3.4 Extended and Contracted Ideals in Rings of Fractions

Let A be a ring and let S be a multiplicatively closed subset of A. Throughout this section let

$$f : A \to S^{-1}A\,, \qquad a \mapsto a/1\,.$$

We study extension and contraction of ideals with respect to this homomorphism.

Put

$$\mathcal{C} \;=\; \{\, I \mid I \lhd A \quad \text{and} \quad I \;=\; I^{\mathsf{ec}} \,\}\,,$$

the set of **contracted ideals** of A and

$$\mathcal{E} \;=\; \{\, J \mid J \lhd S^{-1}A \quad \text{and} \quad J \;=\; J^{\mathsf{ce}} \,\}\,,$$

the set of **extended ideals** of $S^{-1}A$.

Consider $I \lhd A$. Then I may be regarded as an A-submodule of A , so we may form the module of fractions $S^{-1}I$, and make the identification:

$$S^{-1}I \;\equiv\; \{\, a/s \in S^{-1}A \mid a \in I\,,\; s \in S \,\}\,.$$

$$\boxed{\textbf{Claim:} \quad I^{\mathsf{e}} \;=\; S^{-1}I\ .}$$

Proof: $$I^{\mathsf{e}} \;=\; \langle\, f(I)\,\rangle_{\mathsf{ideal}}$$

$$= \left\{ \sum_{i=1}^{n} y_i\, f(x_i) \;\middle|\; n \in \mathbb{Z}^+ ,\; y_i \in S^{-1}A ,\; x_i \in I \quad (\forall i) \right\}$$

$$= \left\{ \sum_{i=1}^{n} (a_i/s_i)\,(x_i/1) \;\middle|\; n \in \mathbb{Z}^+ ,\; a_i \in A ,\; s_i \in S ,\; x_i \in I \quad (\forall i) \right\} .$$

Clearly, if $a \in I$, $s \in S$ then

$$a/s \;=\; (1/s)\,(a/1) \;\in\; I^{\mathsf{e}} \,,$$

so $S^{-1}I \;\subseteq\; I^{\mathsf{e}}$.

Conversely, if

$$\alpha \;=\; \sum_{i=1}^{n} (a_i/s_i)\,(x_i/1) \;\in\; I^{\mathsf{e}}$$

then, putting $s \;=\; s_1 \,\ldots\, s_n$ and

$$t_i \;=\; s_1 \ldots s_{i-1} s_{i+1} \ldots s_n \qquad\qquad (\forall i) \,,$$

$$\alpha = \sum \left(a_i x_i \,/\, s_i \right) = \sum \left(a_i x_i t_i \,/\, s \right)$$

$$= \left(\sum a_i x_i t_i \right) / \, s$$

$$\in S^{-1} I \,,$$

since $\sum a_i x_i t_i \in I$. Thus $I^{\mathsf{e}} \subseteq S^{-1} I$, and the claim is proved.

Theorem:

(1) $\mathcal{E} \;=\; \{$ all ideals of $S^{-1}A \}$.

(2) If $I \lhd A$ then

$$I^{\mathsf{ec}} \;=\; \bigcup_{s \in S} (I : s)\;,$$

in which case,

$$I^{\mathsf{ec}} = A \qquad \Longleftrightarrow \qquad I \cap S \neq \emptyset\;.$$

Theorem continued:

(3) $\mathcal{C} = \{\, I \lhd A \mid (\forall s \in S) \quad s + I$ is not a zero divisor in $A/I \,\}$.

(4) Prime ideals of $S^{-1}A$ are in a one-one correspondence with prime ideals of A disjoint from S , under

$$P \mapsto S^{-1}P \qquad (\forall \text{ prime ideals } P) .$$

(5) S^{-1} commutes with formation of finite sums, products, intersections and radicals.

Proof: **(1)** Consider $J \lhd S^{-1}A$. If $\alpha \in J$ then

$$\alpha = x/s \quad (\exists x \in A\,,\ s \in S)\ ,$$

so

$$f(x) = x/1 = (s/1)(x/s) = (s/1)\,\alpha \in J\ ,$$

giving $x \in f^{-1}(J) = J^{\mathsf{c}}$, so that

$$\alpha = (1/s)(x/1) = (1/s)f(x) \in (J^{\mathsf{c}})^{\mathsf{e}} = J^{\mathsf{ce}}\ .$$

Thus $J \subseteq J^{\mathsf{ce}}$, and, of course, reverse set containment holds, proving all ideals of $S^{-1}A$ are extended.

(2) Suppose $I \lhd A$. Then, for all $x \in A$,

$$x \in I^{\mathsf{ec}} \iff x \in (S^{-1}I)^{\mathsf{c}} \qquad \text{(by the Claim)}$$

$$\iff x/1 \in S^{-1}I$$

$$\iff x/1 = a/s \qquad (\exists a \in I\ ,\ s \in S)$$

$$\iff (xs - a)t = 0 \qquad (\exists a \in I\ ,\ s, t \in S)$$

$$\iff \quad xst \ = \ at \qquad (\exists a \in I \ , \ s, t \in S)$$

$$\iff \quad xu \ \in \ I \qquad (\exists u \in S)$$

(since S is multiplicatively closed)

$$\iff \quad x \ \in \ \bigcup_{u \in S} (I : u) \ .$$

Hence

$$I^{\mathsf{ec}} = \bigcup_{u \in S} (I : u) \,,$$

and thus

$$\begin{aligned} I^{\mathsf{ec}} = A \iff & A = \bigcup_{u \in S} (I : u) \\ \iff & u = 1\,u \in I \qquad (\exists u \in S) \\ \iff & I \cap S \neq \emptyset \,. \end{aligned}$$

(3) If $I \lhd A$ then

$$I \in \mathcal{C} \iff I = I^{\mathsf{ec}} \iff I \supseteq I^{\mathsf{ec}}$$

$$\iff I \supseteq (S^{-1}I)^{\mathsf{c}}$$

(by the earlier Claim)

$$\iff f^{-1}(S^{-1}I) \subseteq I$$

$$\iff \quad (\forall x \in I \ , \ s \in S \ , \ y \in A)$$

$$x/s \ = \ y/1 \qquad \Longrightarrow \qquad y \in I$$

$$\iff \quad (\forall x \in I \ , \ s \in S \ , \ y \in A)$$

$$(\exists t \in S) \ \ (x - ys)t \ = \ 0 \qquad \Longrightarrow \qquad y \in I$$

$$\iff \quad (\forall u \in S\,,\ y \in A) \qquad yu \in I \quad \Longrightarrow \quad y \in I$$

$$\iff \quad (\forall u \in S\,,\ y \in A)$$

$$(y+I)(u+I) \ = \ I \qquad \Longrightarrow \qquad y+I = I$$

$$\iff \quad (\forall u \in S) \qquad u+I \quad \text{is not a zero divisor of} \quad A/I\ ,$$

which proves that **(3)** holds.

(4) Let

$$\mathcal{P}_1 = \{ P \lhd A \mid P \ \text{prime} \ , \ P \cap S = \emptyset \} ,$$

$$\mathcal{P}_2 = \{ Q \lhd S^{-1}A \mid Q \ \text{prime} \} ,$$

and

$$\Phi : \mathcal{P}_1 \longrightarrow \mathcal{P}_2$$

where

$$\Phi(P) = P^{\mathsf{e}} = S^{-1}P \qquad (\forall P \in \mathcal{P}_1) .$$

$$\boxed{\text{WTS } \Phi \text{ is sensibly defined.}}$$

Let $P \in \mathcal{P}_1$. Certainly $S^{-1}P \lhd S^{-1}A$.

Suppose

$$\alpha\ ,\ \beta \in S^{-1}A \qquad \text{and} \qquad \alpha\,\beta \in S^{-1}P\ .$$

$$\boxed{\text{WTS} \quad \alpha \in S^{-1}P \quad \text{or} \quad \beta \in S^{-1}P\ .}$$

Now

$$\alpha = a/s\ ,\ \beta = b/t \qquad (\exists a, b \in A\ ,\ s, t \in S)\ .$$

Then

$$ab \,/\, st \;=\; \alpha\beta \;=\; c/u \qquad (\exists c \in P\,,\; u \in S)\;,$$

so

$$(abu - cst)v \;=\; 0 \qquad (\exists v \in S)\;,$$

yielding

$$abuv \;=\; cstv \;\in\; P\;.$$

But P is prime and $u\,,v \notin P$, so either

$$a \in P \quad \text{or} \quad b \in P\;,$$

which proves

$$\alpha = a/s \in S^{-1}P \quad \text{or} \quad \beta = b/t \in S^{-1}P \; .$$

Thus $S^{-1}P$ is prime, so that Φ is sensibly defined.

$$\boxed{\text{WTS} \quad \Phi \quad \text{is one-one.}}$$

Suppose $P_1 \, , \; P_2 \; \in \; \mathcal{P}_1$ and $\Phi(P_1) \; = \; \Phi(P_2)$, that is,

$$S^{-1}P_1 \; = \; S^{-1}P_2 \; .$$

If $x \in P_1$ then

$$x/1 \ = \ y/s \qquad (\exists y \in P_2 \, , \ s \in S) \ ,$$

so

$$(xs - y)t \ = \ 0 \qquad (\exists t \in S)$$

yielding

$$xst \ = \ yt \ \in \ P_2$$

whence $x \in P_2$ (since $s, t \notin P_2$). This shows $P_1 \subseteq P_2$ and similarly $P_2 \subseteq P_1$, whence equality. Thus Φ is one-one.

$$\boxed{\text{WTS} \quad \Phi \quad \text{is onto.}}$$

Suppose $Q \in \mathcal{P}_2$, so

$$Q^{\mathsf{c}} = f^{-1}(Q) \lhd A$$

is prime, being the preimage of a prime ideal by a ring homomorphism.

By **(1)**, $Q^{\mathsf{ce}} = Q$. If $s \in Q^{\mathsf{c}} \cap S$ then

$$1/1 = (1/s)(s/1) = (1/s)f(s) \in Q^{\mathsf{ce}} = Q ,$$

so $Q = S^{-1}A$, contradicting that Q is prime.

Hence $Q^{\mathsf{c}} \cap S = \emptyset$, so

$$Q^{\mathsf{c}} \in \mathcal{P}_1$$

and

$$\Phi(Q^{\mathsf{c}}) = Q^{\mathsf{c\,e}} = Q .$$

Thus Φ is onto, and the proof of **(4)** is complete.

(5) Let I_1 , $I_2 \in A$. Then, by earlier exercises,

$$S^{-1}(I_1 + I_2) = (I_1 + I_2)^{\mathsf{e}} = I_1^{\mathsf{e}} + I_2^{\mathsf{e}}$$

$$= S^{-1}I_1 + S^{-1}I_2 ,$$

and

$$S^{-1}(I_1\,I_2) \;=\; (I_1\,I_2)^{\mathsf{e}} \;=\; I_1^{\mathsf{e}}\,I_2^{\mathsf{e}}$$

$$\;=\; (S^{-1}I_1)\,(S^{-1}I_2)\ .$$

Further, regarding I_1 , I_2 as A-modules,

$$S^{-1}(I_1 \cap I_2) \;=\; (S^{-1}I_1) \cap (S^{-1}I_2)\ ,$$

by a recently proved theorem.

Suppose $I \lhd A$. Then, by an early exercise,

$$S^{-1}\, r(I) \;=\; \big[r(I)\big]^{\mathsf{e}} \;\subseteq\; r(I^{\,\mathsf{e}}) \;=\; r(S^{-1}I)\ .$$

If $\alpha \in r(S^{-1}I)$ then

$$\alpha^n \in S^{-1}I \qquad (\exists n \geq 1)$$

so that, writing $\alpha = a/s \quad (\exists a \in A\,,\ s \in S)$,

$$a^n\,/s^n \;=\; b/t \qquad (\exists b \in I)(\exists t \in S)$$

so

$$(a^n t - bs^n)u \;=\; 0 \qquad (\exists u \in S)$$

so

$$(atu)^n \;=\; (a^n tu)(t^{n-1}u^{n-1}) \;=\; (bs^n u)t^{n-1}u^{n-1} \;\in\; I\ ,$$

so $atu \;\in\; r(I)$ and

$$\alpha \;=\; a/s \;=\; (atu)/(stu) \;\in\; S^{-1}\big(r(I)\big)\ .$$

Thus $r(S^{-1}I) \;\subseteq\; S^{-1}\big(r(I)\big)$, so equality holds, finally completing the proof of the Theorem.

Corollary: The nilradical of $S^{-1}A$ is $S^{-1}N$ where N is the nilradical of A .

Proof: By **(5)** of the previous Theorem,

$$S^{-1}N \;=\; S^{-1}\big(r(\{0\})\big) \;=\; r(S^{-1}\{0\})$$

$$=\; r(\{0/1\})\ ,$$

which is the nilradical of $S^{-1}A$.

Corollary: Let P be a prime ideal of A .

Then the prime ideals of the local ring A_P are in a one-one correspondence with the prime ideals of A contained in P .

Proof: By part **(4)** of the previous Theorem this correspondence arises, because

an ideal avoids $S = A\backslash P$

iff

the ideal is contained in P .

These considerations tell us that

constructing A_P focuses attention on prime ideals **contained** in P .

On the other hand

constructing A/P focuses attention on prime ideals **containing** P .

Suppose also that Q is a prime ideal and $P \supseteq Q$.

To focus attention on prime ideals **between** P and Q suggests contructing the hybrid

$$\boxed{\quad S^{-1}A \,/\, S^{-1}Q \;\cong\; T^{-1}\big(A/Q\big) \quad} \qquad \cdots (*)$$

(isomorphism proved below)

where $S = A \backslash P$ and $T = (A/Q) \backslash (P/Q)$.

In particular, if $P = Q$ then

$$S^{-1}Q \;=\; \{\, x/s \mid x \in P\,,\; s \in S \,\}$$

is the unique maximal ideal of $A_P \;=\; S^{-1}A$ and

$$T \;=\; \{\, \text{nonzero elements of } A/P \,\}\,,$$

so $(*)$ becomes

residue field of A_P

$\cong$ field of fractions of the integral domain A/P .

Proof of $(*)$ **:** Let $\phi : S^{-1}A \longrightarrow T^{-1}(A/Q)$ where

$$a/s \;\mapsto\; a + Q \;/\; s + Q \qquad (a \in A\,,\; s \in S)\,.$$

It is easy to check that ϕ is a well-defined, onto ring homomorphism.

Clearly $S^{-1}Q \subseteq \ker\phi$.

Suppose $a/s \in \ker\phi$. Then

$$a+Q \;/\; s+Q \;=\; Q \;/\; 1+Q \;,$$

so

$$(a+Q)(t+Q) \;=\; Q \qquad (\exists t \in S) \;.$$

Hence

$$Q \;=\; at+Q \qquad \text{yielding} \qquad at \in Q \;.$$

But Q is prime and $t \notin P \supseteq Q$.

Thus $a \in Q$, so

$$a/s \in S^{-1}Q \ ,$$

which proves $\ker \phi \ = \ S^{-1}Q$.

The result $(*)$ now follows by the Fundamental Homomorphism Theorem.