4.2 Chain Conditions

Imposing **chain conditions** on the

$\boxed{\text{poset of submodules of a module,}}$

or on the

$\boxed{\text{poset of ideals of a ring,}}$

makes a module or ring more tractable and facilitates the proofs of deep theorems.

Proposition: Let Σ be a poset with respect to $\leq$. TFAE

(i) Every increasing sequence

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$$

in Σ is **stationary**, that is,

$$(\exists n)(\forall m \geq n) \qquad x_m = x_n \,;$$

(ii) Every nonempty subset of Σ has a maximal element.

Proof: (i) $\Longrightarrow$ (ii): If (ii) is false, then there is a nonempty subset X of Σ with no maximal element, so $\exists x_1 \in X$;

$$(\exists x_2 \in X) \qquad x_1 < x_2 \,;$$
$$(\exists x_3 \in X) \qquad x_1 < x_2 < x_3 \,;$$

so continuing, X contains

$$x_1 < x_2 < x_3 < \ldots < x_n < \ldots$$

which is strictly increasing, so (i) fails.

(ii) $\implies$ (i): If (ii) holds and

$$x_1 \le x_2 \le \dots \le x_n \le \dots \qquad (*)$$

is an increasing sequence in Σ , then

$$\{ x_1, \dots , x_n , \dots \}$$

has a maximal element, say x_k , so for every $m \ge k$,

$$x_m \ge x_k \ge x_m ,$$

whence equality, proving $(*)$ is stationary, and (i) holds.

Let Σ be the set of submodules of a module M. Regarding Σ as a poset with respect to $\subseteq$, we refer to

(i) as the **ascending chain condition (a.c.c.)**

and

(ii) as the **maximal condition**.

Any module satisfying the a.c.c. or equivalently the maximal condition is called **Noetherian**.

On the other hand regarding Σ as a poset with respect to $\supseteq$, we refer to

(i) as the **descending chain condition (d.c.c.)**

and

(ii) as the **miminal condition**.

Any module satisfying the d.c.c. or equivalently the minimal condition is called **Artinian**.

Examples: **(1)** Any finite module satisfies both the a.c.c. and d.c.c.

These include all finite abelian groups, regarded as $\mathbb{Z}$-modules.

(2)

The ring $\mathbb{Z}$ (regarded as a $\mathbb{Z}$-module) satisfies the a.c.c but not the d.c.c.

This was proved in the Overview (page 18).

(3) Consider the group $\mathbb{Q}$ under addition. Then $\mathbb{Z}$ is a subgroup and we may form the quotient group

$$\mathbb{Q}/\mathbb{Z} \;=\; \{\, q+\mathbb{Z} \mid q \in \mathbb{Q} \,\} \,.$$

Fix a prime number p, and put

$$G \;=\; \{\, a/p^n \,+\, \mathbb{Z} \mid n \geq 0 \,,\;\; a \in \mathbb{Z} \,\}$$

and, for $i \geq 0$,

$$G_i \;=\; \{\, a/p^i \,+\, \mathbb{Z} \mid a \in \mathbb{Z} \,\} \,.$$

Clearly G is a subgroup of $\mathbb{Q}/\mathbb{Z}$ and each G_i is a subgroup of G.

Moreover,

$$G_0 \subset G_1 \subset \ldots G_n \subset \ldots \qquad (*)$$

is a strictly increasing sequence, so, regarded as a $\mathbb{Z}$-module,

$$\boxed{G \text{ does not satisfy the a.c.c.}}$$

Exercise: Prove that the only subgroups of G are G and G_i for $i \geq 0$.

By $(*)$ and this exercise, there are no infinite strictly descending chains of subgroups of G, so, as a $\mathbb{Z}$-module,

$$\boxed{G \text{ satisfies the d.c.c.}}$$

(4) Fix a prime number p and put

$$H = \{ m/p^n \mid m \in \mathbb{Z} , \ n \geq 0 \} .$$

Then clearly H is a subgroup of $\mathbb{Q}$ and

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow H/\mathbb{Z} = G \longrightarrow 0$$

is exact, where the second mapping is inclusion and G is the group of **(3)**. Thus

$$\boxed{H \text{ doesn't satisfy the d.c.c.}}$$

because it has a subgroup $\mathbb{Z}$ which doesn't, and

$$\boxed{H \text{ doesn't satisfy the a.c.c.}}$$

because it has a quotient G which doesn't.

(5)

$\boxed{\text{The polynomial ring } F[x] \text{, where } F \text{ is a field, satisfies the a.c.c. but not the d.c.c. on ideals.}}$

The proof is left as an **exercise**, using the fact that $F[x]$ is a PID, and copying the details of **(2)**.

(6) The polynomial ring $F[x_1\ ,\ x_2\ ,\ \ldots\ ]$ using infinitely many indeterminates does not satisfy the d.c.c. on ideals (as for **(5)**), but also does not satisfy the a.c.c. since

$$\langle x_1 \rangle \ \subset \ \langle x_1\,,\ x_2 \rangle \ \subset \ \ldots \ \subset \ \langle x_1\,,\ \ldots\ x_n \rangle \ \subset \ \ldots$$

is an infinite strictly increasing chain of ideals.

Proposition: Let M be an A-module.

Then M is Noetherian iff every submodule of M is finitely generated.

Proof: $(\Longrightarrow)$ Suppose M is Noetherian and let N be a submodule of M. Let

$$\Sigma = \{ \text{ finitely generated submodules of } N \} .$$

Then Σ has a maximal element N_0.

If $N \neq N_0$ then

$$\exists x \in N \backslash N_0 ,$$

so $\langle N_0 \cup \{x\} \rangle$ is a finitely generated submodule of N bigger than N_0, contradicting maximality.

Hence $N_0 = N$, so N is finitely generated.

$(\Longleftarrow)$ Suppose all submodules of M are finitely generated. Let

$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots \qquad (*)$$

be an ascending chain of submodules.

Then $\bigcup_{i=1}^{\infty} M_i$ is easily seen to be a submodule of M ,

so is generated by finitely many elements, say

$$x_1 , \ldots , x_r .$$

Then

$$(\forall j = 1, \ldots, r)\,(\exists\, i_j\,) \qquad x_j \in M_{i_j}\,.$$

Put

$$m = \max\{\, i_1\,,\ \ldots\ ,\ i_r\,\}$$

so

$$(\forall j) \qquad x_j \in M_m\,.$$

But then

$$\bigcup_{i=1}^{\infty} M_i \subseteq M_m \subseteq \bigcup_{i=1}^{\infty} M_i\,,$$

so equality holds, and $(*)$ is stationary.

Theorem: Let

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''; \longrightarrow 0$$

be an exact sequence of A-modules.

Then M is Noetherian [Artinian] iff M' and M'' are.

Proof: We prove the result for Noetherian, the argument for Artinian being similar.

$(\Longrightarrow)$ Suppose M is Noetherian.

Because α is injective, any ascending chain of submodules of M' corresponds to an ascending chain of submodules of M, so the former is stationary, since the latter is.

Hence M' is Noetherian.

Because β is surjective,

$$M'' \cong M/\ker\beta$$

so that submodules of M'' correspond to submodules of M containing $\ker\beta$, and the correspondence is inclusion preserving.

Hence any ascending chain of submodules of M'' corresponds to an ascending chain of submodules of M, so the former is stationary, since the latter is.

Hence M'' is Noetherian.

$(\Longleftarrow)$ Suppose M', M'' are Noetherian. Let

$$L_1 \subseteq L_2 \subseteq \ldots \subseteq L_n \subseteq \ldots \qquad (*)$$

be an ascending chain of submodules of M. Then

$$\alpha^{-1}(L_1) \subseteq \alpha^{-1}(L_2) \subseteq \ldots \subseteq \alpha^{-1}(L_n) \subseteq \ldots$$

is an ascending chain of submodules of M' , and

$$\beta(L_1) \subseteq \beta(L_2) \subseteq \ldots \subseteq \beta(L_n) \subseteq \ldots$$

is an ascending chain of submodules of M'' .

Since these sequences are stationary,

$$(\exists n_1)(\forall m \geq n_1) \qquad \alpha^{-1}(L_m) = \alpha^{-1}(L_{n_1})$$

$$(\exists n_2)(\forall m \geq n_2) \qquad \beta(L_m) = \beta(L_{n_2})$$

Put $n = \max\{ n_1 , n_2 \}$, so $(\forall m \geq n)$,

$$\alpha^{-1}(L_m) = \alpha^{-1}(L_n) \quad \text{and} \quad \beta(L_m) = \beta(L_n) .$$

$$\boxed{\begin{array}{l}\text{We will prove that} \\[1em] \qquad (\forall m \geq n) \qquad L_m = L_n .\end{array}}$$

Let $m \geq n$ and $x \in L_m$. Then

$$\beta(x) \in \beta(L_m) = \beta(L_n) ,$$

so, for some $y \in L_n$, $\beta(x) = \beta(y)$, so, by exactness,

$$x - y \in \ker\beta = \operatorname{im}\alpha \ .$$

But $L_n \subseteq L_m$, so $x - y \in L_m$, giving

$$x - y = \alpha(z) \qquad \bigl(\exists z \in \alpha^{-1}(L_m)\bigr) \ .$$

But $\alpha^{-1}(L_m) = \alpha^{-1}(L_n)$, so $\alpha(z) \in L_n$, giving

$$x = y + \alpha(z) \in L_n \ .$$

Hence

$$L_m \subseteq L_n \subseteq L_m ,$$

whence equality. This proves $(*)$ is stationary, so M is Noetherian, and the Theorem is proved.

Corollary: If $M_1 , \dots , M_n$ are Noetherian [Artinian] A-modules then so is

$$M_1 \oplus \dots \oplus M_n .$$

Proof: This follows by induction and the previous Theorem applied to the exact sequence

$$0 \longrightarrow M_n \xrightarrow{\alpha} \bigoplus_{i=1}^{n} M_i \xrightarrow{\beta} \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0$$

where

$$\alpha : x \mapsto (0, \ldots, 0, x)$$

$$\beta : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}) .$$

Call a ring A **Noetherian [Artinian]** if it is so as an A-module, that is,

if it satisfies the a.c.c. [d.c.c.] on ideals.

Examples: **(1)** Any ring with only finitely many ideals (such as a finite ring or a field) is certainly both Noetherian and Artinian.

(2) The ring $\mathbb{Z}$ is Noetherian but not Artinian.

(3) Any PID is Noetherian (by the Proposition on page 752) since all ideals are finitely generated.

(4) The ring $F[x_1, \ldots, x_n, \ldots]$ where F is a field is neither Noetherian nor Artinian, but is an integral domain, so has a field of fractions, which is both Noetherian and Artinian. Thus

> subrings of Noetherian [Artinian] rings need not be Noetherian [Artinian].

However quotients are well-behaved. By the earlier Theorem (on page 756) we get immediately

Corollary: Any homomorphic image of a Noetherian [Artinian] ring is Noetherian [Artinian].

Ring and module properties can be linked:

Theorem: Let A be a Noetherian [Artinian] ring and M a finitely generated A-module. Then M is Noetherian [Artinian].

Proof: By general theory (a Proposition on page 324),

$$M \;\cong\; A^n \,/\, N$$

for some $n > 0$ and some submodule N of A^n.

But A^n is Noetherian [Artinian], being a direct sum of Noetherian [Artinian] modules.

Hence, by the previous Corollary, M is Noetherian [Artinian].