## **1.3 Local Rings and Residue Fields**

Call a ring A local if A has exactly one maximal ideal M , and call  $A/M\,$  the residue field of A .

## **Examples:**

(1) Any field F is local since  $\{0\}$  is the only maximal ideal

(  $\{0\}$  and F are the **only** ideals of F )

and  $F \cong F/\{0\}$  is its own residue field.

(2) Let R be any (possibly noncommutative) ring and G any group written multiplicatively. The **group ring** R[G] comprises formal linear combinations

$$\sum_{e \in G} \alpha_g g$$

where  $\alpha_g \in R$  for each  $g \in G$  and only finitely many  $\alpha_g$  are nonzero, with componentwise addition:

g

$$\sum \alpha_g g + \sum \beta_g g = \sum (\alpha_g + \beta_g) g,$$

and **convolution** product:

$$\left(\sum_{g \in G} \alpha_g g\right) \left(\sum_{h \in G} \beta_h h\right) = \sum_{k \in G} \left(\sum_{gh=k} \alpha_g \beta_h\right) k ,$$

It is routine to check that R[G] is a (possibly noncommutative) ring.

e.g. 
$$\mathbb{Z}_2[C_2] = \{ 1, x, 0, 1+x \}$$
,

where  $C_2 = \{ x , x^2 = 1 \}$  is the cyclic group of order 2 , written multiplicatively,

has unique maximal ideal

$$M = \{ 0, 1+x \}$$
 (easy to verify)

so that  $\mathbb{Z}_2[C_2]$  is local, with residue field

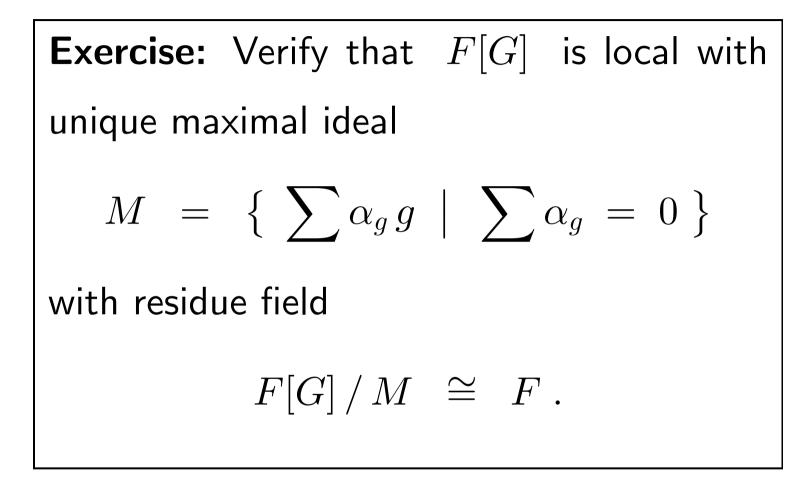
$$\mathbb{Z}_2[C_2]/M \cong \mathbb{Z}_2.$$

**Example:** Let F be a field of prime characteristic p, that is,

$$\underbrace{1+1+\ldots+1}_{p \text{ times}} = 0.$$

Let G be any abelian p-group, that is,

each element of  $\,G\,$  has order a power of  $\,p$  .



A ring with only finitely many maximal ideals is called **semi-local**.

**Exercises:** 

(1) Prove that

$$A_1 = \{ a/b \mid a, b \in \mathbb{Z}, 2 \not| b \}$$

is a local ring with residue field  $\mathbb{Z}_2$ . (2) Prove that

$$A_2 = \{ a/b \mid a, b \in \mathbb{Z}, 2 \not\mid b, 3 \not\mid b \}$$
 is a semi-local ring.

(3) Exhibit a semi-local ring with exactly n maximal ideals, where  $n \in \mathbb{Z}^+$ .

(4) Prove that

$$A_3 = \{ p(x)/q(x) \mid \\ p(x), q(x) \in \mathbb{R}[x], q(0) \neq 0 \}$$

is a local ring with residue field  $\ \mathbb R$  .

(5) Prove that  $\mathbb{R}[[x]]$  is a local ring with residue field  $\mathbb{R}$ .

The following result may be useful:

## **Proposition:**

(i) Let A be a ring and M ≠ A an ideal such that all elements of A\M are units.
Then A is local and M is maximal.
(ii) Let A be a ring and M a maximal ideal such that all elements of 1 + M are units.

Then A is local.

**Proof:** (i) If  $A \neq I \lhd A$  then no element of I is a unit, so  $I \subseteq M$ .

Thus M is the unique maximal ideal of A, so A is local.

(ii) Let 
$$x \in A \setminus M$$
. Since  $M$  is maximal,  
 $A = \langle M \cup \{x\} \rangle$   
 $= \{ ax + m \mid a \in A, m \in M \}.$ 

In particular

$$1 = ax + m \qquad (\exists a \in A)(\exists m \in M)$$

SO

$$ax = 1 - m \in 1 + M$$
.

By hypothesis ax is a unit. Hence x also is a unit.

Thus all elements of  $A \setminus M$  are units.

By (i), A is local, and the proof is complete.