Lemma: Let M be an $S^{-1}A$-module, which may be regarded as an A-module by restriction of scalars. Then

$$S^{-1}M \;\cong\; M$$

as $S^{-1}A$-modules, under the map

$$\Phi \;:\; \alpha/s \;\mapsto\; (1/s)\,\alpha$$

for all $\alpha \in M$ and $s \in S$.

Proof: Observe that Φ is well-defined and one-one because

$$\alpha/s \ = \ \alpha'/s' \quad \Longleftrightarrow \quad t \cdot (s' \cdot \alpha - s \cdot \alpha') \ = \ 0 \quad (\exists t \in S)$$

$$\Longleftrightarrow \quad (t/1) \left((s'/1)\, \alpha - (s/1)\, \alpha' \right) \ = \ 0$$

$$\Longleftrightarrow \quad (s'/1)\, \alpha - (s/1)\, \alpha' \ = \ 0 \ ,$$

so

$$\alpha/s \;=\; \alpha'/s' \quad\Longleftrightarrow\quad (s'/1)\,\alpha \;=\; (s/1)\,\alpha'$$

$$\Longleftrightarrow\quad (1/s)\,\alpha \;=\; (1/s')\,\alpha'\;,$$

and Φ is onto because

$$(\forall \alpha \in M) \qquad \Phi(\alpha/1) \;=\; (1/1)\alpha \;=\; \alpha\;.$$

It is routine to check that Φ preserves module operations, completing the proof of the Lemma.

Now we can prove that taking fractions "commutes" with taking tensors:

Theorem: Let M, N be A-modules. Then there is a unique isomophism of $S^{-1}A$-modules

$$f \;:\; S^{-1}M \;\otimes_{S^{-1}A}\; S^{-1}N \;\longrightarrow\; S^{-1}(M \otimes_A N)$$

such that

$$f\big(\alpha/s \;\otimes\; \beta/t\big) \;=\; \alpha \otimes \beta \;/\; st\;.$$

Proof: Define

$$f' : S^{-1}M \times S^{-1}N \longrightarrow S^{-1}(M \otimes_A N)$$

by

$$(\alpha/s \, , \, \beta/t) \mapsto \alpha \otimes \beta \, / \, st \, .$$

This is well-defined, because if

$$\alpha/s = \alpha'/s' \, , \, \beta/t = \beta'/t'$$

then

$$\begin{cases} u(s'\alpha - s\alpha') \;=\; 0 & (\exists u \in S) \\ \\ v(t'\beta - t\beta') \;=\; 0 & (\exists v \in S) \end{cases}$$

so

$$u\,s'\,\alpha \;=\; u\,s\,\alpha' \qquad \text{and} \qquad v\,t'\,\beta \;=\; v\,t\,\beta'\;,$$

yielding

$$\alpha \otimes \beta \;/\; s\,t \;\;=\;\; \frac{u\,s'\,v\,t'}{u\,s'\,v\,t'}\left(\frac{\alpha \otimes \beta}{st}\right)$$

$$= \frac{u\,s'\,v\,t'\,(\alpha \otimes \beta)}{u\,s'\,v\,t'\,s\,t} = \frac{(u\,s'\,\alpha) \otimes (v\,t'\,\beta)}{u\,s'\,v\,t'\,s\,t}$$

$$= \frac{(u\,s\,\alpha') \otimes (v\,t\,\beta')}{u\,s\,v\,t\,s'\,t'} = \frac{u\,s\,v\,t\ (\alpha' \otimes \beta')}{u\,s\,v\,t\,s'\,t'}$$

$$= \frac{u\,s\,v\,t}{u\,s\,v\,t}\left(\frac{\alpha' \otimes \beta'}{s'\,t'}\right) = \alpha' \otimes \beta' \ / \ s'\,t' \ .$$

It is routine to check that f' is $S^{-1}A$-bilinear.

Hence there is a unique $S^{-1}A$-module homomorphism f which makes the following diagram commute:

$$\begin{array}{ccc} S^{-1}M \times S^{-1}N & \longrightarrow & S^{-1}M \otimes_{S^{-1}A} S^{-1}N \\ & f' \searrow \quad \swarrow f & \\ & S^{-1}(M \otimes_A N) & \end{array}$$

that is,

$$f\big(\alpha/s \otimes \beta/t\big) \;=\; f'(\alpha/s, \beta/t) \;=\; \alpha \otimes \beta \,/\, st \;.$$

It remains to show f is an isomorphism.

Let

$$h' : M \times N \longrightarrow S^{-1}M \otimes_{S^{-1}A} S^{-1}N$$

where

$$(\alpha, \beta) \mapsto \alpha/1 \otimes \beta/1 .$$

It is easy to check that h' is A-bilinear.

Hence there is a unique A-module homomorphism h such that the following diagram commutes:

$$
\begin{array}{ccc}
M \times N & \longrightarrow & M \otimes_A N \\
 & h' \searrow \quad \swarrow h & \\
 & S^{-1}M \otimes_{S^{-1}A} S^{-1}N &
\end{array}
$$

that is,

$$h(\alpha \otimes \beta) \;=\; h'(\alpha, \beta) \;=\; \alpha/1 \otimes \beta/1 \,,$$

and so

$$S^{-1}h \;:\; S^{-1}(M \otimes_A N) \longrightarrow S^{-1}\left(S^{-1}M \otimes_{S^{-1}A} S^{-1}N\right)$$

where

$$S^{-1}h\left(\frac{\alpha \otimes \beta}{s}\right) \quad = \quad \frac{(\alpha/1 \;\otimes\; \beta/1)}{s}\,.$$

Now let

$$\Phi \;:\; S^{-1}\left(S^{-1}M \;\otimes_{S^{-1}A}\; S^{-1}N\right) \longrightarrow$$

$$S^{-1}M \;\otimes_{S^{-1}A}\; S^{-1}N$$

be the isomorphism of the previous lemma, so

$$\Phi \ : \ \theta/s \ \mapsto \ (1/s)\,\theta \ .$$

Then

$$\Phi \ \circ \ S^{-1}h \ : \ \ S^{-1}(M \otimes_A N) \ \ \longrightarrow \ \ S^{-1}M \ \otimes_{S^{-1}A} \ S^{-1}N$$

where

$$\big(\Phi \ \circ \ S^{-1}h\big)\left(\frac{\alpha \otimes \beta}{s}\right) \ = \ \Phi\left(\frac{\big(\alpha/1 \ \otimes \ \beta/1\big)}{s}\right)$$

$$= \ 1/s \ \left(\alpha/1 \otimes \beta/1\right)$$

$$= \ \alpha/s \otimes \beta/1 \ = \ \alpha/1 \otimes \beta/s \, .$$

Put $k \ = \ \Phi \circ S^{-1}h$.

Thus

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \ \underset{k}{\overset{f}{\rightleftarrows}} \ S^{-1}\left(M \otimes_A N\right)$$

Observe that

$$(k \circ f)\left(\alpha/s \ \otimes \beta/t\right) \ = \ k\left(\frac{\alpha \otimes \beta}{s\,t}\right)$$

$$= \ \frac{1}{s\,t}\left(\alpha/1 \ \otimes \beta/1\right) \ = \ \left(\frac{1}{s}\right)\left(\frac{1}{t}\right)\left(\alpha/1 \ \otimes \beta/1\right)$$

$$= \ \left(\frac{1}{s}\right)(\alpha/1) \ \otimes \ \left(\frac{1}{t}\right)(\beta/1) \ = \ \alpha/s \ \otimes \ \beta/t\ ,$$

and

$$(f \circ k)\big(\alpha \otimes \beta \;/\; s\big) \;=\; f\big(\alpha/s \;\otimes\; \beta/1\big)$$

$$= \;\alpha \otimes \beta \;/\; s \;,$$

so, since they fix generators,

$$k \circ f \;\text{ and }\; f \circ k$$

are identity mappings on their respective domains, which proves that f is an isomorphism.