 0$; *lemons* for $\epsilon = \gamma$ [10]; *stadiums* for $\gamma = \pi/2$ [6]. The Benettin-Strelcyn line [2] $\epsilon = \gamma - \pi/4$ connects *circles* and *stadiums.*

   Lemons might in fact be extended to the region $\gamma < \epsilon < (\gamma + \pi)/2$, as a two-parameter family of its own, but these other *lemons* differ from those on the line $\epsilon = \gamma$ only by their size. It is easy to see that if (γ', ϵ') are the parameters of such a *lemon*, then the transformation

   $$\sin\epsilon \ = \ \frac{\sin\epsilon'}{\cos(\epsilon' - \gamma')} \tag{7}$$

   maps it to a similar *lemon* with $\epsilon = \gamma$. This observation can be expressed by saying that

   $$d_L \ := \ \frac{\sin\epsilon}{\cos\rho} \ = \ -\frac{c_y}{r_y} \qquad (0 \le d_L \le 1) \tag{8}$$

is an invariant of similar *lemons*. We shall call d_L the *lemon* parameter of any *oval* billiard. It provides a natural parametrization not only of *lemon* shapes, but also of *lemon* orbits, i. e. orbits in *ovals* that never reach the circular arcs with centers $C_{1,3}$, which are the two arcs that are farthest away from each other.

In a similar way, we may introduce a *stadium* parameter d_S as

$$d_S := \frac{\sin\epsilon}{\sin\rho} = \frac{c_x}{r_x} \qquad (0 \le d_S < \infty) \tag{9}$$

in order to characterize orbits that never reach the arcs with centers $C_{2,4}$, which are closer to each other. They behave like orbits in a true *stadium*, and therefore cannot be elliptic.

2. *Peanuts:* $\pi/2 < \gamma \le \pi, \quad \gamma - \pi/2 \le \epsilon \le \gamma/2 + \pi/4.$
   They develop from *stadiums* as γ increases beyond $\pi/2$, and turn into two circles as $\gamma \to \pi$. Along the lower line $\epsilon = \gamma - \pi/2$ *peanuts* develop cusps on the y-axis. At the upper line $2\epsilon = \gamma + \pi/2$, they disintegrate into two *concave drops*.

3. *Violins:* $\pi/2 < \gamma \le \pi, \quad \gamma/2 - \pi/4 \le \epsilon \le \gamma - \pi/2.$
   Their inner loops develop from the cusps of the *peanuts*, as ϵ is lowered. When the loops touch each other, for $2\epsilon = \gamma - \pi/2$, *violins* break up into two *moons*.

In addition, there are four kinds of billiards that consist of two disconnected parts. Individual parts have lost the reflection symmetry with respect to the y-axis, but as a pair they respect it.

4. *Concave drops:* $\pi/2 < \gamma \le \pi, \quad \gamma/2 + \pi/4 \le \epsilon \le \gamma.$
   Their cusps are at $(\pm s_x, 0)$ with

$$s_x^2 = r_y^2 - c_y^2 \ . \tag{10}$$

   The normalization

$$a = 1/(c_x + r_x - s_x) \tag{11}$$

   ensures that the two pieces have unit horizontal width. Note, however, that $a \to \infty$ as the line $\epsilon = \gamma$ is approached, but $ar_x \to 1$.

5. *Convex drops:* $\pi/2 < \gamma \le \pi, \quad \gamma \le \epsilon \le \gamma/2 + \pi/2.$
   They are a natural extension of *concave drops*, with cusps on the other side of the centers C_1, C_3. The normalization is given by

$$a = 1/(s_x - c_x + |r_x|) \ . \tag{12}$$

6. *Flippers:* $\pi/2 < \gamma \le \pi, \quad \gamma/2 + \pi/2 \le \epsilon \le \pi.$
   These shapes have been shown in Fig. 2. Their horizontal extent is from $x = 0$ to $x = \pm s_x$ which implies the normalization condition

$$a = 1/s_x \ . \tag{13}$$

   As $\epsilon \to \pi$, the two *flippers* approach the shape of a semicircular arc.

7. *Moons:* $\pi/2 < \gamma \le \pi, \quad 0 \le \epsilon \le \gamma/2 - \pi/4.$
   These shapes, except for an interchange of x and y, have also been shown in Fig. 2. They develop from *violins* when their inner loops overlap. The normalization is the same as in (6). In the limit as $\epsilon \to 0$, the *moons* develop into semicircular arcs.

Fig. 3 *about here*

The boundary of the parameter region shown in Fig. 3 is a natural boundary in the sense that except for the *lemon* line $0 \leq \epsilon = \gamma \leq \pi/2$, it describes integrable limiting cases, of three different kinds: full circles, semicircular arcs, and infinitely thin rods. Depending on their location along the boundary, these integrable cases are perturbed in characteristically different ways, giving rise to a number of alternatives for the transition to chaos.

(i) *Circles* occur along the parameter segment $\epsilon = 0$, $0 \leq \gamma < \pi/2$, and along the line $\gamma = \pi$. The first segment is a limiting case of *oval* shapes; small perturbations are fairly smooth so that a KAM-like scenario may be expected (even though the discontinuities in curvature contradict the assumption of the KAM theorem). The situation is different along the line $\gamma = \pi$. For $\epsilon < \pi/4$ and $\epsilon > 3\pi/4$, the circles are perturbed by small but non-smooth insertions, developing into *moons* and *concave drops* respectively. Intuitively, this should severely disrupt the foliation of energy surfaces into invariant tori, since every orbit with an irrational winding number encounters the perturbation. For $\pi/4 < \epsilon < 3\pi/4$ the perturbations are also strong: two circles merge through a small opening which allows the mass point to pass from one side to the other. This generates considerable chaos even for tiny perturbations.

(ii) *Semicircular arcs* are the limits of *moons* and *flippers* along the parameter lines $\pi/2 < \gamma < \pi$, $\epsilon = 0$ and $\epsilon = \pi$ respectively. The motion occurs between infinitesimally close semicircular lines. This implies that the Poincaré map (to be discussed in the subsequent section) becomes the identity. It is not intuitively clear what this implies for the existence or nonexistence of invariant structures on the energy surface, as the small but non-smooth perturbations are added. Numerical evidence has been inconclusive; it appears that there are innumerably many higher order stable resonances surrounded by invariant tori, but no KAM tori separating them into distinct layers.

(iii) *Rods* of zero thickness occur as limits of *flippers* and *convex drops* along the line $\gamma = \pi/2$, $\pi/2 < \epsilon < \pi$. Again the system becomes trivially integrable in the sense that the Poincaré map approaches the identity. Yet it appears that no invariant structures can be identified for γ ever so slightly larger than $\pi/2$.

3 Poincaré sections

Our configuration space for given parameter values (γ, ϵ) is the closed set of accessible points (x, y) from the corresponding billiard, and will be denoted by $\mathcal{B}_{\gamma,\epsilon}$. The four-dimensional phase space is trivially foliated into three-dimensional energy surfaces

$$\mathcal{E}_{\gamma,\epsilon} := \{(x, y, p_x, p_y)|\ (x, y) \in \mathcal{B}_{\gamma,\epsilon},\ p_x^2 + p_y^2 = 2E = \text{const}\} = \mathcal{B}_{\gamma,\epsilon} \times \mathrm{S}^1\ . \qquad (14)$$

As the energy E may be scaled away, we choose $2E = 1$ for convenience.

The partitioning of the energy surfaces $\mathcal{E}_{\gamma,\epsilon}$ into regular and chaotic motion is best studied in terms of two-dimensional Poincaré surfaces of section. A natural choice for the section condition is to consider orbits immediately after reflection, i. e. to take $(x, y) \in \partial\mathcal{B}_{\gamma,\epsilon}$, with (p_x, p_y) from the half circle $p_\perp > 0$. Here we assume $\partial\mathcal{B}_{\gamma,\epsilon}$ to be oriented anticlockwise, and parametrized by its arc length s (normalized to 1); the momentum components $p_\parallel$ and $p_\perp$ are taken with respect to the tangent direction. As usual, the surface of section is represented in its projection onto the cylinder

$$\mathcal{P}_{\gamma,\epsilon} = \{(s, p_\parallel)|\ s \in \mathrm{S}^1,\ -1 \leq p_\parallel \leq 1\} \tag{15}$$

of canonically conjugate variables. The Poincaré map P: $\mathcal{P}_{\gamma,\epsilon} \rightarrow \mathcal{P}_{\gamma,\epsilon}$, induced by the dynamics in phase space, is then area preserving.

The Poincaré map is C^∞ almost everywhere. There are two types of lines where the map is singular. At s-values corresponding to boundary points P_i, the map is continuous but not differentiable; this is the only type of singularity in *ovals* and *peanuts*. At s-values where the tangents to $\partial\mathcal{B}_{\gamma,\epsilon}$ change discontinuously, the Poincaré map is also discontinuous; there is one such s-value in the two *drops*, two in *lemons*, three in *flippers* and *moons*, four in *violins*.

Fig. 4 *about here*

In the following illustrative examples, the singular s-values are indicated as vertical lines. The value $s = 0$ is chosen to correspond to point P_1 on $\partial\mathcal{B}_{\gamma,\epsilon}$ except for *violins* and *moons* where $s = 0$ corresponds to P_4.

Consider Fig. 4 as a typical Poincaré surface of section for an *oval*. Its main features are the two large areas of regularity and irregularity around the two orbits of period 2. There is an elliptic periodic orbit corresponding to motion along the y-axis, and a hyperbolic orbit corresponding to motion along the x-axis (see Sec. 5). The elliptic centers at $(s, p_\parallel) = (s_e, 0)$ and $(s_e + 1/2, 0)$ are surrounded by invariant circles (tori in phase space), whereas the hyperbolic points at $(s_h, 0)$ and $(s_h + 1/2, 0)$ are at the heart of the major chaotic band. The relative size of this chaotic region as well as the number of secondary resonances turns out to be strongly dependent on the choice of parameters (γ, ϵ).

Fig. 4 reflects three obvious symmetries of the system: time reversal invariance, and reflection symmetry with respect to the x- and y-axes. The action of time reversal T on $(s, p_\parallel)$-space and on the Poincaré map P is

$$\mathrm{T} : (s, p_\parallel) \mapsto (s, -p_\parallel)\ , \qquad \mathrm{P} \mapsto \mathrm{P}^{-1} = \mathrm{TPT}\ . \tag{16}$$

The last equality can be expressed as $(\mathrm{PT})^2 = \mathbf{1}$ which implies that $\mathrm{P} = (\mathrm{PT}) \circ \mathrm{T}$ is a combination of two involutions. Analogous statements can be obtained from the two spatial reflection symmetries if we combine them with time reversal. Thus we define X to be the combination of T and reflection at the x-axis; the corresponding symmetry operation is

$$\mathrm{X} : (s, p_\parallel) \mapsto ((2s_h - s) \bmod 1, p_\parallel)\ , \qquad \mathrm{P} \mapsto \mathrm{P}^{-1} = \mathrm{XPX}\ . \tag{17}$$

Similarly, defining Y to be the combination of T and reflection at the y-axis, we have

$$\mathrm{Y} : (s, p_\parallel) \mapsto ((2s_e - s) \bmod 1, p_\parallel)\ , \qquad \mathrm{P} \mapsto \mathrm{P}^{-1} = \mathrm{YPY}\ . \tag{18}$$

Thus, with S being T, X, or Y, we have three different decompositions of the Poincaré map P into involutions $\mathrm{P}_S := \mathrm{P} \circ \mathrm{S}$ and S:

$$\mathrm{P} = \mathrm{P}_S \circ \mathrm{S}\ , \qquad \mathrm{S}^2 = \mathrm{P}_S^2 = \mathbf{1}\ , \quad \det \mathrm{S} = \det \mathrm{P}_S = -1\ . \tag{19}$$

This representation of P is helpful in discussions of symmetric periodic orbits, as these may be obtained from the invariant sets of the various involutions ([4], [13]). But note that individual orbits need not possess the full symmetry of the dynamical laws. There are examples of periodic orbits which share all three symmetries (the orbits of period 2 mentioned above), and others which have only some of them or none at all. It is then possible to generate new orbits by applying the symmetry operation(s) which the orbit is lacking.

As a consequence of these symmetries, Fig. 4 may be considered to be eightfold redundant. Sufficient information about the dynamical characteristics (up to symmetry) is contained in the rectangle $s_h \leq s \leq s_e$, $0 \leq p_\parallel \leq 1$ which is one eighth of the complete section $\mathcal{P}_{\gamma,\epsilon}$.

Lemon, *peanut* and *violin* shaped billiards have the same symmetry properties as the *ovals*. Figs. 5-7 show typical examples of Poincaré surfaces of section. The *lemons*, as a special case of *ovals*, exhibit similar complexity in mixing regular and chaotic motion, see Fig. 5. *Peanut* shaped billiards, on the other hand, are completely ergodic; as far as numerical evidence can tell, there is no sign of stable periodic motion anywhere, see Fig. 6. This had to be expected as a consequence of a theorem by Bunimovich [5]. *Violins* tend to be strongly chaotic, but depending on the values of γ and ϵ, there are small islands of regular motion, see Fig. 7.

Fig. 5 *about here*

Fig. 6 *about here*

Fig. 7 *about here*

The other four shapes of billiard lack the Y symmetry (their partner under Y having been discarded by the rules of construction). Typical Poincaré surfaces of section are shown in Figs. 8-11. *Moons* behave similarly as *violins*, cf. Figs. 8 and 7. Some stable periodic orbits of *violins* survive the transition from *violins* to *moons*. *Flippers* may be considered a natural extension of *moons*, and again it is possible to observe corresponding stable periodic orbits. In the limit where both shapes degenerate into half circular arcs, the Poincaré surfaces of section develop a very rich structure, with innumerably many elliptic islands.

Crossing over from *flippers* to *convex drops*, the picture again changes smoothly, some stable islands surviving the transition. Yet in all three cases, *moons*, *flippers*, and *convex drops*, the dynamics seems to be predominately chaotic. It is completely chaotic in the case of *concave drops*, as Fig. 11 illustrates by way of example. This is again in line with Bunimovich's general theorem on the ergodicity of billiards with piecewise smooth focusing and scattering components [5].

Fig. 8 *about here*

Fig. 9 *about here*

Fig. 10 *about here*

Fig. 11 *about here*

4 Global results

The two main results of our computer experiments are Figs. 12 and 13. Together they give a comprehensive survey on the degree of chaoticity in our two-parameter family of generalized Benettin-Strelcyn billiards. For some 62,000 different values of parameters (γ, ϵ), we determined the relative area A of the main chaotic band in the Poincaré surface of section, and its Lyapunov exponent λ.

Fig. 12 *about here*

Fig. 13 *about here*

Fig. 12 contains our results on the relative area A. Values from 0 (no chaos) to 1 (full chaos) are indicated in terms of grey-values for each individual set of parameters. The most conspicuous feature of the picture is that billiards in the parameter range of *peanuts* and *concave drops*, i. e. $\pi/2 < \gamma < \pi$ and $\gamma - \pi/2 < \epsilon < \gamma$, are completely chaotic. This amounts to 40 % of all Benettin-Strelcyn billiards. The feature responsible for this behavior, according to a theorem by Bunimovich [5], is that the complement of the convex circular arcs (around $C_{1,3}$) which is not part of the boundary, is contained in the billiard's interior. In the remaining 60 % of all billiards of our family, we observe a mixture of chaotic and regular motion, with systematic variations of their relative weights. The resolution of our analysis is still insufficient to show all details of this very complex behavior. This is particularly true for the *ovals* which will be analyzed to some depth in the subsequent section.

Fig. 13 gives the Lyapunov exponents λ of the dominant chaotic region, for the same set of parameters (γ, ϵ). The Lyapunov exponent measures the average rate of exponential divergence of orbits along a typical chaotic orbit. The values of λ are again given in terms of a color code. They range from 0 to about 1.10.

Lyapunov exponents and relative chaotic area were calculated in the same run of 100,000 − 400,000 iterations of between 5 and 10 initial points, depending on whether sufficient convergence was reached for the different initial points. All initial points were chosen inside the main chaotic band. Depending on whether the hyperbolic orbit corresponding to motion along the x-axis exists in the domain of continuity of the Poincaré map or not, these points were taken from the immediate neighborhood of that orbit, or from the neighborhood of a discontinuity of the map. The relative area A was determined by dividing the Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ into 320 × 320 pixels, and counting the number of different pixels hit during the iteration. The number of iterations being at least 5 times larger than the number of pixels, this would ensure quite reliable measurements for a random process. To compute the Lyapunov exponents λ, each initial point was given a companion in terms of an arbitrary point close by, and the difference vector was iterated with the linearized Poincaré map. Whenever its length grew too large, it was reset to its original length. λ was determined as the average over all iteration steps of the logarithmic growth. The 5 to 10 values of λ from the different initial points were then compared and averaged.

Fig. 14 *about here*

Fig. 15 *about here*

Fig. 16 *about here*

The dynamic information contained in λ complements the purely static information given by the computation of relative area A. This is most clearly seen in the cases of *peanuts* and *concave drops*. If λ is taken as a measure for the rate of mixing (per iteration step, not per unit time!), we see a systematic increase of that rate, from 0 at the boundary of the parameter region to 1.1 in the neighborhood of $(\gamma, \epsilon) = (0.60\pi, 0.33\pi)$.

The most interesting behavior occurs in the parameter range of *ovals*. Comparison of Figs. 12 and 13 shows a general agreement in the sense that regions of small (large) chaotic area and low (high) Lyapunov exponents roughly coincide; this is particularly true near the limits of *circles* and *stadiums*. Closer inspection of the fine structure, however, reveals interesting differences, especially near the limit of *lemons*. Local minima (maxima) of A seem to coincide with local maxima (minima) of λ. This observation is corroborated by Fig. 14 where results of a calculation with higher resolution are presented for the line $\rho = 0.3(\pi/2 - \gamma)$, $3/13 \leq \gamma \leq \pi/2$. For reasons to be discussed in Sec. 6, the line is parametrized with the *lemon* parameter d_L instead of γ. $d_L = 0$ corresponds to the limit of a *circular* billiard whereas $d_L = 1$ characterizes an infinitely long box. Both λ and A increase on the average, as d_L is raised from 0 to 1/2; at larger d_L, the area decreases with a regular pattern of peaks superimposed, whereas λ continues to grow (before it collapses at the very end), with the same pattern superimposed, but of opposite sign. The fine structure is less pronounced, and seems to be less correlated, in the low d_L range.

Figs. 15 and 16 combine additional information for five cross-sections $\rho = \mathrm{m}(\pi/2 - \gamma)$ through the region of *ovals*, m taking the values 0, 0.1, 0.2, 0.3 and 0.4, starting with the limit of *lemons* m = 0 at the lowest curve. These plots make it obvious that the transition from regular to globally chaotic behavior does not proceed in a monotonous way; rather there is a pronounced "breathing" as discussed previously in connection with the wedge billiard [13]. The figures also suggest that a considerable portion of the global features of chaos should be understandable in terms of the properties of *lemon* orbits.

5 Periodic orbits and their stability

A good portion of the global features discussed in the previous section can be understood by studying the properties of periodic orbits and their stability [13]. Determination of the location of periodic orbits in our billiards is an exercise in elementary geometry; the analysis of their stability requires knowledge of the linearized Poincaré map [11], [12].

A general remark about bifurcations of periodic orbits concerns the influence of the discontinuities of our billiard system. In *smooth* Hamiltonian systems, new periodic orbits are born parabolic, with eigenvalue +1, and develop into pairs of elliptic and hyperbolic orbits as parameters are varied. (We ignore here the special bifurcations that are possible at critical energy surfaces.) An important modification introduced by the discontinuities of curvature in our billiard, and even more so by the discontinuities in the direction of tangents, is the possibility of sudden appearance or sudden death of periodic orbits of any character, elliptic, hyperbolic, or inverse hyperbolic. When this happens upon parameter variation, the corresponding orbit hits the billiard boundary at such a point of discontinuity. This leads to severe differences in the stability diagrams of smooth and non-smooth systems [7].

5.1 Linearized Poincaré maps

Assume a segment of an orbit to be given. Let l be its length, r_1 and r_2 the radii of curvature at the two end points 1 and 2 of the segment, positive or negative depending on whether the boundary is convex or concave. Let the components of the normalized momentum vectors $\vec{p}_i$ with respect to the tangent direction be $p_{i\,\|}$ and $p_{i\perp}$ (i = 1,2). It is then straightforward to show that the linearized Poincaré map is ([12], [11])

$$\left(\begin{array}{c} \delta s_2 \\ \delta p_{2\,\|} \end{array}\right) = \mathbf{B}_{2,1} \left(\begin{array}{c} \delta s_1 \\ \delta p_{1\,\|} \end{array}\right) \quad \text{with} \quad \mathbf{B}_{2,1} = \left(\begin{array}{cc} \dfrac{l - r_1 p_{1\perp}}{r_1 p_{2\perp}} & -\dfrac{l}{p_{1\perp} p_{2\perp}} \\ \dfrac{r_1 p_{1\perp} + r_2 p_{2\perp} - l}{r_1 r_2} & \dfrac{l - r_2 p_{2\perp}}{r_2 p_{1\perp}} \end{array}\right) . \tag{20}$$

As the map is area preserving, $\det \mathbf{B}_{2,1} = 1$. Time reversal invariance is expressed as

$$\mathbf{B}_{2,1}{}^{-1} = \mathbf{T}\, \mathbf{B}_{T(1),T(2)}\, \mathbf{T} = \mathbf{T}\, \mathbf{B}_{1,2}\, \mathbf{T} \qquad \text{where} \qquad \mathbf{T} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) . \tag{21}$$

Let us discuss some special cases. If both points 1 and 2 belong to the same circle, then $r_1 = r_2 =: r$, $p_{1\perp} = p_{2\perp} =: p_\perp$, and $l = 2rp_\perp$. The matrix $\mathbf{B}_{2,1} = \mathbf{B}_{1,2} =: \mathbf{B}$ is then

$$\mathbf{B} = \left(\begin{array}{cc} 1 & -2r/p_\perp \\ 0 & 1 \end{array}\right) . \tag{22}$$

The monodromy matrix $\mathbf{M}$ for periodic orbits inside a circle of radius r can be expressed as the appropriate power of $\mathbf{B}$. For orbits of period n and winding number k/n, k and n coprime, we have $p_\perp = \sin(k\pi/n)$ and

$$\mathbf{M} = \mathbf{B}^n = \left(\begin{array}{cc} 1 & -2nr/p_\perp \\ 0 & 1 \end{array}\right) . \tag{23}$$

The two eigenvalues $\lambda_{1,2}$ of this matrix being 1, all periodic orbits of such a billiard are parabolic.

This situation occurs in a large part of the parameter space of our billard system, most notably along the lines where the billiard assumes circular shape. The boundary of the region of *ovals*, $\epsilon = 0,\ 0 < \gamma < \pi/2$, is an isolated line of this kind. In contrast, the circles from the parameter line $\gamma = \pi$ are perturbed at only one point, as γ is decreased below π. As a result, an interval of parabolic orbits of period 2 (around the vertical orbit through point C_1) survives in the whole region $\pi/2 < \gamma < \pi$, $\epsilon_1(\gamma) < \epsilon < \epsilon_2(\gamma)$, where ϵ_1 is determined from the requirement $c_x > r_y$ (if $\rho > 0$), and ϵ_2 from $c_x > -r_x - c_x$ (if $\rho < 0$). Evaluating these requirements, we find

$$\tan \epsilon_1 = \frac{\sin \gamma}{1 - \sin \gamma \tan \gamma} \qquad \text{and} \qquad \tan \epsilon_2 = \frac{-\sin \gamma}{2 - \cos \gamma} . \tag{24}$$

The result of this analysis is that all *drops* and *peanuts* contain intervals of parabolic orbits of period 2; in *violins* and *moons* these orbits exist only if $\epsilon > \epsilon_1(\gamma)$, and in *flippers* if $\epsilon < \epsilon_2(\gamma)$. The existence of similar parabolic orbits of higher period depends on the size of the unperturbed part of the circle around C_1. Observation of Poincaré surfaces of section shows that these lines of parabolic orbits are always embedded in a chaotic neighborhood.

Another important special case of the matrix $\mathbf{B}_{2,1}$ occurs when the curvature in point 2 is zero, $r_2 = \infty$, and the orbit segment is perpendicular to the boundary in 2: $p_{2\perp} = 1$. The linearized Poincaré map is then

$$\mathbf{B}_{\infty,1} \;=\; \left(\begin{array}{cc} \dfrac{l}{r_1} - p_{1\perp} & -\dfrac{l}{p_{1\perp}} \\ \dfrac{1}{r_1} & -\dfrac{1}{p_{1\perp}} \end{array}\right) \;. \tag{25}$$

The time reversed situation is described by the matrix $\mathbf{B}_{1,\infty} = \mathbf{T}\,\mathbf{B}_{\infty,1}^{-1}\,\mathbf{T}$.

These matrices are useful whenever we consider a periodic orbit with X- or Y-symmetry, and a segment 1-3 of length $2l$ which is perpendicular to the y- or x-axis. We can then formally treat the symmetry axis as a boundary, and take the intersection point as point 2. With $r_1 = r_3 =: r$, $p_{1\perp} = p_{2\perp} =: p_\perp$, the matrix $\mathbf{B}_{3,1}$ can be decomposed as

$$\mathbf{B}_{3,1} \;=\; -\mathbf{B}_{3,\infty}\,\mathbf{B}_{\infty,1} \;=\; -\mathbf{T}\,\mathbf{B}_{\infty,1}^{-1}\,\mathbf{T}\,\mathbf{B}_{\infty,1} \;=\; \left(\begin{array}{cc} \dfrac{2l}{rp_\perp} - 1 & -\dfrac{2l}{p_\perp^2} \\ \dfrac{2rp_\perp - 2l}{r^2} & \dfrac{2l}{rp_\perp} - 1 \end{array}\right) \tag{26}$$

where the minus sign takes care of the reflection (including change of orientation) at the symmetry axis.

In general, if we have $\mathbf{B}_{2,1}$ for a given segment, the three symmetry operations T, X, and Y provide us with the corresponding tangent maps for different segments. Time reversal has already been discussed in Eq. (21). The analogous statements for reflection at x- and y-axis derive from Eqs. (17) and (18). With

$$\mathbf{U} = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right) \tag{27}$$

as the tangent map of both X and Y, we have

$$\mathbf{B}_{X(1),X(2)} \;=\; \mathbf{B}_{Y(1),Y(2)} \;=\; \mathbf{B}_{1,2} \;=\; \mathbf{U}\,\mathbf{B}_{2,1}^{-1}\,\mathbf{U} \;. \tag{28}$$

We remark here that when the monodromy matrix $\mathbf{M}$ of a periodic orbit is constructed from matrices $\mathbf{B}$ of type (20), its trace can be expressed by the lengths l_i and the "perpendicular curvatures" K_i defined as

$$K_i \;:=\; \frac{1}{r_i p_{i\perp}} \;. \tag{29}$$

5.2 Orbits of period 2

The dominant periodic orbits are those of period 2, will full symmetry. Requiring differentiability of the Poincaré map, we only consider *ovals*, *peanuts*, and *violins*. Depending on whether the particle moves along the x- or the y-axis, we distinguish "horizontal" and "vertical" orbits. In both cases, they consist of two identical segments with $r_1 = r_2 =: r$ and $p_{1\perp} = p_{2\perp} = 1$. The monodromy matrix is

$$\mathbf{M} \;=\; \mathbf{B}^2 \qquad \text{with} \qquad \mathbf{B} \;=\; \begin{pmatrix} (l-r)/r & -l \\ (2r-l)/r^2 & (l-r)/r \end{pmatrix} . \tag{30}$$

We shall discuss stability in terms of Greene's residues [8] $R := (2 - \mathrm{Tr})/4$, where Tr is the trace of the monodromy matrix. Its eigenvalues are $\lambda_{1,2} = 1 - 2R \pm 2\sqrt{R(R-1)}$, elliptic for $0 < R < 1$, hyperbolic and positive for $R < 0$, hyperbolic and negative for $R > 1$. The cases $R = 0$, $\lambda_{1,2} = +1$, and $R = 1$, $\lambda_{1,2} = -1$, are parabolic.

From Eq. (30) we obtain the residues $R = (l/r)(2 - l/r)$ in agreement with the statement at end of the last subsection. Evaluating this for vertical orbits first, we find

$$R \;=\; \begin{cases} \;\;\,4d_L(1-d_L) & \text{for } \textit{ovals} \text{ and } \textit{peanuts} \\ -4d_L(1+d_L) & \text{for } \textit{violins} \end{cases} \tag{31}$$

where d_L is the *lemon* parameter introduced in Eq. (8). This orbit is the prototype of a *lemon* orbit as it only sees that part of the billiard boundary which remains unchanged if the circles around $C_{2,4}$ were continued beyond the points of discontinuous curvature. The orbits are stable in the regions of *ovals*, i. e. for $\gamma < \pi/2$, and unstable with negative residues, or positive eigenvalues, in the regions of *peanuts* and *violins*, see Fig. 17. The lines of marginal stability with $R = 0$, or $\lambda = +1$, are the boundaries $\epsilon = 0$ (*circles*) and $\gamma = \pi/2$ (*stadiums*) of the region of *ovals*, and the boundaries $\epsilon = \gamma/2 \pm \pi/4$ of the regions of *peanuts* and *violins*. A line of marginal stability with $R = 1$, or $\lambda = -1$, runs through the region of *ovals*: $\tan\epsilon = \cos\gamma/(2 - \sin\gamma)$, where $d_L = 1/2$. The boundary of *peanuts* and *violins* has residue $R \to -\infty$. Essentially all stable orbits of this kind can already be found along the line of *lemons* where $d_L = \sin\epsilon$. Along the right boundary of *peanuts* and *violins*, $\gamma = \pi$, we have $R = 4\tan\epsilon(1 - \tan\epsilon)$; thus R decreases from 0 towards $-\infty$ as ϵ increases from $\pi/4$ to $\pi/2$, then R increases back to 0 as ϵ goes to $3\pi/4$.

Fig. 17 *about here*

The situation is somewhat simpler for horizontal orbits: for all four shapes we find

$$R \;=\; -4d_S(1-d_S) \tag{32}$$

where d_S is the *stadium* parameter introduced in Eq. (9). This orbit is the prototype of a *stadium* orbit as it only sees the circular arcs around $C_{1,3}$ which could be part of a *stadium*. Therefore the orbit cannot possibly be elliptic. Except for the line of circles, $\epsilon = 0$, where $d_S = 0$ and $R = 0$, d_S is everywhere > 0, and so $R < 0$; the orbits are hyperbolic with positive eigenvalues. The residue is $R = -8$ along the lines $\epsilon = \gamma/2$, and $\gamma = \pi$. In the limit of *lemons*, $\epsilon \to \gamma$, $R \to -\infty$.

5.3 Orbits of period 4

Of all orbits of period 4 the most conspicuous have two symmetries: T,X or T,Y or X,Y (cf. Fig. 18).

Fig. 18 *about here*

Let us first consider orbits with symmetries T and Y. They come in pairs of V or Λ-shaped trajectories, their two legs being reflected back into themselves in the end points. This requires the trajectories to pass through the centers C_1 and C_3. It is not hard to see that only *ovals* (including *lemons*) and *peanuts* allow for this kind of orbits. As they always have points on circular arcs with radii r_x and r_y, they are neither *lemon* nor *stadium* orbits; we call them *mixed* orbits.

Their two legs are characterized by matrices $\mathbf{B}_{2,1}$ and $\mathbf{B}_{1,2} = \mathbf{U}\,\mathbf{B}_{2,1}^{-1}\,\mathbf{U}$ with

$$r_1 = r_x,\ p_{1\perp} = 1; \quad r_2 = r_y,\ p_{2\perp} = \cos\varphi \ \text{ where } \ \tan\varphi = \frac{c_x}{c_y + r_y}; \quad l = r_x + \frac{c_x}{\sin\varphi}\ . \tag{33}$$

The residue of the monodromy matrix $\mathbf{M} = (\mathbf{B}_{1,2}\mathbf{B}_{2,1})^2$ is

$$R \;=\; 4B(1-B) \tag{34}$$

whith

$$B \;=\; \left(\frac{l}{r_1} - 1\right)\left(\frac{l}{r_2 p_{2\perp}} - 1\right) \;=\; (lK_1 - 1)(lK_2 - 1), \tag{35}$$

where the K_i of Eq. (29) have been introduced. The results of a computation of R are shown in Fig. 19. Note that these orbits do not exist in the low ϵ region of *ovals*, where $r_y < -2c_y$. At the limit of their existence, the legs of these trajectories go to the points $\mathrm{P_i}$ of discontinuous curvature, and the centers C_2, C_4 lie on the boundary of the billiard. When this happens, a family of parabolic *lemon* orbits of residue $R = 0$ exists which connects the V- or Λ-shaped orbits to the marginally stable ($R = 1$) *lemon* orbits of period 2.

Next we consider orbits with symmetries T and X. They are >-shaped in *moons* and *violins*, and <-shaped in *flippers* (where they exist for $2c_x < -r_x$ only), see Fig. 18. Their residues are again given by formulas (34) and (35), with

$$r_1 = -r_y; \quad r_2 = r_x,\ p_{2\perp} = \cos\varphi \ \text{ where } \ \tan\varphi = \frac{c_y}{c_x + r_x}; \quad l = -r_y + \frac{c_y}{\sin\varphi}\ . \tag{36}$$

As they exist in complementary regions of parameter space, the results are included in Fig. 19.

Fig. 19 *about here*

The stability diagram shows that all orbits with symmetries T,Y or T,X, just as the fully symmetric period 2 orbits, have residues $R \le 1$. There are lines in parameter space where $B = 1/2$ and thus $R = 1$, indicating that the orbits are about to bifurcate via period doubling. These lines are the center of regions of ellipticity (residues $0 < R < 1$). Stability is lost along lines where $R = 0$ and the orbits become hyperbolic with positive eigenvalues (i. e. residues $R < 0$).

Fig. 20 *about here*

Fig. 21 *about here*

Let us now consider orbits with X,Y but no T-symmetry. They are either rectangles or diamonds, see Fig. 18. We begin with the rectangles which are either *lemon* or *stadium* orbits. They consist of four segments of the kind discussed in Eq. (26). Their monodromy matrix is $\mathbf{M} = (\mathbf{B}^v\mathbf{B}^h)^2$ with $\mathbf{B}^h$ and $\mathbf{B}^v$ like $\mathbf{B}_{3,1}$ in Eq. (26), for the horizontal and vertical segments respectively. It is obvious that there is a distinction to be made for $\gamma < \pi/4$ and $\gamma > \pi/4$. In the former case, the corners of the rectangle lie on circles with radii r_y: the rectangular orbit is then a *lemon* orbit. All $p_\perp = \cos(\pi/4) = 1/\sqrt{2}$, and $l = l^h = r_y/\sqrt{2}$ for the horizontal segments, $l = l^v = l^h + c_y$ for the vertical. The result for the residues is again Eq. (34) with

$$B \; = \; 2\sqrt{2}\, d_L \; , \tag{37}$$

i. e. it is given by the *lemon* parameter d_L. When $\gamma > \pi/4$, the corners lie on circles with radii r_x, and the orbit is of *stadium* type. Interchanging horizontal and vertical segments with respect to the case $\gamma < \pi/4$, we immediately get Eq. (34) again with

$$B \; = \; -2\sqrt{2}\, d_S \; . \tag{38}$$

Whereas Eq. (37) is only applicable to *ovals*, the latter result holds for *peanuts* and *violins* as well, provided the orbit is not intercepted by the concave parts of the boundary. This requires r_x to be smaller than $\sqrt{2}(c_y \pm r_y)$, where the plus sign holds for *peanuts*, the minus sign for *violins*.

Fig. 20 is the stability diagram for these orbits. For $\gamma < \pi/4$ and sufficiently small ϵ, or $d < 1/2\sqrt{2}$, they are stable. For $d = 1/4\sqrt{2}$ the residue is 1. When $\gamma > \pi/4$, these rectangular orbits are unstable with negative residues, i. e. positive eigenvalues.

The case of diamonds is more complicated because the orbits are of *mixed* type, but everything can be computed from a matrix $\mathbf{B}_{2,1}$ of type (20) with

$$r_1 = r_y,\; r_2 = r_x;\;\; l^2 = (c_x + r_x)^2 + (c_y + r_y)^2;\;\; p_{1\perp} = \frac{c_y + r_y}{l},\; p_{2\perp} = \frac{c_x + r_x}{l} \; . \tag{39}$$

The monodromy matrix is the same as for the V-Orbits, and the residue can be obtained from Eq. (34) and (35) by inserting the above values for l, K_1 and K_2. This result holds for *ovals*, *peanuts*, and *violins* (if r_y is replaced by $-r_y$ in the latter case). The stability diagram is presented in Fig. 21. The diamond shaped orbits are stable in part of the *oval* region, hyperbolic with positive eigenvalues everywhere else.

5.4 Orbits of period 3

Orbits of any odd period cannot have time reversal symmetry; they appear in pairs which transform into each other under T. Likewise, they cannot have both X and Y symmetry. In discussing orbits of period 3, we therefore concentrate on orbits with symmetry X or Y. As the calculations are getting more and more cumbersome, we shall from now on restrict the discussion to the region of *ovals* and *lemons*. It will be sufficient to consider orbits with Y symmetry if we allow the angle ρ to vary from 0 to $\pi/2$, or ϵ from γ down to $\gamma - \pi/2$. Billiards with $\rho > \gamma$, or $\epsilon < 0$, are the mirror images of standard billiards under reflection at the main diagonal, and their Y symmetry transforms into X symmetry of the standard billiards. Consider Fig. 22 for location of the orbits and notation used.

Fig. 22 *about here*

There are two types of symmetric orbits of period 3: those which have all three corners on circles with the same curvature (*lemon* or *stadium*), and those which meet circles of both radii r_x and r_y (*mixed*). The former orbits are *lemon* orbits if $\rho < \gamma$, and *stadium* orbits if $\rho > \gamma$; their stability should depend on either d_L or d_S only.

In all cases the monodromy matrix can be written as

$$\mathbf{M} \;=\; \mathbf{B}_{1,3}\,\mathbf{U}\,\mathbf{B}_{2,1}{}^{-1}\,\mathbf{U}\,\mathbf{B}_{2,1}\ , \tag{40}$$

with $\mathbf{B}_{2,1}$ as in Eq. (20), and $\mathbf{B}_{1,3}$ as in Eq. (26). Let us begin with *lemon/stadium* type orbits. As points 1 and 3 belong to the same circle, the matrix $\mathbf{B}_{1,3}$ has the simple form of Eq. (22), with $r = r_y$ and $p_\perp = \cos\varphi_1$. Matrix $\mathbf{B}_{2,1}$ is

$$\begin{aligned}\mathbf{B}_{2,1} \;&=\; \begin{pmatrix} \dfrac{l_1 - r_y\cos\varphi_1}{r_y\cos\varphi_2} & -\dfrac{l_1}{\cos\varphi_1\cos\varphi_2} \\[2ex] \dfrac{r_y(\cos\varphi_1+\cos\varphi_2)-l_1}{r_y^2} & \dfrac{l_1 - r_y\cos\varphi_2}{r_y\cos\varphi_1}\end{pmatrix} \\ &=\; \begin{pmatrix} 1-2d & -\dfrac{r_y}{2\cos\varphi_1\sin\varphi_1}(1+2(1-2d)\sin\varphi_1) \\[2ex] \dfrac{4\cos\varphi_1\sin\varphi_1}{r_y}d & 1-4d\sin\varphi_1\end{pmatrix}\end{aligned} \tag{41}$$

where d is the *lemon* or *stadium* parameter, depending on whether $\rho < \gamma$ or $\rho > \gamma$. Use has been made of the relation $\cos\varphi_2 = 2\sin\varphi_1\cos\varphi_1$ as well as of the sine theorem for the triangle $12\mathrm{C}_2$,

$$\frac{l_1}{\cos\varphi_1} \;=\; \frac{2c_y + r_y}{\sin\varphi_1} \;=\; \frac{r_y}{\cos 2\varphi_1}\ . \tag{42}$$

One of its consequences is the expression

$$\sin\varphi_1 \;=\; \frac{1}{4(1-2d)}\left(-1+\sqrt{1+8(1-2d)^2}\right) \tag{43}$$

from which the relation

$$\frac{1-2d}{\sin\varphi_1} \;=\; 1+2(1-2d)\sin\varphi_1 \tag{44}$$

can be derived and has been substituted in $\mathbf{B}_{2,1}$.

It is now straightforward to compute the residue of $\mathbf{M}$:

$$R \;=\; 2d\,(1+4(1-2d)\sin\varphi_1)) \;=\; 2d\sqrt{1+8(1-2d)^2}\ . \tag{45}$$

As expected, this result depends only on the parameter d. Its limit of validity is reached when the orbit meets the points P_2, P_3 of discontinuous curvature. This happens when

$$2\tan\rho \;=\; \tan 2\gamma + \tan\gamma - \frac{1}{\cos\gamma}\ . \tag{46}$$

$d = d_L$ if $\rho < \gamma$, or $\epsilon > 0$. For billiards with $\epsilon < 0$ we now apply the transformation $(\gamma, \epsilon) \to (\pi/2 - \gamma, -\epsilon)$ to obtain standard shaped billiards with $\epsilon > 0$, and $d = d_S$. The result is shown in Fig. 23 where the orbits just discussed are immediately recognizable from the fact that their lines of constant R are lines of constant *lemon* parameter in the case of Y symmetry (a), and of constant *stadium* parameter in the case of X symmetry (b). The separation line from the *mixed* type period 3 orbits runs from $(\gamma, \epsilon) = (\pi/10, \pi/10)$ to $(\pi/3, 0)$ in Fig. 23a where the *lemon* orbits are always stable. In Fig. 23b, the separation line runs from $(\pi/4, \pi/4)$ down to $(2\pi/3, 0)$, with *stadium* orbits to its right. They are always hyperbolic, with positive eigenvalues.

Fig. 23 *about here*

We now turn to orbits *mixed* type. Both matrices $\mathbf{B}_{2,1}$ and $\mathbf{B}_{1,3}$ in Eq. (40) are now considerably more complicated. They must be employed in their original forms (20) and (26), with

$$r_1 = r_x,\ r_2 = r_y;\ p_{1\perp} = \cos\varphi_1,\ p_{2\perp} = \cos\varphi_2;\ l = l_1 = r_x \sin\varphi_1 - c_y \tag{47}$$

in $\mathbf{B}_{2,1}$, and

$$r = r_x,\ p_\perp = \cos\varphi_1,\ l = l_2 = r_x \cos\varphi_1 + c_x \tag{48}$$

in $\mathbf{B}_{1,3}$. The angles φ_1 and φ_2 are related by $\varphi_2 = \pi/2 - 2\varphi_1$, and φ_1 is given implicitly by

$$\tan\rho \ = \ \tan\gamma\,\frac{\sin(\gamma + 2\varphi_1) - \cos 2\varphi_1}{\sin(\gamma + 2\varphi_1) - \cos\varphi_1}\ . \tag{49}$$

The residue is computed using

$$R \ = \ (1 - 2l_1K_1)(1 - (l_1 + l_2)K_2 - 2l_2K_1 + 2l_1l_2K_1K_2) \tag{50}$$

which is a general consequence of the monodromy matrix (40). The K_i are the "perpendicular curvatures" introduced in Eq. (29).

The results are also shown in Fig. 23. There are regions of stability for both X- and Y-symmetric orbits of *mixed* type. In addition to hyperbolic regions with positive eigenvalues, there are also regions with negative eigenvalues. It is interesting to consider the three limiting cases of *ovals*: *circles*, *lemons*, and *stadiums*. The line of *circles* is divided in three equal parts, with X- and Y-symmetric orbits alternating in stability, one being elliptic, the other hyperbolic (with positive eigenvalues). The only stable orbits along the line of *lemons* are the Y-symmetric *lemon* orbits, with the exception of just two singular cases $\gamma = \pi/10$ and $\gamma = \pi/2$ of *mixed* type orbits, with X and Y symmetry respectively. In *stadiums*, all orbits are of course unstable, X-symmetric orbits of *stadium* type with positive eigenvalues, Y-symmetric orbits of *mixed* type with negative eigenvalues.

5.5 Orbits of period 6 and higher

We derive here the range of existence, and the stability properties, of three different types of symmetric *lemon* orbits of period 6. One has symmetries T,Y, the other two X,Y. The last of these orbits belongs to a sequence with periods 2N that will be analyzed to infinite order. This family also exists as unstable *stadium* orbits. Fig. 24 gives a survey on the various trajectories, and introduces the notation.

Fig. 24 *about here*

We begin with orbits of symmetries T and Y. Their stability matrix is

$$\mathbf{M} \ = \ (\mathbf{U}\,\mathbf{B}_{2,1}^{-1}\,\mathbf{U}\,\mathbf{B}_{3,2}\,\mathbf{B}_{2,1})^2\ , \tag{51}$$

where $\mathbf{B}_{2,1}$ is of type (20), with $p_{1\perp} = 1$, and $\mathbf{B}_{2,3}$ of type (26). Orbits of this kind, in general, have the residue

$$R = (1-2l_1K_2)(K_1+2K_2-2l_1K_1K_2)(2l_1+l_2+2l_1l_2K_2)(2-(2l_1+l_2)K_1-2l_2K_2+2l_1l_2K_1K_2). \tag{52}$$

It is a matter of elementary geometry to express the lengths l_1, l_2 in terms of the *lemon* parameter d_L, the result being

$$l_2 \;=\; \frac{\sqrt{2}}{4d_L}\sqrt{-1+16d_L^2-\sqrt{1+32d_L^2}}\;, \qquad l_1 \;=\; 1+\frac{l_2}{l_2^2-2}\;. \tag{53}$$

The K_i are obtained from

$$r_1 = r_2 = r_y; \;\; p_{1\perp} = 1, \;\; p_{2\perp} = \cos\varphi_2 = \frac{l_2}{2r_y}\;. \tag{54}$$

This leads to the residues

$$R \;=\; -(2-2l_1+l_2)(4-4l_1+l_2)(1-\frac{2l_1}{l_2})(1-\frac{4l_1}{l_2}) \tag{55}$$

where the l_i are understood to be given by Eq. (53). The condition of existence of these orbits is $\varphi_1 < (\pi-\gamma)/2$.

Fig. 25 shows the result. The orbit exists in a fairly small region of parameter space, near the *lemon* line $\epsilon = \gamma$. Changing parameters along a path parallel to this line, the orbit appears and disappears at a point of discontinous curvature, with residue $R \neq 0$. On the *lemon* line, the orbit is unstable at the boundaries of its existence, but runs through a region of stability in between.

Fig. 25 *about here*

Next we analyze orbits with symmetries X and Y of which there are two kinds. The orbit of Fig. 24b is characterized by a stability matrix of the same form as (51), if points 1, 2, 3 are chosen as indicated. The residue is therefore given by the general result (52), but the different geometry leads to different expressions for the l_i and K_i. We find

$$r_1 = r_2 = r_y; \;\; p_{2\perp} = \frac{1}{4d_L-2}, \;\; p_{1\perp} = 2{p_{2\perp}}^2-1; \;\; l_1 = 2d_L-1\;, l_2 = \frac{1+2d_L-4d_L^2}{2d_L-1}\;. \tag{56}$$

The residue can then be given directly in terms of d_L:

$$\begin{aligned} R \;&=\; -\;\frac{32d_L^3(4d_L-1)(4d_L-3)^3}{(1-8d_L(1-d_L))^2}\;\cdot \\ &\quad\cdot\;\left(-1+6d_L-32d_L^2(1-d_L)\right)\left(-1+2d_L+4d_L^2+32d_L^2(1-3d_L+2d_L^2)\right) \end{aligned} \tag{57}$$

The condition of existence of this *lemon* orbit is

$$d_L \geq \frac{1}{2} \qquad \text{and} \qquad 0 \leq \frac{(4d_L-1)(4d_L-3)}{4(2d_L-1)^2} \leq \cos^2\gamma\;. \tag{58}$$

Fig. 25 contains the corresponding stability diagram. The orbit exists above the curve $d_L = 3/4$ where $R = 0$; it is born along this curve in a natural bifurcation of the vertical orbit of period 2 (whose residue is then 3/4, corresponding to winding number 1/3). The upper boundary in parameter space is reached when the orbit hits the points of discontinuous curvature. The orbit is stable wherever it exists.

The last type of period 6 orbits is the simplest. It develops out of the regular hexagon in the limit of circular billiard shape. There is in fact a whole family of orbits of period

2N which behaves in a similar way, starting with the vertical period 2 orbit (N=1) and the rectangular period 4 orbit (N=2). It is easy to treat the general case because the monodromy matrix is

$$\mathbf{M} \;=\; (\mathbf{B}^{N-1}\,\mathbf{C})^2 \;, \tag{59}$$

where $\mathbf{B}$ is of type (22) and $\mathbf{C}$ of type (26). The curvature is everywhere given by $r_i = r_y$, the angles by $p_\perp = \cos\varphi = \sin\pi/2N$, the segment length in matrix $\mathbf{C}$ by $l = r_y(p_\perp - d_L)$. The residue of these orbits turns out to be

$$R \;=\; R_N \;:=\; 4\,\frac{N d_L}{p_\perp}(1 - \frac{N d_L}{p_\perp})\;. \tag{60}$$

The range of existence in parameter space is given by $\gamma = \gamma_N := \pi/2N$ where the *lemon* parameter has the value $d_L = d_N := \sin\pi/2N$. The residue R_N starts with the value 0 at $d_L = 0$ and reaches its maximum $R_N = 1$ at $d_L = d_N/2N$.

6 Discussion

We have presented a comprehensive picture of the global features of chaoticity for a two-parameter family of billiards. There is an immense richness in the parameter dependence of the relative area A of the main chaotic part of the energy surface, and of its Lyapunov exponent λ. Only part of this complexity can possibly be described in quantitative terms, let alone be explained by analytic methods. Nevertheless, a fair amount of the most conspicuous features can be understood on the basis of a consideration of stability properties of the most important periodic orbits.

The prime observation is that, as far as numerical evidence can tell, the system seems to be ergodic in a good part of parameter space. This is in agreement with a

According to a theorem by Bunimovich [5] a billiard is ergodic, if it is composed of circular arcs and for any convex component of its boundary the complement to a full circle lies inside the billiard. This happens to be true for *peanuts* and *concave drops*, but for no other general shape in the generalized Benettin-Strelcyn family. The largest Lyapunov exponent is found in the region of *peanuts*.

In about half of the parameter space, the system exhibits mixed behavior, i. e., an intricate combination of regular and chaotic motion. This behavior is reminiscent of the typical behavior of smooth Hamiltonian systems with two degrees of freedom, even though the isolated discontinuities in curvature (and even worse: in the direction of tangents) introduce severe modifications with a tendency to enhance chaoticity. This aspect has been carefully discussed by Hénon and Wisdom [11] who showed that the existence of KAM tori is much more delicate in Benettin-Strelcyn billiards than it is in smooth systems. On the other hand, the absence of a large portion of KAM tori which would normally exist, helps to make our strategy meaningful to measure the degree of global stochasticity by looking only at the main chaotic region: in many cases, as was shown in Ref. [11], there exists no other (apart, of course, from the unobservably thin bands surrounding elliptic centers).

A system is completely chaotic if all its periodic orbits are unstable. Conversely, the existence of elliptic periodic orbits reduces the relative area of the chaotic part of the energy surface. Very roughly, it may also be stated that the Lyapunov exponent is reduced by the existence of elliptic islands, for two reasons. One is that small chaotic bands tend to be only weakly hyperbolic even at their center (by continuity of the linearized Poincaré map), the other reason is that when chaotic orbits spend a large portion of their time near the outskirts of elliptic islands, they cannot develop strong exponential divergence on the average. Closer inspection, however, shows that the argument must be refined in connection with bifurcations of the elliptic orbits. It is well known [1] that when the winding number around non-degenerate elliptic orbits is 1/3 (corresponding to residue R =3/4), then nonlinear stability analysis gives zero radius of stability, i. e., the size of the elliptic island shrinks to zero, and the chaotic area tends to be large. Nevertheless the Lyapunov exponent may be small because the parabolic traces of the vanishing island may reach far into the chaotic region. At bifurcations with winding number 1/4 (corresponding to residue R =1/2) the picture can be similar, but with a characteristic asymmetry on the two sides of the bifurcation. For winding numbers $1/n$ with $n > 4$, the asymmetry is even more pronounced: "daughter orbits" of period n (relative to the elliptic "mother orbit") exist only on one side; they are born in the bifurcation and move out from the center into the chaos band. When their hyperbolic part merges with the main chaos, its relative area is momentarily increased, and its Lyapunov exponent reduced, as in the case of bifurcations with winding numbers 1/3 and 1/4, although with smaller and smaller amplitude as n gets larger.

This scenario seems to be the explanation for some of the features that we observe. By non-degeneracy of the elliptic orbit we mean that the symmetry properties of the central "mother" orbit, and of the "daughters" participating in the bifurcation, should be the same. When this is not the case, the picture is different. The bifurcation scheme of the elliptic *lemon* orbit of period 2 is such an exception: its symmetries are T, X, and Y, and when its winding number is $W = 1/n$, $(n > 2)$, *two* elliptic orbits of period $2n$ (i. e. n relative to period 2) are born, with symmetries X and Y only (and with their hyperbolic partners). Together these two orbits behave rather as *one* orbit of period $2n$, relative to period 2, moving out from the elliptic center towards the main chaos where they are finally absorbed.

The case $n = 2$ is particularly different from the non-degenerate case (where winding numbers 1/2 are the beginning of a period doubling scenario). When d_L increases to 1/2, R reaches the value 1 and $W = 1/2$, but no period doubling takes place; rather R decreases again while two orbits of period 4 with symmetries T and X are born (the V- and Λ-shaped orbits). This birth is quite dramatic in that it is accompanied by the existence of two parabolic families of *lemon* orbits of period 4. The behavior is reminiscent of a bifurcation with winding number 1/4, and numerical evidence shows that the parabolic character of the Poincaré map affects a large part of the energy surface. This explains the very strong increase in chaotic area and the drop of the Lyapunov exponent seen in Figs. 15 and 16.

Let us now compare the results of Secs. 4 and 5. Evidently, none of the periodic orbits analyzed has been found to be stable in the parameter regions of *peanuts* and *concave drops*. Moreover, as some of the orbits are marginally stable on the boundaries of these regions, it is at least plausible that the Lyapunov exponent has its maximum deep in the center of the *peanut* region.

The stability of some T-symmetric orbits of period 4 in the regions of *violins, moons*, and *flippers* correlates with the global results on chaotic area and Lyapunov exponents. With some additional effort we could have demonstrated the stability of some non-symmetric orbits of period 2 as well as of other orbits with higher period. But let us concentrate on the region of *ovals* which shows the most interesting behavior anyway. Its dominant feature is the linear stability of the *lemon* orbit of period 2. This gives the global measures of chaos in Figs. 12, 13, and in the more detailed computations of Figs. 15 and 16, a pattern along lines of constant *lemon* parameter d_L. Consider first $d_L = 1/2$, where the orbit has residue $R = 1$, and winding number $1/2$. As was already mentioned, instead of a period doubling scenario, we observe here the X-symmetry breaking birth of two period 4 orbits of *mixed* type. In addition, there exist two entire families of parabolic orbits ($R = 0$) of *lemon* type right at the bifurcation. This explains the very high relative area A and the low Lyapunov exponent λ along this line in parameter space.

Starting with *lemon* parameter $d_L = 3/4$, residue $R = 3/4$, and winding number $W = 1/3$, we have a series of T-symmetry breaking bifurcations with winding numbers $W = W_n := 1/n$ at

$$d_L \;=\; d_n \;:=\; \frac{1}{2}(1+\cos\frac{\pi}{n}) \quad \text{with} \quad R = R_n := (1-\cos 2\pi W_n)/2 = \sin\pi/n\ . \tag{61}$$

The two new orbits behave like a combined orbit of period $2n$. They move out from the elliptic center into the main chaotic band; when they reach it (at d_L-increments somewhat dependent on n), they enhance its relative area, and at the same time reduce its Lyapunov exponent. This regular sequence of d_L-values describes the oscillations on the high-d_L side of Figs. 15 and 16 fairly well. Of course, there are more stable orbits than just those of this particular series, e. g. the orbit of period 6 with broken X symmetry whose contribution to A and λ is recognizable as little tongues in Figs. 12 and 13.

There are indications of similar patterns on the low-d side, $d_L = (1 - \cos\pi/n)/2$ having the same residue as $d_L = d_n$; but these are blurred by interference from other orbits. Remember the sequence of *lemon* orbits of period 2N in the range $\gamma < \gamma_N$ with residues given by Eq. (60), and furthermore the *lemon* orbit of period 3 with residue (45). Of course, there are still more *lemon* orbits with higher odd periods. All these have similar stability properties in d_L-ranges of decreasing width, including bifurcations with winding numbers $1/n$, ($n = 2, 3, \ldots$). Much of the complicated picture of Figs. 15 and 16 can be explained in this way, but as more and more orbits interfere in their contribution to A and λ, the global picture of Figs. 12 and 13 becomes less and less meaningful. A local analysis of the energy surface is then called for.

In addition to what can be explained in terms of *lemon* orbits, there are features that originate from stable *mixed* orbits, such as the elliptic orbits of period 3 (cf. Fig. 23), or the V and diamond shaped orbits of period 4 (cf. Figs. 19 and 21). These *mixed* orbits tend to explain features that run across the lines of constant d_L which are also observed in the global results, notably in the range $\gamma > \pi/4$, and ϵ small, far away from the pure *lemons*. There are again repetitive patterns in these parts of Figs. 12 and 13, indicating another "breathing of chaos", but the scenario is more complicated than can be explained by our stability results for orbits of low periods. It appears that more and more KAM-like lines develop in this region of parameter space, and the analysis should focus on the pattern of bifurcation of homoclinic orbits. This is beyond the scope of the present investigation.

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Figure captions

Fig. 1: Definition of a generalized BS billiard. Points P_i ($i = 1, 2, 3, 4$) form the basic rectangle, defined by size a and angle ρ. The four circular arcs $\widehat{P_i P_{i+1}}$ (we identify P_5 as P_1) have centers C_i with location determined by the angle γ. The radii are r_x for centers $C_{1,3}$ and r_y for centers $C_{2,4}$. The angle $\epsilon = \gamma - \rho$ is also indicated; we shall use the set of (γ, ϵ) pairs as a convenient description of parameter space.

Fig. 2: Alternative choice of circular arcs, for the same γ and ρ as in Fig. 1. The figure can be partitioned into five billiards: a *lemon* at the center, two *flippers* with X symmetry and two *moons* with Y symmetry.

Fig. 3: Survey of (γ, ϵ)-parameter space. There are seven regions of different shapes in the generalized Benettin-Strelcyn system, three of them with Y symmetry and the boundary in a single piece – *ovals*, including *circles, stadiums*, and *lemons* as limiting cases, *peanuts*, and *violins* –, the other four without that symmetry and as symmetric pairs: *concave* and *convex drops, flippers*, and *moons*.

Fig. 4: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *oval* $(\gamma, \epsilon) = (1, 0.5)$. The horizontal coordinate is the arc length s along $\partial\mathcal{B}_{\gamma,\epsilon}$, starting ($s = 0$) and ending ($s = 1$) at point P_1; the other points P_i are indicated as vertical lines. The vertical coordinate is the tangential component $p_{\|}$ of the velocity at the moment after reflection. The dominant feature of *ovals* is a pair of orbits of period 2: an elliptic orbit with points at $(s, p_{\|}) = (s_e, 0)$ and $(s_e{+}1/2, 0)$, and a hyperbolic orbit with points $(s_h, 0)$ and $(s_h{+}1/2, 0)$. The relative size of regular and chaotic portions of $\mathcal{P}_{\gamma,\epsilon}$ varies with parameters (γ, ϵ). Here, the small islands belong to three kinds of periodic orbits; a: V-shaped orbit of period 4, with symmetries T and X, see Fig. 18; b: diamond shaped orbit of period 4, with symmetries X and Y; c: orbit of period 6 with symmetries X and Y. All these orbits have a partner obtained by applying the symmetry they are lacking.

Fig. 5: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *lemon* $(\gamma, \epsilon) = (0.9, 0.9)$. As points P_1, P_2 as well as P_3, P_4 coincide, there are only two lines of discontinuity left. These lines are now the heart of the main chaotic region, but a lot of regularity is found between them. At this particular choice of γ, there exist two strong resonances of period 6, related by symmetry T. The corresponding orbits are of the type shown in Fig. 24b, with symmetries X and Y.

Fig. 6: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *peanut* $(\gamma, \epsilon) = (2.4, 1.5)$. The motion is ergodic. The conspicuous horizontal lines are neighborhoods of parabolic orbits of periods 2 and 3, as discussed in Sec. 5.1, with monodromy matrices of type (23).

Fig. 7: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *violin* $(\gamma, \epsilon) = (2.5, 0.65)$. The motion is predominately chaotic. Apart from two parabolic orbits of period 2, there are two kinds of islands with stable orbits at their centers; a: period 10 with symmetries T,X; b: period 8 with only the combined symmetry TX.

Fig. 8: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *moon* $(\gamma, \epsilon) = (2.34, 0.31)$. The two conspicuous resonances have periods 4 (in the center) and 6, both with symmetries T and X. The period 4 orbit is of the type shown in 18b.

Fig. 9: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *flipper* $(\gamma, \epsilon) = (2.46, 2.95)$. To the left a period 6 orbit with TX symmetry can be observed, at the right and the center a period 4 orbit with the same symmetry. This type of orbit is shown in Fig. 18d.

Fig. 10: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *convex drop* $(\gamma, \epsilon) = (1.7, 2.4)$. There are two stable resonances of period 5 and symmetry X, related by T.

Fig. 11: Poincaré surface of section $\mathcal{P}_{\gamma,\epsilon}$ for the *concave drop* $(\gamma, \epsilon) = (2.75, 2.36)$. The motion is ergodic. Parabolic orbits with periods up to 5 can be recognized.

Fig. 12: Relative area A of the main chaotic band in the Poincaré surface of section. For each point (γ, ϵ) in parameter space, a value is indicated in terms of the grey scale shown as the inset at left (running from 0 at the top to 1 at the bottom). A=0 means complete regularity of the motion, or integrability. A=1 means complete irregularity in the sense that a single chaotic orbit is dense in the energy surface; this is numerical evidence for ergodicity of the motion. The picture gives a static measure of the extent of chaos. – The resolution is 0.01 in γ and ϵ.

Fig. 13: Lyapunov exponent λ of the main chaotic band. For each point (γ, ϵ) in parameter space, the value of λ is indicated in terms of the grey scale shown as the inset. λ ranges from 0 (top of the scale) to 1.10 (at bottom); values near 0 mean slow mixing,

high values of λ mean fast mixing. The Lyapunov exponent allows for a distinction of different degrees of chaoticity where the relative chaotic area is uniformly 1. The white maximum is attained in the peanut region, to the right of the lower left triangle.

Fig. 14: Relative chaotic area A and Lyapunov exponent λ along the line $\rho = 0.3(\pi/2 - \gamma)$. The *lemon* parameter d_L is used to parametrize the line; $d_L = 0$ is the limit of a *circle*, $d_L = 1$ the limit of an infinitely long box. At $d_L = 1/2$, the area A has a pronounced peak indicating almost complete chaos whereas the Lyapunov exponent λ, on the contrary, develops a sharp minimum. A similar anticorrelation is clearly observed in the whole range $d_L > 1/2$.

Fig. 15: Relative chaotic area A along five lines $\rho = \mathrm{m}(\pi/2 - \gamma)$ in the region of *ovals*; m = 0, 0.1, 0.2, 0.3, 0.4, starting with m=0 (*lemons*) at the lowest curve. The maxima are quite well aligned because the area is plotted vs. the *lemon* parameter d_L.

Fig. 16: Lyapunov exponent λ along the same lines as in Fig. 15. The agreement of curves with different m is not as perfect as in the case of area A, but nevertheless there is a clear correlation of extrema with the same d_L but different m.

Fig. 17: Stability of orbits of period 2. (a) vertical, or *lemon* orbits; (b) horizontal, or *stadium* orbits. The shading gives the values of Greene's residue R, with independent scales for elliptic and hyperbolic regions. The region $\gamma < \pi/2$ of part (a) is elliptic, with residues $0 < R < 1$. Here the grey scale identifies R-values from $R = 0$ (black) to $R = 1$ (light-grey); lines of constant $R = 0.1, 0.2, \ldots, 1.0$ are marked white. All other regions of (a) and (b) are hyperbolic, with $R < 0$. The grey scale is then for negative R, from $R = 0$ (light-grey) to $R \to -\infty$ (black); lines $R = -1, -2, \ldots, -10$ are drawn white. In this way a change of stability is indicated by a jump from light-grey to black. Regions where the corresponding orbit does not exist are white. This code is used for all pictures of this type.

Fig. 18: Trajectories of period 4. a: V-shaped orbit with T and Y symmetries. This type occurs in *ovals* and *peanuts*. b - d: Orbits with T and X symmetries in *moons*, *violins*, and *flippers*. e: Rectangular orbit. f: Diamond shaped orbit; both types e and f have symmetries X and Y. They occur in *ovals*, *peanuts*, and *violins*.

Fig. 19: Stability of orbits of period 4 with symmetries T,Y and T,X. There are three elliptic regions with $0 < R < 1$: the central part in the region of *ovals* (symmetries T,Y), the bottom of the region of *moons* and *violins*, and the upper part of the region of *flippers* (both with symmetries T,X). The other shaded regions are for negative R.

Fig. 20: Stability of orbits of period 4 with symmetries X, Y, and rectangular shape. Elliptic orbits occur only in the lower part of the *lemon* type region $\gamma < \pi/4$. The rest is hyperbolic, with $R < 0$. The break at $\gamma = \pi/4$ results from the points of discontinuous curvature.

Fig. 21: Stability of orbits of period 4 with symmetries X, Y, and diamond shape. Orbits are elliptic in the horn shaped part of the region $\pi/4 < \gamma < \pi/2$. The rest is hyperbolic, with $R < 0$.

Fig. 22: Trajectories of period 3. Point 2 is always on the y-axis. The segments 12 and 23 are of equal length l_1, the segment 13 is horizontal and of length $2l_2$. The angles φ_1 and φ_2 are related by $\varphi_2 = \pi/2 - 2\varphi_1$. *lemon/stadium* orbits (a) have points 1 and 3 on the upper circle of radius r_y, *mixed* orbits (b) have them on the two circles with radius r_x.

Fig. 23: Stability diagram of symmetric orbits of period 3, for *ovals*; (a) orbits with Y-symmetry, (b) orbits with X-symmetry. *Lemon* type orbits are in the lower left part of (a); their stability depends only on d_L, cf. Fig. 17(a). *Stadium* type orbits are in the right part of (b); their stability depends on d_S only, cf. Fig. 17(b). The line separating these

two from the *mixed* orbits is given by Eq. (46). In the inverse hyperbolic parts next to the *stadium* in (a), and in the upper central region in (b), the grey values indicate $R = 1$ (black) to $R \to \infty$ (light grey). The values $R = 1, 2, \ldots, 10$ are marked as white lines.

Fig. 24: Trajectories of period 6. a: Orbit with symmetries T and Y; at point 1 it hits the boundary at right angle. b: Orbit with symmetries X and Y. c: Hexagon type orbit, also with symmetries X and Y.

Fig. 25: Stability diagram of symmetric *lemon* orbits of period 6. (a) Orbits with symmetries T and Y are elliptic in the central part of their region of existence, hyperbolic with $R < 0$ on the two flanks. (b) Orbits with symmetries X and Y bifurcate from the vertical period 2 orbit along the line $d_L = 3/4$, and are elliptic wherever they exist. (c) Orbits with symmetries X and Y that develop from regular hexagons on the line of *circles* are part of a family that includes the vertical orbits of period 2, cf. Fig. 17(a), and the rectangular *lemon* orbits of period 4, cf. Fig. 20. They are elliptic in the low-ϵ range, hyperbolic with $R < 0$ elsewhere.