(A)

Commutative Algebra Lecture 35 Tue 26/5/09

Recall $Q \underset{\text{primary}}{\lhd} A$ means $Q \neq A$ and

$xy \in Q \Rightarrow x \in Q$ or $y^n \in Q \ \exists n \geq 1$.

Example: (primary ideals need not be powers of prime ideals)

Let $A = F[x,y]$, $Q = \langle x, y^2 \rangle \underset{\text{primary}}{\lhd} A$,

so

$$r(Q) = \langle x, y \rangle \underset{\text{prime}}{\lhd} A$$

facts checked last time

and

$$r(Q)^2 \subsetneq Q \subsetneq r(Q)$$

$\boxed{\text{Claim: } Q \text{ is not the power of any prime ideal.}}$

Proof: If $Q = P^k \ \exists P \underset{\text{prime}}{\lhd} A, \ \exists k$,

then

$$r(Q) = r(P^k) = P \quad \text{(first assignment)}$$

so

$$P^2 = r(Q)^2 \subsetneq Q = P^k \subsetneq r(Q) = P$$

$\times$ if $k \geq 2$ $\qquad$ $\times$ if $k = 1$

$P^2 \subsetneq P^k \subseteq P^2$ $\qquad$ $P \subsetneq P$

$\square$

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Example: (powers of prime ideals need not be primary)

Let $A = F[x,y,z]$, $I = (xy - z^2)A$,

$B = A/I$, $P = \langle x+I, z+I \rangle$.

"$xy = z^2$" in B, so B is not a UFD

$\boxed{\text{Claim: } P \underset{\text{prime}}{\lhd} B \text{ but } P^2 \text{ is not primary.}}$

Proof: Notice first the following decomposition of $\alpha = \alpha(x,y,z) \in A$:

$$\boxed{\begin{array}{l} \alpha = x\beta + z\gamma + \alpha(0,y,0) \\ \quad \exists \beta \in A, \ \exists \gamma \in F[z,y] \end{array}} \quad (*)$$

Certainly $P \neq B$ since $1 + I \notin P$.

Suppose $\pi_1, \pi_2 \in A$ and

$$(\pi_1 + I)(\pi_2 + I) = \pi_1\pi_2 + I \in P$$

Then $\pi_1\pi_2 + I = x\rho + z\sigma + I \quad (\exists \rho, \sigma \in A)$

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so

$$\pi_1\pi_2 - x\rho - z\sigma \in (xy - z^2)A.$$

In particular

$$(\pi_1\pi_2 - x\rho - z\sigma)(0,y,0) = 0,$$

since $(xy - z^2)(0,y,0) = 0$.

If $\pi_1(0,y,0), \pi_2(0,y,0) \neq 0$ then L.H.S. $\neq 0$, $\times$.

Hence

$$\pi_1(0,y,0) = 0 \quad \text{or} \quad \pi_2(0,y,0) = 0,$$

so, from (*),

$$\pi_1 \in xA + zA \quad \text{or} \quad \pi_2 \in xA + zA,$$

so that

$$\pi_1 + I \in P \quad \text{or} \quad \pi_2 + I \in P.$$

This proves $\boxed{P \underset{\text{prime}}{\lhd} A.}$ ✓

But $P^2 = \langle x + I,\ z + I \rangle^2$

$$= \langle x^2 + I,\ xz + I,\ z^2 + I \rangle$$

Suppose P^2 is primary.

Observe that $(x+I)(y+I) = xy + I = z^2 + I \in P^2$.

Hence
$$x+I \in P^2 \quad \text{or} \quad (y+I)^k = y^k + I \in P^2 \quad (\exists k \geq 1),$$

so
$$x \text{ or } y^k \in \underbrace{\langle x^2, xz, z^2, xy-z^2 \rangle}$$

every monomial has degree > 1 and has x or z as a factor

Hence P^2 is not primary. $\square$

Primary and prime ideals are closely related via the radical operator:

> Theorem: If $Q \underset{\text{primary}}{\lhd} A$ then $r(Q) \underset{\text{prime}}{\lhd} A$, and $r(Q)$ is the smallest prime ideal containing Q.

Proof: If $Q \underset{\text{primary}}{\lhd} A$ then, by earlier work,

$r(Q)$ = intersection of prime ideals containing Q

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so it suffices to check $r(Q) \underset{\text{prime}}{\triangleleft} A$:

certainly $r(Q) \neq A$ and

$$xy \in r(Q) \implies (xy)^n \in Q \quad \exists n \geq 1$$

$$\implies x^n \in Q \text{ or } (y^n)^k \in Q \quad \exists k \geq 1$$

$$\implies x \in r(Q) \text{ or } y \in r(Q).$$

✓ □

We prove that all powers of <u>maximal</u> ideals are primary, and the key is the radical:

<u>Theorem</u>: If $Q \triangleleft A$ and $r(Q) \underset{\max}{\triangleleft} A$ then Q is primary.

<u>Corollary</u>: If $M \underset{\max}{\triangleleft} A$ then $M^k \underset{\text{primary}}{\triangleleft} A$ $\forall k \geq 1$.

<u>Proof</u>: $r(M^k) = M$ by first assignment.

□

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<u>Proof of Theorem</u>: Suppose $Q \lhd A$, $r(Q) \lhd_{max} A$.

Certainly $Q \neq A$. Suppose $x, y \in A$ and

$$\boxed{xy \in Q, \quad x \notin Q, \quad y^k \notin Q \ \forall k \geq 1}.$$

If $y + Q$ is a unit in A/Q then

$$x + Q = (x+Q)(y+Q)(y+Q)^{-1}$$

$$= (xy + Q)(y+Q)^{-1} = Q(y+Q)^{-1}$$

$$= Q,$$

so $x \in Q$ "$\times$". ✓

Hence $y + Q$ is not a unit in A/Q, so, by Zorn's Lemma, $y + Q \in M/Q$ for some $Q \subseteq M \lhd_{max} A$.

But $r(Q)$ is the intersection of all prime ideals containing Q, so $r(Q) \subseteq M$.

Hence $r(Q) = M$ since $r(Q) \lhd_{max} A$.

In particular $y \in M = r(Q)$ so

$$y^k \in Q \quad \exists k \geq 1$$

"$\times$". ✓ $\square$