3.2 Modules of Fractions

Let A be a ring, S a multiplicatively closed subset of A , and M an A-module.

Define a relation $\equiv$ on

$$M \times S \ = \ \{ \ (m, s) \mid m \in M, \ s \in S \ \}$$

by, for $m, m' \in M$, $s, s' \in S$,

$$\boxed{\begin{array}{l} \qquad\qquad\qquad (m, s) \;\equiv\; (m', s') \\[2em] \text{iff} \quad (\exists t \in S) \qquad t(sm' - s'm) \;=\; 0\;. \end{array}}$$

As before it is straightforward to check that

$$\boxed{\equiv \ \text{ is an equivalence relation.}}$$

If $m \in M$ and $s \in S$ then write

$$m/s \;=\; \text{equivalence class of } (m, s) \,.$$

and put

$$S^{-1}M \;=\; \{\, m/s \mid m \in M \,,\; s \in S \,\} \,.$$

Define addition and **scalar** multiplication on $S^{-1}M$ by, for $m, m' \in M$, $s, s' \in S$, $a \in A$, $t \in S$,

$$\boxed{\begin{gathered}(m/s) \; + \; (m'/s') \; = \; (sm' + s'm) \,/\, ss' \\[2em] (a/t)\,(m/s) \; = \; am \,/ts \; .\end{gathered}}$$

member of $S^{-1}A$.

Exercise: Prove that addition and scalar multiplication are well-defined.

It is now routine to check that

$S^{-1}M$ is an $S^{-1}A$-module, referred to as the **module of fractions with respect to** S .

Since the mapping $a \mapsto a/1$ is a ring homomorphism: $A \rightarrow S^{-1}A$, by restriction of scalars we have

$S^{-1}M$ is an A-module with scalar multiplication

$$(\forall a \in A \ , \ m \in M \ , \ s \in S)$$

$$a \cdot (m/s) \ = \ (a/1)(m/s) \ = \ am/s \ .$$

The mapping: $M \ \rightarrow \ S^{-1}M \ , \ x \ \mapsto \ x/1$ is an A-module homomorphism and injective iff

$$(\forall x \in M \ , \ x \neq 0 \ ) \quad \text{Ann}\,(Ax) \cap S \ = \ \emptyset \ .$$

Some notation: Let M be an A-module.

(1) Write
$$M_P = S^{-1}M$$
if $S = A \backslash P$ where P is a prime ideal of A .

(2) Write
$$M_x = S^{-1}M$$
if $S = \{ x^n \mid n \geq 0 \}$ for some $x \in A$.

Think of S^{-1} as an "operator" which manufactures $S^{-1}A$-modules from A-modules.

Also S^{-1} "operates" on module homomorphisms. Let $u : M \to N$ be an A-module homomorphism. Define

$$\boxed{\begin{array}{c} S^{-1}u \ : \ S^{-1}M \ \to \ S^{-1}N \qquad \text{by} \\ m/s \ \mapsto \ u(m)/s \qquad (m \in M \ , \ s \in S) \ . \end{array}}$$

It is routine to check that $S^{-1}u$ is well-defined. We check S^{-1} preserves addition:

$$(S^{-1}u)\big(m_1/s_1 + m_2/s_2\big) \;=\; (S^{-1}u)\big(s_2m_1 + s_1m_2 \;/\; s_1s_2\big)$$

$$= u(s_2m_1 + s_1m_2) \,/\, s_1s_2 \;=\; \big[s_2\, u(m_1) + s_1u(m_2)\big] \,/\, s_1s_2$$

$$= u(m_1)/s_1 + u(m_2)/s_2 = (S^{-1}u)\big(m_1/s_1\big) + (S^{-1}u)\big(m_2/s_2\big).$$

Similarly S^{-1} preserves scalar multiplication.

Hence

$$\boxed{S^{-1}u \text{ is an } S^{-1}A\text{-module homomorphism}}$$

(and also, by restriction of scalars, an A-module homomorphism).

Further, if

$$M_1 \xrightarrow{u} M_2 \xrightarrow{v} M_3$$

are A-module homomorphisms, then,

for all $x \in M_1$, $s \in S$,

$$\big[S^{-1}(v \circ u)\big](x/s) \;=\; (v \circ u)(x) \,/\, s \;=\; v(u(x)) \,/\, s$$

$$=\; (S^{-1}v)\big((S^{-1}u)(x/s)\big) \;=\; \big[(S^{-1}v) \circ (S^{-1}u)\big](x/s) \;,$$

which shows

$$\boxed{S^{-1}(v \circ u) \;=\; (S^{-1}v) \circ (S^{-1}u) \;.}$$

(We call S^{-1} a **functor**.)

Theorem: Suppose

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is exact at M . Then

$$S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$$

is exact at $S^{-1}M$.

(We call S^{-1} an **exact** functor.)

Proof: We have $g \circ f = 0$ the zero homomorphism, so

$$(S^{-1}g)\circ(S^{-1}f) \;=\; S^{-1}(g\circ f) \;=\; S^{-1}(0) \;=\; 0\;,$$

which proves $\mathrm{im}\,(S^{-1}f) \subseteq \ker(S^{-1}g)$.

Suppose $m/s \in \ker(S^{-1}g)$, so $g(m)/s$ is the zero of $S^{-1}M''$. Hence $(g(m), s) \equiv (0, 1)$, so

$$0 \;=\; t\,g(m) \;=\; g(tm) \qquad (\exists t \in S)\;,$$

yielding $tm \in \ker g = \mathrm{im}\, f$.

Hence

$$tm \;=\; f(m') \qquad\quad (\exists m' \in M') \ ,$$

whence

$$(S^{-1}f)(m' \,/\, st) \;=\; f(m') \,/\, st \;=\; tm \,/\, ts \;=\; m/s \ ,$$

proving $m/s \in \mathrm{im}\,(S^{-1}f)$.

Thus $\ker(S^{-1}g) \;\supseteq\; \mathrm{im}\,(S^{-1}f)$, completing the proof of exactness at $S^{-1}M$.

In particular, if M' is a submodule of M then

$$0 \to M' \to M$$

is exact (where the mapping on the right is the inclusion embedding), so, by the Theorem,

$$0 = S^{-1}0 \to S^{-1}M' \to S^{-1}M$$

is exact, so that

inclusion induces an embedding of $S^{-1}M'$ in $S^{-1}M$.

Thus we may regard $S^{-1}M'$ as a submodule of $S^{-1}M$, identifying each

$$x/s \text{ in } S^{-1}M'$$

with

$$x/s \text{ in } S^{-1}M .$$

With this identification we can prove that formation of fractions "commutes" with formation of sums, finite intersections and quotients:

Theorem: Let N and P be submodules of an A-module M. Then

(1) $S^{-1}(N+P) \;=\; (S^{-1}N) \;+\; (S^{-1}P)$;

(2) $S^{-1}(N \cap P) \;=\; (S^{-1}N) \;\cap\; (S^{-1}P)$;

(3) $S^{-1}(M/N) \;\cong\; (S^{-1}M)/(S^{-1}N)$

(as $S^{-1}A$-modules).

Proof: (1) Clearly $S^{-1}N\ ,\ S^{-1}P \subseteq S^{-1}(N+P)\,,$

so $S^{-1}N \ + \ S^{-1}P \ \subseteq \ S^{-1}(N+P)\ ,$

since $S^{-1}(N+P)$ is a submodule of $S^{-1}M$.

Also $S^{-1}(N+P) \ = \ \{\ (x+y)/s \mid x \in N\ ,\ y \in P\ ,\ s \in S\ \}$

$$= \ \{\ x/s \ + \ y/s \ \mid \ x \in N\ ,\ y \in P\ ,\ s \in S\ \}$$

$\subseteq \quad S^{-1}N \ + \ S^{-1}P\ ,$ whence equality holds.

(2) Clearly $N \cap P \subseteq N\ ,\ P$, so

$$S^{-1}(N \cap P) \subseteq S^{-1}N\ ,\ S^{-1}P\ ,$$

so

$$S^{-1}(N \cap P) \subseteq S^{-1}N \cap S^{-1}P\ .$$

Suppose $\alpha \in (S^{-1}N) \cap (S^{-1}P)$, so

$$\alpha = x/s = y/t$$

for some $x \in N\ ,\ y \in P\ ,\ s, t \in S$.

Then

$$u(sy - tx) = 0 \qquad (\exists u \in S)$$

so

$$usy \;=\; utx \;\in\; N \cap P \; .$$

Hence

$$\alpha \;=\; x/s \;=\; (ut)x \;/\; (ut)s \;\in\; S^{-1}(N \cap P) \; .$$

Thus

$$(S^{-1}N) \;\cap\; (S^{-1}P) \;\;\subseteq\;\; S^{-1}(N \cap P) \; ,$$

whence equality holds.

(3) Observe that

$$0 \to N \to M \to M/N \to 0$$

is exact, where the second mapping is inclusion, and the third mapping is natural. By the previous Theorem,

$$0 \to S^{-1}N \to S^{-1}M \to S^{-1}(M/N) \to 0$$

is exact, whence

$$S^{-1}(M/N) \cong S^{-1}M \,/\, S^{-1}N \,.$$

Theorem: Let M be an A-module. Then

$$S^{-1}M \;\cong\; S^{-1}A \otimes_A M$$

as $S^{-1}A$-modules, under the unique isomorphism

$$f : S^{-1}A \otimes_A M \to S^{-1}M$$

with the property that

$$(a/s) \otimes m \mapsto am \,/\, s \qquad \ldots (*)$$

Proof: Easy to see

$$f' \;:\; S^{-1}A \times M \;\to\; S^{-1}M \;, \quad (a/s, m) \;\mapsto\; am \,/\, s$$

is A-bilinear, so there is a unique A-module homomorphism f making the following diagram commute yielding the rule $(*)$:

$$\begin{tikzcd}
S^{-1}A \times M \arrow[rr] \arrow[dr, "f'"'] & & S^{-1}A \otimes M \arrow[dl, "f"] \\
& S^{-1}M &
\end{tikzcd}$$

It remains to check f is bijective and preserves scalar multiplication by $S^{-1}A$.

Certainly f is onto because

$$(\forall m \in M\ ,\ s \in S) \qquad m/s \ = \ f(1/s \ \otimes \ m)\ .$$

Before proving f is one-one, we prove:

Claim:

$$S^{-1}A \otimes M \ = \ \{\ 1/s \otimes m \ \mid \ s \in S\ ,\ \ m \in M\ \}\ .$$

Let

$$\alpha \;=\; \sum_{i=1}^{n} (a_i/s_i) \otimes m_i \;\in\; S^{-1}A \otimes M \;.$$

Put

$$s \;=\; s_1 \ldots s_n$$

and

$$t_i \;=\; s_1 \ldots s_{i-1} s_{i+1} \ldots s_n$$

for each $i = 1, \ldots, n$.

Then

$$\alpha = \sum (a_i t_i / s) \otimes m_i = \sum \Big[(a_i t_i / 1)(1/s) \Big] \otimes m_i$$

$$= \sum (a_i t_i / 1) \Big[(1/s) \otimes m_i \Big] = \sum (a_i t_i) \cdot \Big[(1/s) \otimes m_i \Big]$$

$$= \sum (1/s) \otimes (a_i t_i m_i) = 1/s \otimes m ,$$

where $m = \sum a_i t_i m_i$, proving the Claim.

We now prove that f is one-one.

Suppose $\alpha \in \ker f$. Then, by the Claim,

$$\alpha \; = \; 1/s \; \otimes \; m \qquad\qquad (\exists s \in S)(\exists m \in M) \; ,$$

so

$$0 \; = \; f(\alpha) \; = \; f(1/s \; \otimes \; m) \; = \; m/s \; .$$

Thus $(m, s) \; \equiv \; (0, 1)$ so

$$tm = 0 \text{ for some } t \in S \; .$$

Hence

$$
\begin{aligned}
\alpha \;&=\; 1/s \otimes m \;=\; t/ts \otimes m \\
&=\; \big[(t/1)(1/st)\big] \otimes m \;=\; (t/1)\big[1/st \otimes m\big] \\
&=\; t \cdot \big[1/st \otimes m\big] \;=\; 1/st \otimes (tm) \\
&=\; 1/st \otimes 0 \;=\; 0\,.
\end{aligned}
$$

Hence $\ker f = \{0\}$, so f is one-one.

That f preserves scalar multiplication by elements of $S^{-1}A$ follows from the following Lemma, whose proof is left as an **exercise**. This completes the proof that f is an $S^{-1}A$-module isomorphism.

Lemma: Let $f : M_1 \longrightarrow M_2$ be an A-module homomorphism, where M_1 and M_2 are $S^{-1}A$-modules, regarded as A-modules by restriction of scalars.

Then f is also an $S^{-1}A$-module homomorphism.