Lemma: Let M be an $S^{-1}A$ -module, which may be regarded as an A-module by restriction of scalars. Then

$$S^{-1}M \cong M$$

as $S^{-1}A$ -modules, under the map

$$\Phi : \alpha/s \mapsto (1/s) \alpha$$

for all $\alpha \in M$ and $s \in S$.

Proof: Observe that Φ is well-defined and one-one because

$$\alpha/s = \alpha'/s' \iff t \cdot (s' \cdot \alpha - s \cdot \alpha') = 0 \quad (\exists t \in S)$$

$$\iff$$
 $(t/1)\left((s'/1)\alpha - (s/1)\alpha'\right) = 0$

$$\iff$$
 $(s'/1) \alpha - (s/1) \alpha' = 0$,

SO

$$\alpha/s = \alpha'/s' \iff (s'/1) \alpha = (s/1) \alpha'$$

$$\iff (1/s) \alpha = (1/s') \alpha',$$

and Φ is onto because

$$(\forall \alpha \in M)$$
 $\Phi(\alpha/1) = (1/1)\alpha = \alpha$.

It is routine to check that Φ preserves module operations, completing the proof of the Lemma.

Now we can prove that taking fractions "commutes" with taking tensors:

Theorem: Let M , N be A-modules. Then there is a unique isomorphism of $S^{-1}A$ -modules

$$f: S^{-1}M \otimes_{S^{-1}A} S^{-1}N \longrightarrow S^{-1}(M \otimes_A N)$$

such that

$$f(\alpha/s \otimes \beta/t) = \alpha \otimes \beta / st.$$

Proof: Define

$$f': S^{-1}M \times S^{-1}N \longrightarrow S^{-1}(M \otimes_A N)$$

by

$$(\alpha/s, \beta/t) \mapsto \alpha \otimes \beta / st$$
.

This is well-defined, because if

$$\alpha/s = \alpha'/s'$$
 , $\beta/t = \beta'/t'$

then

$$\begin{cases} u(s'\alpha - s\alpha') = 0 & (\exists u \in S) \\ v(t'\beta - t\beta') = 0 & (\exists v \in S) \end{cases}$$

SO

$$u s' \alpha = u s \alpha'$$
 and $v t' \beta = v t \beta'$,

yielding

$$\alpha \otimes \beta / st = \frac{u s' v t'}{u s' v t'} \left(\frac{\alpha \otimes \beta}{st} \right)$$

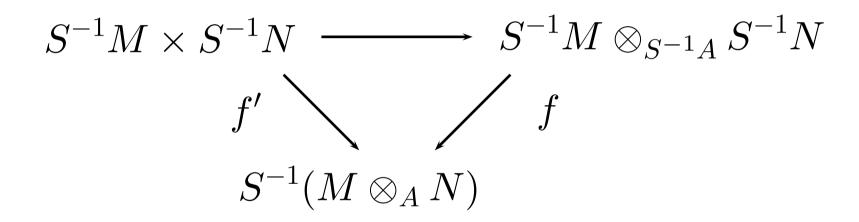
$$= \frac{u \, s' \, v \, t' \, (\alpha \otimes \beta)}{u \, s' \, v \, t' \, s \, t} = \frac{(u \, s' \, \alpha) \otimes (v \, t' \, \beta)}{u \, s' \, v \, t' \, s \, t}$$

$$= \frac{(u s \alpha') \otimes (v t \beta')}{u s v t s' t'} = \frac{u s v t (\alpha' \otimes \beta')}{u s v t s' t'}$$

$$= \frac{u s v t}{u s v t} \left(\frac{\alpha' \otimes \beta'}{s' t'} \right) = \alpha' \otimes \beta' / s' t'.$$

It is routine to check that f' is $S^{-1}A$ -bilinear.

Hence there is a unique $S^{-1}A$ -module homomorphism f which makes the following diagram commute:



that is,

$$f(\alpha/s \otimes \beta/t) = f'(\alpha/s, \beta/t) = \alpha \otimes \beta/st$$
.

It remains to show f is an isomorphism.

Let

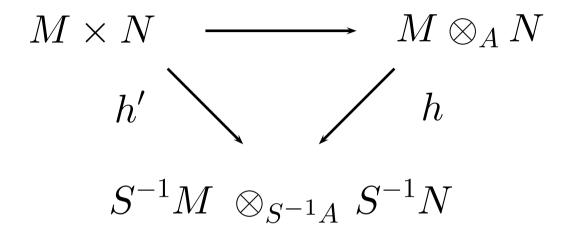
$$h': M \times N \longrightarrow S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$

where

$$(\alpha,\beta) \mapsto \alpha/1 \otimes \beta/1$$
.

It is easy to check that h' is A-bilinear.

Hence there is a unique A-module homomorphism h such that the following diagram commutes:



that is,

$$h(\alpha \otimes \beta) = h'(\alpha, \beta) = \alpha/1 \otimes \beta/1$$
,

and so

$$S^{-1}h: S^{-1}(M\otimes_A N) \longrightarrow S^{-1}(S^{-1}M\otimes_{S^{-1}A} S^{-1}N)$$

where

$$S^{-1}h\left(\frac{\alpha\otimes\beta}{s}\right) = \frac{\left(\alpha/1\otimes\beta/1\right)}{s}.$$

Now let

$$\Phi : S^{-1} \left(S^{-1}M \otimes_{S^{-1}A} S^{-1}N \right) \longrightarrow S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$

be the isomorphism of the previous lemma, so

$$\Phi : \theta/s \mapsto (1/s) \theta$$
.

Then

$$\Phi \circ S^{-1}h : S^{-1}(M \otimes_A N) \longrightarrow S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$

where

$$(\Phi \circ S^{-1}h)\left(\frac{\alpha \otimes \beta}{s}\right) = \Phi\left(\frac{(\alpha/1 \otimes \beta/1)}{s}\right)$$

$$= 1/s (\alpha/1 \otimes \beta/1)$$

$$= \alpha/s \otimes \beta/1 = \alpha/1 \otimes \beta/s.$$

Put
$$k = \Phi \circ S^{-1}h$$
.

Thus

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$
 $\sum_{k}^{f} S^{-1}(M \otimes_A N)$

Observe that

$$(k \circ f) \left(\alpha/s \otimes \beta/t \right) = k \left(\frac{\alpha \otimes \beta}{st} \right)$$

$$= \frac{1}{st} \left(\alpha/1 \otimes \beta/1 \right) = \left(\frac{1}{s} \right) \left(\frac{1}{t} \right) \left(\alpha/1 \otimes \beta/1 \right)$$

$$= \left(\frac{1}{s}\right)(\alpha/1) \otimes \left(\frac{1}{t}\right)(\beta/1) = \alpha/s \otimes \beta/t ,$$

and

$$(f \circ k)(\alpha \otimes \beta / s) = f(\alpha/s \otimes \beta/1)$$

$$= \alpha \otimes \beta / s$$
,

so, since they fix generators,

$$k \circ f$$
 and $f \circ k$

are identity mappings on their respective domains, which proves that f is an isomorphism.