4.3 Composition Series

Let M be an A-module.

A **series** for M is a strictly decreasing sequence of submodules

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = \{0\}$$

beginning with M and finishing with $\{0\}$.

The **length** of this series is n .

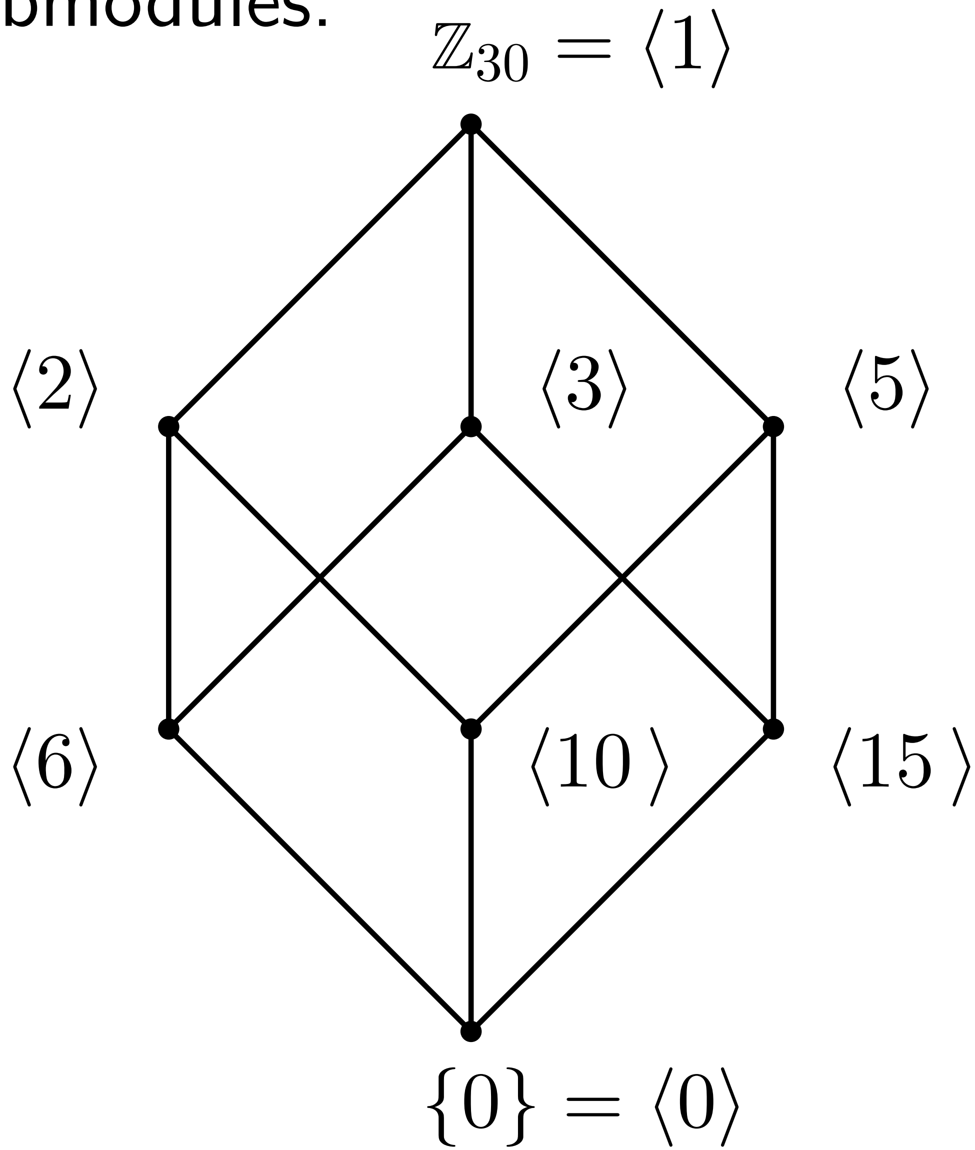
A **composition series** is a series in which no further submodule can be inserted

which, for the above, is equivalent to saying

each **composition factor** M_i/M_{i+1} is **simple**,

that is, each M_i/M_{i+1} is nontrivial and has no submodule except for itself and the trivial submodule.

Example: The $\mathbb{Z}$-module $\mathbb{Z}_{30}$ has the following lattice of submodules:



Any path from the top to the bottom will yield a composition series:

e.g. $\langle 1 \rangle \supset \langle 2 \rangle \supset \langle 6 \rangle \supset \langle 0 \rangle$,

with composition factors:

$$\langle 1 \rangle / \langle 2 \rangle = \mathbb{Z}_{30}/2\mathbb{Z}_{30} = (\mathbb{Z}/30\mathbb{Z})/(2\mathbb{Z}/30\mathbb{Z})$$

$$\cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 ,$$

$$\langle 2\rangle/\langle 6\rangle \;\cong\; 2\mathbb{Z}/6\mathbb{Z} \;\cong\; \mathbb{Z}_3 \, ,$$

$$\langle 6\rangle/\langle 0\rangle \;\cong\; 6\mathbb{Z}/30\mathbb{Z} \;\cong\; \mathbb{Z}_5$$

In fact, all composition series for $\mathbb{Z}_{30}$ produce composition factors $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$

(in accordance with the Jordan-Holder Theorem below).

Notation: If N is a module, let $\ell(N)$ denote the least length of a composition series of N, if one exists, and put $\ell(N) = \infty$ if no composition series for N exists.

Theorem: Suppose M has a composition series of length n.

Then every composition series of M has length n, and every series can be refined (that is, submodules can be inserted) to yield a composition series.

Proof: Here $\ell(M) \leq n$. Consider a submodule N of M where $N \neq M$. We first show

$$\boxed{\ell(N) \; < \; \ell(M) \; .} \qquad (*)$$

Put $\ell \; = \; \ell(M)$ and let

$$M \; = \; M_0 \; \supset \; M_1 \; \supset \; \ldots \; \supset \; M_\ell \; = \; \{0\}$$

be a composition series of length ℓ . Then

$$N \; = \; N \cap M_0 \; \supseteq \; N \cap M_1 \; \supseteq \; \ldots \; \supseteq \; N \cap M_\ell \; = \; \{0\}$$

is a chain of submodules from N to $\{0\}$.

By a module isomorphism theorem, for each i ,

$$\frac{N \cap M_i}{N \cap M_{i+1}} = \frac{N \cap M_i}{(N \cap M_i) \cap M_{i+1}}$$

$$\cong \frac{(N \cap M_i) + M_{i+1}}{M_{i+1}} ,$$

the last of which is a submodule of the simple module $M_i \,/\, M_{i+1}$. Hence

$$\frac{N \cap M_i}{N \cap M_{i+1}} \quad \text{is trivial or simple.}$$

Thus, deleting repetitions from the above chain from N to $\{0\}$ must yield a composition series for N, which proves

$$\boxed{\ell(N) \ \leq \ \ell(M) \ .}$$

Suppose $\ell(N) \ = \ \ell$.

Then, no repetitions occurred in the previous process, so, by the earlier isomorphism,

$$\boxed{(N \cap M_i) + M_{i+1} \ = \ M_i \qquad (\forall i) \ .} \qquad (**)$$

We observe, by induction, that

$$\boxed{N \cap M_i \ = \ M_i \qquad\quad (\forall i)\ ,}$$

since, clearly

$$N \cap M_\ell \ = \ N \cap \{0\} \ = \ \{0\} \ = \ M_\ell \ ,$$

which starts the induction,

and, for $i \leq \ell - 1$, using $(**)$ and an inductive hypothesis,

$$N \cap M_i \;=\; (N \cap M_i) + (N \cap M_{i+1})$$

$$= \;(N \cap M_i) + M_{i+1} \;=\; M_i \;.$$

In particular,

$$M \;=\; M_0 \;=\; N \cap M_0 \;=\; N \cap M \;=\; N \;,$$

which contradicts that $N \neq M$.

Hence $\ell(N) \;<\; \ell$ and $(*)$ is proved.

Now consider any series

$$M = M'_0 \supset M'_1 \supset \ldots \supset M'_k = \{0\} \qquad (\dagger)$$

of length k . By $(*)$,

$$\ell = \ell(M) > \ell(M'_1) > \ldots > \ell(M'_k) = 0 \, ,$$

so

$$\ell \geq k \, .$$

If $(\dagger)$ is a composition series, then $\ell \leq k$, by definition of ℓ , so $\ell = k$.

This proves

$$\boxed{\text{all composition series of } M \text{ have the same length } n\,.}$$

If $(\dagger)$ is not a composition series, then $k < n$, because if $k = n$, then we can insert another module somewhere to get another series of length $n+1 \leq \ell = n$, which is nonsense.

Hence any series which is not a composition series can be successively refined until its length is n, in which case it becomes a composition series, and the Theorem is proved.

> **Corollary:** A module M has a composition series iff M satisfies the a.c.c. and d.c.c.

Proof: $(\Longrightarrow)$ If M has a composition series of length n, then, by the previous Theorem,

all series have length $\leq n$,

so all ascending and descending chains must be stationary,

that is, M satisfies both the a.c.c. and the d.c.c.

$(\Longleftarrow)$ Suppose M satisfies both the a.c.c. and the d.c.c.

If $M = \{0\}$ then certainly M has a composition series.

Suppose $M \neq \{0\}$.

Since M satisfies the maximal condition (equivalent to the a.c.c),

the set of **proper** submodules of M

has a maximal element M_1 , so M/M_1 is simple.

Suppose we have found submodules

$$M \ = \ M_0 \ \supset \ M_1 \ \supset \ \ldots \ \supset \ M_k \qquad (k \geq 1)$$

where M_i/M_{i+1} is simple for $i = 0, \ldots, k-1$.

$\boxed{\text{If } M_k = \{0\} \text{ then we have a composition series for } M .}$

If $M_k \neq \{0\}$, then, again since M satisfies the maximal condition, we can find a submodule M_{k+1} of M_k such that M_k/M_{k+1} is simple.

Either M has a composition series, or by induction we have an infinite strictly descending chain

$$M = M_0 \supset M_1 \supset \ldots \supset M_k \supset \ldots$$

The latter is excluded because M satisfies the d.c.c.

Hence M has a composition series and the Corollary is proved.

Say that a module M has **finite length** if it has a composition series

(equivalently satisfies both the a.c.c. and d.c.c.)

in which case all composition series have the same length $\ell(M)$, called the **length** of M .

We now prove a uniqueness result concerning the composition factors:

Jordan-Holder Theorem: There is a one-one correspondence between the composition factors of any two composition series of a module of finite length such that

corresponding factors are isomorphic.

Proof: Let M be a module of finite length ℓ .

If $\ell = 0$ then the set of composition factors is always empty, so the result is vacuously true, which starts an induction.

Suppose $\ell > 0$ and let

$$M = M_0 \supset M_1 \supset \ldots \supset M_\ell = \{0\}$$

$$M = M'_0 \supset M'_1 \supset \ldots \supset M'_\ell = \{0\}$$

be two composition series.

If $M_1 = M_1'$ then, by an inductive hypothesis

(since $\ell(M_1) = \ell - 1$),

there is an appropriate correspondence between

$$\{ M_i/M_{i+1} \mid i = 1, \ldots, \ell - 1 \}$$

and

$$\{ M_i'/M_{i+1}' \mid i = 1, \ldots, \ell - 1 \} ,$$

so, since

$$M_0/M_1 = M_0'/M_1' ,$$

there is an appropriate correspondence between

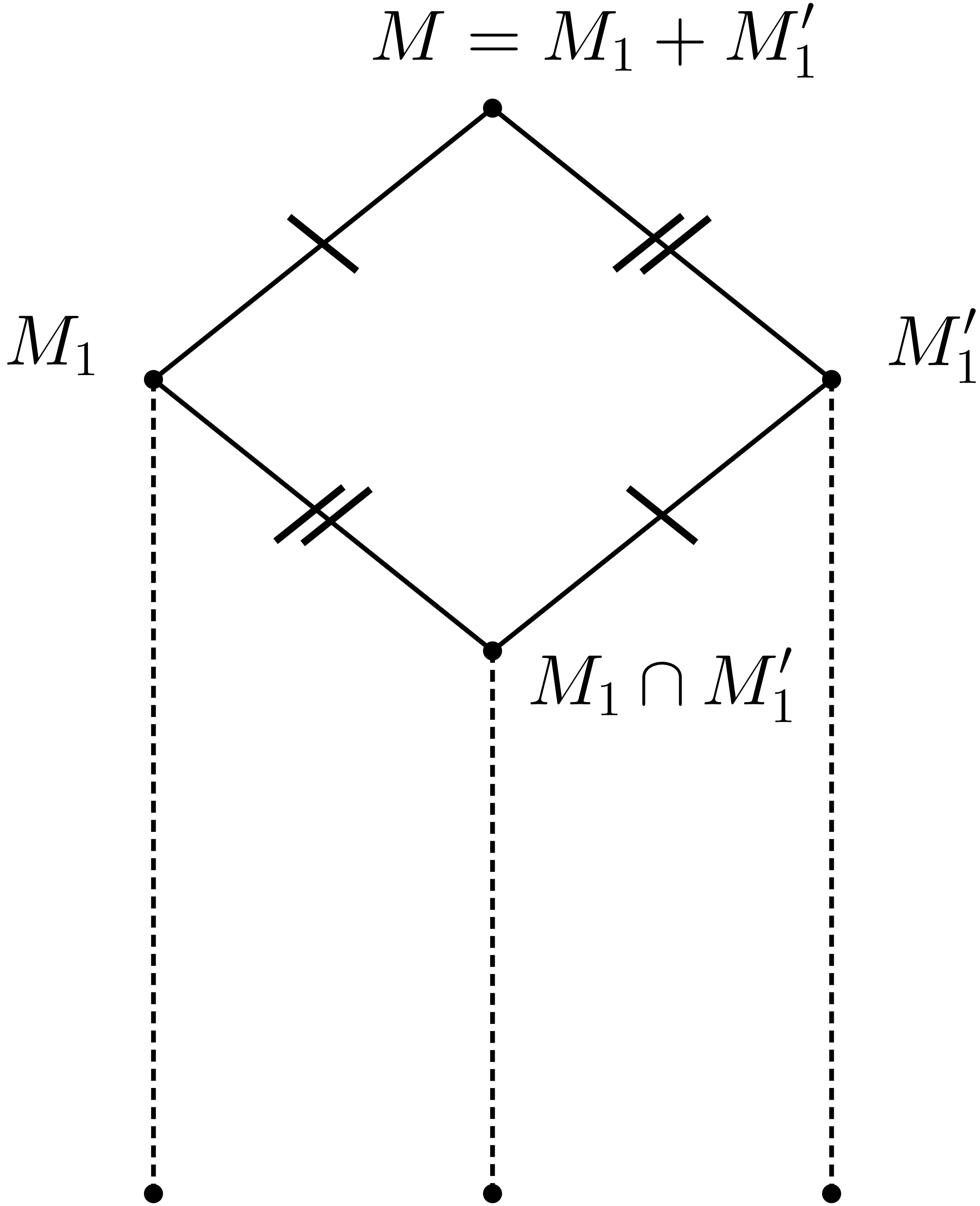
$$\{ M_i/M_{i+1} \mid i = 0, \ldots, \ell - 1 \}$$

and

$$\{ M'_i/M'_{i+1} \mid i = 0, \ldots, \ell - 1 \} ,$$

and we are done.

Suppose then $M_1 \neq M'_1$, so $M_1 + M'_1 = M$, since M_1 is maximal in M .


$M = M_1 + M_1'$
M_1
M_1'
$M_1 \cap M_1'$

By module isomorphism theorems,

$$M/M_1 \;=\; M_1 + M_1' \,/\, M_1 \;\cong\; M_1' \,/\, M_1 \cap M_1'$$

and

$$M/M_1' \;=\; M_1 + M_1' \,/\, M_1' \;\cong\; M_1 \,/\, M_1 \cap M_1' \,.$$

But

$$\ell(M_1 \cap M_1') \;\;<\;\; \ell(M_1)\,,\; \ell(M_2) \;\;<\;\; \ell$$

so we can apply an inductive hypothesis to composition series of these modules.

Let F_1 , F_2 , F_1' be the collections of composition factors for M_1 , $M_1 \cap M_1'$, M_1' respectively. Then there are appropriate correspondences between

$$F_1 \cup \{ \, M/M_1 \, \}$$

$$\text{and} \qquad F_2 \cup \{ \, M_1/M_1 \cap M_1' \, , \; M/M_1 \, \}$$

$$\text{and} \qquad F_2 \cup \{ \, M/M_1' \, , \; M_1'/M_1 \cap M_1' \, \}$$

$$\text{and} \qquad F_1' \cup \{ \, M/M_1' \, \}$$

which proves the Theorem.

Theorem: The length $\ell(M)$ of a module M defines an additive function on the class of all A-modules of finite length.

Proof: Suppose

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

is exact, where all modules have finite length. We need to show

$$\boxed{\ell(M) \ = \ \ell(M') + \ell(M'') \ .}$$

Let

$$M' = M'_0 \supset M'_1 \supset \ldots \supset M'_{\ell(M')} = \{0\}$$

$$M'' = M''_0 \supset M''_1 \supset \ldots \supset M''_{\ell(M'')} = \{0\}$$

be composition series for M' and M'' respectively. Since α is injective,

$$\alpha(M') = \alpha(M'_0) \supset \alpha(M'_1) \supset \ldots \supset \alpha(M'_{\ell(M')}) = \{0\}$$

is a composition series for $\alpha(M') = \ker \beta$.

Since β is onto,

$$
\begin{aligned}
M/\ker\beta &= \beta^{-1}(M''_0)/\ker\beta \\
&\supset \beta^{-1}(M''_1)/\ker\beta \supset \\
&\ldots \supset \beta^{-1}(M''_{\ell(M'')})/\ker\beta = \{\ker\beta\}
\end{aligned}
$$

is a composition series for $M/\ker\beta$. Combining these two series produces the following series for M :

$$M \;=\; \beta^{-1}(M''_0) \;\supset\; \beta^{-1}(M''_1) \;\supset$$

$$\ldots \;\supset\; \beta^{-1}(M''_{\ell(M'')}) \;=\; \ker\beta$$

$$\|$$

$$\{0\} \;=\; \alpha(M'_{\ell(M')}) \;\subset\; \ldots \;\subset\; \alpha(M'_0) \;=\; \alpha(M')$$

But this is a composition series, because, by another isomorphism theorem, for $i = 0\,,\;\ldots\;,\ell(M'') - 1$,

$$\beta^{-1}(M_i'') \;/\; \beta^{-1}(M_{i+1}'')$$

$$\cong \quad \left(\beta^{-1}(M_i'') \;/\; \ker\beta\right) \;/\; \left(\beta^{-1}(M_{i+1}'') \;/\; \ker\beta\right)$$

which is simple. Hence

$$\ell(M) \;=\; \ell(M') \;+\; \ell(M'') \;,$$

and the Theorem is proved.

Interpreting the theory for vector spaces yields:

Proposition: Let V be a vector space over a field F. TFAE

(i) V is finite dimensional.

(ii) V has finite length.

(iii) V satisfies the a.c.c.

(iv) V satisfies the d.c.c.

If any of these hold, then dimension equals length.

Proof: (i) $\Longrightarrow$ (ii) If $\{ x_1 , \ldots , x_n \}$ is a basis for V then

$$V = \langle x_1 , \ldots , x_n \rangle \supset \langle x_1 , \ldots , x_{n-1} \rangle \supset$$

$$\ldots \supset \langle x_1 \rangle \supset \{0\}$$

is a composition series of length n .

(ii) $\Longrightarrow$ (iii), (ii) $\Longrightarrow$ (iv): follow from the earlier Corollary on page 781.

(iii) $\Longrightarrow$ (i), (iv) $\Longrightarrow$ (i): Suppose (i) is false, so V contains an infinite sequence

$$x_1 \, , \; \ldots \; , \; x_n \, , \; \ldots$$

of linearly independent vectors. For each $n \geq 1$ put

$$U_n \;=\; \langle \, x_1 \, , \; \ldots \; , \; x_n \, \rangle \; ,$$

$$V_n \;=\; \langle \, x_{n+1} \, , \; x_{n+2} \, , \ldots \, \rangle \; .$$

Then

$$\{0\} \subset U_1 \subset U_2 \subset \ldots \subset U_n \subset \ldots$$

$$V \supset V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots$$

are strictly ascending and descending chains respectively,

so that both (iii) and (iv) fail to hold.

Corollary: Let A be a ring in which

$$M_1 \, M_2 \, \ldots \, M_N \;=\; \{0\}$$

for some (not necessarily distinct) maximal ideals $M_1 \, , \ldots , M_n$.

Then A is Noetherian iff A is Artinian.

Proof: First note that if

$$I \lhd A\,, \quad M \lhd A\,, \qquad M \text{ maximal}$$

then $I \,/\, IM$ is an A-module, so also

$$I/IM \text{ is an } A/M\text{-module}$$

(since $M \subseteq \text{Ann}\,(I/IM)$), that is,

$$\boxed{I/IM \text{ is a vector space over the field } A/M\ ,}$$

so, by the previous Proposition,

I/IM satisfies the a.c.c. on subspaces (ideals)

iff

I/IM satisfies the d.c.c. on subspaces (ideals).

Consider the chain

$$A \supset M_1 \supseteq M_1 M_2 \supseteq \ldots \supseteq M_1 \ldots M_n = \{0\} .$$

Then, by repeated application of an earlier Theorem about exactness (on page 756),

A satisfies the a.c.c. on ideals

iff each factor

$$A/M_1\ ,\ M_1/M_1M_2\ ,\ \dots\ ,\ M_1\dots M_{n-1}\ /\ M_1\dots M_n$$

satisfies the a.c.c. on ideals

iff

each factor satisfies the d.c.c. on ideals

iff

A satisfies the d.c.c. on ideals,

and the Corollary is proved.