3.3 Some Properties of Localization

Let A be a ring.

If M is an A-module and P a prime ideal of A, recall the notation

$$A_P = S^{-1}A \ , \qquad M_P = S^{-1}M$$

where $S = A \backslash P$.

Special Case of the Previous Theorem:

Let P be a prime ideal of A and M , N be A-modules. Then

$$M_P \otimes_{A_P} N_P \cong \big(M \otimes_A N\big)_P$$

as A_P-modules.

This follows because (putting $S = A \backslash P$)

$$\begin{aligned} M_P \otimes_{A_P} N_P &= S^{-1}M \otimes_{S^{-1}A} S^{-1}N \\ &\cong S^{-1}(M \otimes_A N) = \big(M \otimes_A N\big)_P . \end{aligned}$$

Call a property $\mathcal{P}$ involving rings, modules or homomorphisms "local"

if the property holds whenever every "localized" version of the property holds.

In the next two results we will prove, respectively that "being trivial" is a local property of modules, and

"being injective/surjective" is a local property of module homomorphisms.

Theorem: Let M be an A-module. Then TFAE:

(i) $M = 0$;

(ii) $M_P = 0$ for all prime ideals P ;

(iii) $M_Q = 0$ for all maximal ideals Q .

Proof: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is obvious, so it remains only to prove (iii) $\Longrightarrow$ (i).

Suppose that (iii) holds and M is not the zero module. Hence

$$(\exists x \in M) \qquad x \neq 0 \, .$$

Now

$$\text{Ann}\,(x) = \{\, a \in A \mid ax = 0 \,\} \lhd A$$

and certainly $\text{Ann}\,(x) \neq A$, since $1 \notin \text{Ann}\,(x)$.

Hence

$$\text{Ann}\,(x) \subseteq Q$$

for some maximal ideal Q of A .

But (iii) holds, so $M_Q = 0$. In particular,

$x/1$ is zero in M_Q , so $(x, 1) \equiv (0, 1)$.

Hence

$$0 = u(1 \cdot x - 1 \cdot 0) = ux \qquad (\exists u \in A \backslash Q)$$

so

$$u \in \text{Ann}\,(x) \subseteq Q ,$$

contradicting that $u \notin Q$.

Thus $M = 0$, so (i) holds, and the Theorem is proved.

Theorem: Let $\phi : M \to N$ be an A-module homomorphism, and write $\phi_P = S^{-1}\phi$ where $S = A \backslash P$ for a prime ideal P of A.

Then TFAE:

(i) ϕ is injective [surjective];

(ii) ϕ_P is injective [surjective] for all prime ideals P;

(iii) ϕ_Q is injective [surjective] for all maximal ideals Q.

Proof: We prove the injective case.

(i) $\Longrightarrow$ (ii): If ϕ is injective then

$$0 \longrightarrow M \overset{\phi}{\longrightarrow} N$$

is exact, so, for each prime ideal P ,

$$0 \longrightarrow M_P \overset{\phi_P}{\longrightarrow} N_P$$

is exact, by an earlier theorem, whence ϕ_P is injective.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (i): Suppose (iii) holds, and put

$$M' \;=\; \ker\phi\;.$$

Then

$$0 \;\longrightarrow\; M' \;\longrightarrow\; M \;\stackrel{\phi}{\longrightarrow}\; N$$

is exact, where the second mapping is inclusion.

For each maximal ideal Q of A,

$$0 \longrightarrow M'_Q \longrightarrow M_Q \overset{\phi_Q}{\longrightarrow} N_Q$$

is exact, by an earlier theorem, so

$$M'_Q \;\cong\; \ker \phi_Q \;=\; 0 \,,$$

by (iii), since ϕ_Q is injective.

By the previous Theorem in this section,

$$M' \;=\; 0\,,$$

which proves ϕ is injective, and (i) holds.

The surjective case is left as an **exercise**.

(Hint: reverse arrows in the previous argument, and use the image instead of the kernel.)