

Recall $Q \triangleleft A$ means $Q \neq A$ and
primary

$$xy \in Q \Rightarrow x \in Q \text{ or } y^n \in Q \quad \exists n \geq 1.$$

Example: (primary ideals need not be powers of prime ideals)

Let $A = F[x, y]$, $Q = \langle x, y^2 \rangle \triangleleft A$
primary

so

$$r(Q) = \langle x, y \rangle \triangleleft A$$

prime

and

$$r(Q)^2 \subsetneq Q \subsetneq r(Q)$$

facts checked last time

Claim: Q is not the power of any prime ideal.

Proof: If $Q = P^k \quad \exists P \triangleleft A \text{ prime}, \exists k$

then

$$r(Q) = r(P^k) = P \quad (\text{first assignment})$$

so

$$P^2 = r(Q)^2 \subsetneq Q = P^k \subsetneq r(Q) = P$$

~~X~~ if $k \geq 2$

$$P^2 \subsetneq P^k \subseteq P^2$$

~~X~~ if $k = 1$

$$P \subsetneq P$$

□

(B)

Example: (powers of prime ideals need not be primary)

$$\text{Let } A = F[x, y, z], \quad I = (xy - z^2)A,$$

$$B = A/I, \quad P = \langle x+I, z+I \rangle.$$

" $xy = z^2$ " in B , so B is not a UFD

Claim: $P \triangleleft B$ but P^2 is not primary.
prime

Proof: Notice first the following decomposition

of $x = x(x, y, z) \in A$:

$$x = x\beta + z\gamma + x(0, y, 0)$$

$$\exists \beta \in A, \exists \gamma \in F[z, y]$$

(*)

Certainly $P \neq B$ since $1+I \notin P$.

Suppose $\pi_1, \pi_2 \in A$ and

$$(\pi_1 + I)(\pi_2 + I) = \pi_1\pi_2 + I \in P$$

Then $\pi_1\pi_2 + I = x\beta + z\sigma + I \quad (\exists \beta, \sigma \in A)$

(c)

so

$$\pi_1 \pi_2 - x\rho - z\sigma \in (xy - z^2)A.$$

In particular

$$(\pi_1 \pi_2 - x\rho - z\sigma)(0, y, 0) = 0,$$

$$\text{since } (xy - z^2)(0, y, 0) = 0.$$

If $\pi_1(0, y, 0), \pi_2(0, y, 0) \neq 0$ then L.H.S. $\neq 0$, ~~✗~~

Hence

$$\pi_1(0, y, 0) = 0 \text{ or } \pi_2(0, y, 0) = 0,$$

so, from $\textcircled{*}$,

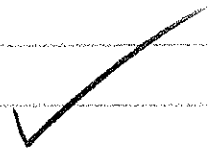
$$\pi_1 \in xA + zA \text{ or } \pi_2 \in xA + zA,$$

so that

$$\pi_1 + I \in P \text{ or } \pi_2 + I \in P.$$

This proves

$$\boxed{P \triangleleft A \text{ prime}}.$$



$$\text{But } P^2 = \langle x+I, z+I \rangle^2$$

$$= \langle x^2+I, xz+I, z^2+I \rangle$$

Suppose P^2 is primary.

$$\text{Observe that } (x+I)(y+I) = xy+I = z^2+I \in P^2.$$

②

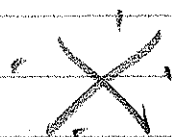
Hence

$$x + I \in P^2 \quad \text{or} \quad (y + I)^k = y^k + I \in P^2 \quad (\exists k \geq 1),$$

so

$$x \quad \text{or} \quad y^k \in \langle x^2, xz, z^2, xy - z^2 \rangle$$

every monomial has degree ≥ 1
and has x or z as a factor



Hence P^2 is not primary.



Primary and prime ideals are closely related
via the radical operator:

Theorem: If $Q \triangleleft_{\text{primary}} A$ then $r(Q) \triangleleft_{\text{prime}} A$,
and $r(Q)$ is the smallest prime ideal containing Q .

Proof: If $Q \triangleleft_{\text{primary}} A$ then, by earlier work,

$r(Q) =$ intersection of prime ideals
containing Q

(E)

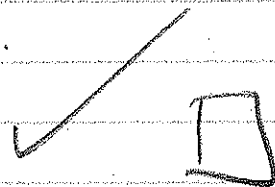
so it suffices to check $r(Q) \trianglelefteq_{\text{prime}} A$:

certainly $r(Q) \neq A$ and

$$xy \in r(Q) \Rightarrow (xy)^n \in Q \quad \exists n \geq 1$$

$$\Rightarrow x^n \in Q \text{ or } (y^n)^k \in Q \quad \exists k \geq 1$$

$$\Rightarrow x \in r(Q) \text{ or } y \in r(Q).$$



We prove that all powers of maximal ideals are primary, and the key is the radical:

Theorem: If $Q \trianglelefteq A$ and $r(Q) \trianglelefteq_{\text{max}} A$
then Q is primary.

Corollary: If $M \trianglelefteq_{\text{max}} A$ then $M^k \trianglelefteq_{\text{primary}} A$
 $\forall k \geq 1$.

Proof: $r(M^k) = M$ by first assignment.



(F)

Proof of Theorem : Suppose $Q \triangleleft A$, $r(Q) \triangleleft_{\max} A$,

Certainly $Q \neq A$. Suppose $x, y \in A$ and

$$\boxed{xy \in Q, \quad x \notin Q, \quad y^k \notin Q \quad \forall k \geq 1}$$

If $y+Q$ is a unit in A/Q then

$$\begin{aligned} x+Q &= (x+Q)(y+Q)(y+Q)^{-1} \\ &= (xy+Q)(y+Q)^{-1} = Q(y+Q)^{-1} \\ &= Q, \end{aligned}$$

so $x \in Q$ ~~✗~~ ✓

Hence $y+Q$ is not a unit in A/Q ,

so, by Zorn's Lemma, $y+Q \in M/Q$

for some $Q \subseteq M \triangleleft_{\max} A$.

But $r(Q)$ is the intersection of all prime ideals containing Q , so $r(Q) \subseteq M$.

Hence $r(Q) \supseteq M$ since $r(Q) \triangleleft_{\max} A$.

In particular $y \in M = r(Q)$ so

$y^k \in Q \quad \exists k \geq 1$ ~~✗~~ ✓ □