1.3 Local Rings and Residue Fields

Call a ring A **local** if A has exactly one maximal ideal M, and call A/M the **residue field** of A.

Examples:

(1) Any field F is local since $\{0\}$ is the only maximal ideal

($\{0\}$ and F are the **only** ideals of F)

and $F \cong F/\{0\}$ is its own residue field.

(2) Let R be any (possibly noncommutative) ring and G any group written multiplicatively. The **group ring** $R[G]$ comprises formal linear combinations

$$\sum_{g \in G} \alpha_g \, g$$

where $\alpha_g \in R$ for each $g \in G$ and only finitely many α_g are nonzero, with componentwise addition:

$$\sum \alpha_g \, g \; + \; \sum \beta_g \, g \; = \; \sum (\alpha_g \, + \, \beta_g) \, g \; ,$$

and **convolution** product:

$$\left(\sum_{g \in G} \alpha_g \, g\right) \left(\sum_{h \in G} \beta_h \, h\right) = \sum_{k \in G} \left(\sum_{gh=k} \alpha_g \beta_h\right) k \; ,$$

It is routine to check that $R[G]$ is a (possibly noncommutative) ring.

e.g. $\mathbb{Z}_2[C_2] = \{ 1 , \ x , \ 0 , \ 1 + x \}$,

where $C_2 = \{ x , \ x^2 = 1 \}$ is the cyclic group of order 2 , written multiplicatively,

has unique maximal ideal

$$M = \{ 0 , \ 1 + x \} \qquad \text{(easy to verify)}$$

so that $\mathbb{Z}_2[C_2]$ is local, with residue field

$$\mathbb{Z}_2[C_2]/M \cong \mathbb{Z}_2 .$$

Example: Let F be a field of prime **characteristic** p , that is,

$$\underbrace{1 + 1 + \ldots + 1}_{p \text{ times}} \;=\; 0\,.$$

Let G be any abelian p-group, that is,

each element of G has order a power of p .

Exercise: Verify that $F[G]$ is local with unique maximal ideal

$$M = \{ \sum \alpha_g \, g \mid \sum \alpha_g = 0 \}$$

with residue field

$$F[G] \, / \, M \cong F \, .$$

A ring with only finitely many maximal ideals is called **semi-local**.

Exercises:

(1) Prove that

$$A_1 = \{ a/b \mid a, b \in \mathbb{Z} , \;\; 2 \not| \, b \}$$

is a local ring with residue field $\mathbb{Z}_2$.

(2) Prove that

$$A_2 = \{ a/b \mid a, b \in \mathbb{Z} , \;\; 2 \not| \, b , \; 3 \not| \, b \}$$

is a semi-local ring.

(3) Exhibit a semi-local ring with exactly n maximal ideals, where $n \in \mathbb{Z}^+$.

(4) Prove that

$$A_3 = \{ p(x)/q(x) \mid p(x), q(x) \in \mathbb{R}[x] , \quad q(0) \neq 0 \}$$

is a local ring with residue field $\mathbb{R}$.

(5) Prove that $\mathbb{R}[[x]]$ is a local ring with residue field $\mathbb{R}$.

The following result may be useful:

Proposition:

(i) Let A be a ring and $M \neq A$ an ideal such that all elements of $A \backslash M$ are units.

Then A is local and M is maximal.

(ii) Let A be a ring and M a maximal ideal such that all elements of $1 + M$ are units.

Then A is local.

Proof: (i) If $A \neq I \lhd A$ then no element of I is a unit, so $I \subseteq M$.

Thus M is the unique maximal ideal of A , so A is local.

(ii) Let $x \in A \backslash M$. Since M is maximal,

$$A \;=\; \langle\, M \,\cup\, \{x\} \,\rangle$$

$$=\; \{\, ax + m \;\mid\; a \in A \,,\;\; m \in M \,\} \,.$$

In particular

$$1 \quad = \quad ax + m \qquad\qquad (\exists a \in A)(\exists m \in M)$$

so

$$ax \quad = \quad 1 - m \ \in 1 + M \ .$$

By hypothesis ax is a unit. Hence x also is a unit.

Thus all elements of $A \backslash M$ are units.

By (i), A is local, and the proof is complete.