Under finite generation assumptions, S^{-1} commutes with other operations involving ideals.

Proposition: Let M be a finitely generated A-module and S a multiplicatively closed subset of A. Then

$$S^{-1}\big(\text{Ann}\,(M)\big) \;=\; \text{Ann}\,(S^{-1}M)\;.$$

Proof: If $a \in \text{Ann}\,(M)$, $m \in M$ and $s, t \in S$

then $am = 0$, so

$$(a/s)(m/t) = am \,/\, st = 0 \,/\, st = 0 \,,$$

which proves $S^{-1}(\mathsf{Ann}\,(M)) \subseteq \mathsf{Ann}\,(S^{-1}M)$.

Suppose conversely that $a \in A$, $s \in S$ and $a/s \in \mathsf{Ann}\,(S^{-1}M)$.

Let the generators of M be $m_1 , \ldots , m_n$.

Then, for $i = 1, \ldots, n$,

$$(a/s)(m_i/1) \;\; = \;\; am_i/s \;\; = \;\; 0/1 \, ,$$

so

$$t_i a m_i \; = \; 0 \qquad\qquad (\exists t_i \in S) \; .$$

Put

$$t \; = \; t_1 \ldots t_n \; ,$$

so

$$(at) m_i \;\; = \;\; 0 \qquad\qquad (\forall i = 1, \ldots, m) \; .$$

Hence $at \; \in$ Ann (M)

(since m_1 , . . . , m_n generate M),

so

$$a/s \;=\; at \;/\; st \;\in\; S^{-1}(\mathsf{Ann}\;(M))$$

(noting $t \in S$),

so

$$\mathsf{Ann}\;(S^{-1}M) \;\subseteq\; S^{-1}(\mathsf{Ann}\;(M))\;,$$

whence equality holds.

Exercise: Find a multiplicatively closed subset S of a ring A and an A-module M such that

$$S^{-1}(\mathsf{Ann}\,(M)) \subset \mathsf{Ann}\,(S^{-1}M)$$

Corollary: Let N , P be submodules of an A-module M , and suppose P is finitely generated. Then

$$S^{-1}(N : P) = (S^{-1}N : S^{-1}P) \ .$$

Proof: By the previous Proposition,

$$S^{-1}(N : P) \;=\; S^{-1}\Big(\mathsf{Ann}\,(N + P \,/\, N)\Big)$$

$$=\; \mathsf{Ann}\,\Big(S^{-1}(N + P \,/\, N)\Big)$$

But, by **(3)** of the Theorem on page 572,

$$S^{-1}\big(N + P \,/\, N\big) \;\cong\; S^{-1}(N + P) \,/\, S^{-1}N$$

as $S^{-1}A$ -modules, so have the same annihilators.

By **(1)** of the same Theorem,

$$S^{-1}(N+P) \;=\; \left(S^{-1}N\right) + \left(S^{-1}P\right) \;,$$

so

$$\begin{aligned} S^{-1}(N:P) \;\; &= \;\; \mathsf{Ann}\left(S^{-1}(N+P) \,/\, S^{-1}N \right) \\ &= \;\; \mathsf{Ann}\left((S^{-1}N) \;+\; (S^{-1}P) \,/\, S^{-1}N \right) \\ &= \;\; \left(S^{-1}N : S^{-1}P\right) \;. \end{aligned}$$

We finish with an application of the theory of fractions:

Theorem: Let $f : A \to B$ be a ring homomorphism and P a prime ideal of A. Then

P is the contraction of a prime ideal of B

iff P is contracted, that is,

$$P^{\text{ec}} = P .$$

Proof: $(\Longrightarrow)$ is obvious.

$(\Longleftarrow)$ Suppose $P^{\mathsf{ec}} = P$, and put

$$S = \{ f(a) \mid a \in A \backslash P \} .$$

Then S is multiplicatively closed in B since $A \backslash P$ is multiplicatively closed in A.

If $\alpha \in P^{\mathsf{e}} \cap S$ then $\alpha = f(a)$ for some $a \in A \backslash P$, so

$$a \in f^{-1}(P^{\mathsf{e}}) = P^{\mathsf{ec}} = P ,$$

a contradiction. Hence

$$P^{\mathsf{e}} \cap S = \emptyset .$$

By the last part of **(2)** of the Theorem on page 616,

$$S^{-1}(P^{\mathsf{e}}) \neq S^{-1}B ,$$

so

$$S^{-1}(P^{\mathsf{e}}) \subseteq M \qquad (\exists \text{ maximal } M \lhd S^{-1}B) .$$

Put $Q = \{ b \in B \mid b/1 \in M \}$.

Then Q is prime, being the preimage of a maximal ideal with respect to a ring homomorphism.

Certainly $Q \supseteq P^{\mathsf{e}}$, so

$$Q^{\mathsf{c}} \supseteq P^{\mathsf{ec}} = P .$$

If $Q \cap S \neq \emptyset$ then, again by **(2)** of the same Theorem,

$$S^{-1}B = S^{-1}Q \subseteq M \subseteq S^{-1}B ,$$

so $S^{-1}B = M$, a contradiction.

Hence

$$Q \cap S = \emptyset ,$$

so

$$Q^{\mathsf{c}} = f^{-1}(Q) \subseteq P ,$$

yielding finally

$$Q^{\mathsf{c}} = P ,$$

and the Theorem is proved.