3.1 Rings of Fractions

Let A be a ring.

Call a subset S of A **multiplicatively closed** if

(i) $1 \in S$;

(ii) $(\forall x, y \in S) \quad xy \in S$.

For example, if A is an integral domain then $A \backslash \{0\}$ is multiplicatively closed.

More generally, if P is a prime ideal of A then

$A \backslash P$ is multiplicatively closed.

Let S be a multiplicatively closed subset of A . Define a relation $\equiv$ on

$$A \times S = \{ (a, s) \mid a \in A, \ s \in S \}$$

as follows:

for $a, b \in A$, $s, t \in S$,

$$(a, s) \equiv (b, t)$$

$$\text{iff} \quad (\exists u \in S) \qquad (at - bs)u = 0 .$$

Claim: $\equiv$ is an equivalence relation.

Proof: Clearly $\equiv$ is reflexive and symmetric.

Suppose $(a, s) \equiv (b, t) \equiv (c, u)$.

Then, for some $v, w \in S$

$$(at - bs)v \;=\; 0 \;=\; (bu - ct)w \;,$$

so

$$\begin{aligned} atv - bsv \;&=\; 0 \\ buw - ctw \;&=\; 0 \end{aligned}$$

so

$$\begin{aligned} atv(uw) - bsv(uw) \;&=\; 0 \\ -ctw(sv) + buw(sv) \;&=\; 0 \end{aligned}$$

so

$$au(tvw) \; - \; cs(tvw) \; = \; (au - cs)(tvw) \; = \; 0 \; ,$$

But $tvw \in S$, since S is multiplicatively closed,

so $(a, s) \; \equiv (c, u)$, which proves $\equiv$ is transitive.

If $a \in A$ and $s \in S$ then write $a/s \; = \;$ equivalence class of (a, s) .

Put

$$S^{-1}A \;=\; \{\, a/s \;\mid\; a \in A\,,\; s \in S \,\}$$

and define addition and multiplication on $S^{-1}A$ by

$$(a/s) \;+\; (b/t) \;=\; (at + bs)\,/\,st$$

$$(a/s)\,(b/t) \;=\; ab\,/st\;.$$

We check that multiplication is well-defined:

Suppose

$$(a_1, s_1) \equiv (a_2, s_2) \qquad \text{and} \qquad (b_1, t_1) \equiv (b_2, t_2) \,.$$

Then, for some $u, v \in S$,

$$(a_1 s_2 - a_2 s_1) u = 0 \quad \text{and} \quad (b_1 t_2 - b_2 t_1) v = 0 \,.$$

WTS $(a_1 b_1, s_1 t_1) \equiv (a_2 b_2, s_2 t_2)$.

$$\Big[(a_1b_1)(s_2t_2) \ - \ (a_2b_2)(s_1t_1)\Big]uv$$

$$\begin{aligned} = \ & (a_1b_1)(s_2t_2)(uv) \ - \ (a_2b_2)(s_1t_1)(uv) \\ & - \ (a_2s_1)(b_1t_2)(uv) \ + \ (a_2s_1)(b_1t_2)(uv) \end{aligned}$$

$$= \ (a_1s_2 \ - \ a_2s_1)u(b_1t_2v) \ + \ (b_1t_2 \ - \ b_2t_1)v(a_2s_1u)$$

$$= \ 0 + 0 \ = \ 0\,.$$

Thus

$$(a_1b_1, s_1t_1) \equiv (a_2b_2, s_2t_2) ,$$

which verifies that multiplication is well-defined.

Exercise: Prove that addition in $S^{-1}A$ is well-defined.

It is now routine to check that $S^{-1}A$ is a ring with identity $1 = s/s \qquad (\forall s \in S)$.

We call $S^{-1}A$ the **ring of fractions of** A **with respect to** S .

If A is an integral domain and $S = A \backslash \{0\}$ then

$S^{-1}A$ is the familiar **field of fractions** of A .

Let $f : A \rightarrow S^{-1}A$ where $f(x) = x/1$.

Clearly f is a ring homomorphism.

Observation: If A is an integral domain and S any multiplicatively closed subset not containing 0 then

$$f \text{ is injective.}$$

Proof: Suppose A is an integral domain, $0 \notin S \subseteq A$, and S multiplicatively closed.

Let $x_1, x_2 \in A$ such that $x_1/1 = x_2/1$.

Then $(x_1, 1) \equiv (x_2, 1)$, so

$$(x_1 - x_2)u = 0 \qquad (\exists u \in S) ,$$

yielding $x_1 - x_2 = 0$, since A is an integral domain and $u \neq 0$.

Thus $x_1 = x_2$, proving f is injective.

If A is not an integral domain then f need not be injective:

Exercise: Let $A = \mathbb{Z}_6$,

$$S_1 = \{ 1, 2, 4 \} \qquad S_2 = \{ 1, 3 \} .$$

Then S_1 and S_2 are multiplicatively closed.

Verify that

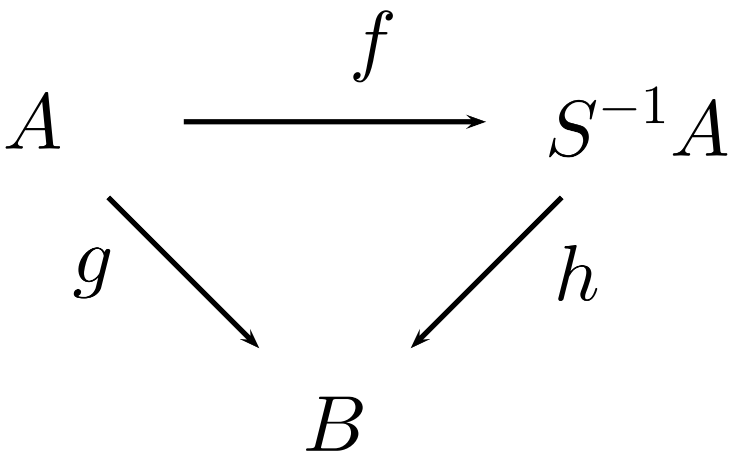
$$S_1^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_3 , \quad S_2^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_2$$

(so certainly, in both cases, f is not injective).

$S^{-1}A$ has the following universal property:

Theorem: Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for each $s \in S$.

Then there is a unique homomorphism h such that



commutes.

Proof: Define $h : S^{-1}A \longrightarrow B$ by

$$h(a/s) \ = \ g(a)\, g(s)^{-1} \qquad\qquad (a \in A \ , \ \ s \in S) \ .$$

WTS h is well defined.

Suppose $a/s \ = \ a'/s'$ so $(a, s) \ \equiv \ (a', s')$, so

$$(as' - a's)t \ = \ 0 \qquad\qquad (\exists t \in S) \ .$$

Thus

$$0 \;=\; g(0) \;=\; g\Big(\,(as' - a's)t\,\Big)$$

$$= \; \big[g(a)g(s') \; - \; g(a')g(s)\big]\, g(t) \; ,$$

so, since $g(t)$ is a unit in B ,

$$g(a)g(s') \; - \; g(a')g(s) \; = \; 0 \; ,$$

so

$$g(a)g(s') \; = \; g(a')g(s) \; ,$$

yielding, since $g(s), g(s')$ are units in B ,

$$g(a)\, g(s)^{-1} \;=\; g(a')\, g(s')^{-1} \;.$$

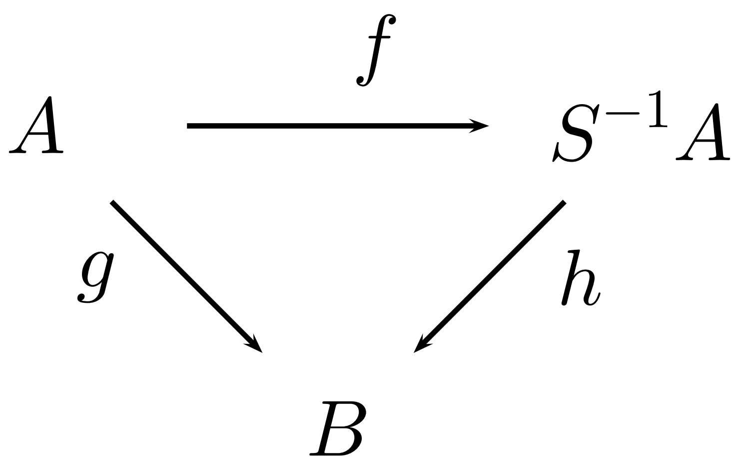
This proves h is well-defined.

It is routine now to check that h is a ring homomorphism.

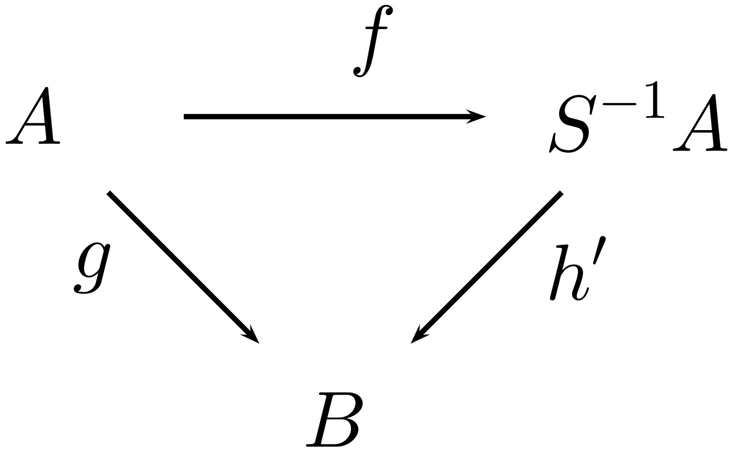
Further, if $a \in A$ then

$$(h \circ f)(a) \ = \ h(a/1) \ = \ g(a)g(1)^{-1} \ = \ g(a) \ ,$$

so that the following diagram commutes:



Suppose also that $h' : S^{-1}A \to B$ is a ring homomorphism such that



commutes.

Then

$$h'(a/s) \;=\; h'(a/1 \cdot 1/s) \;=\; h'(a/1)\, h'(1/s)\ .$$

But $1/s$ is a unit in $S^{-1}A$ with inverse $s/1$, so that $h'(1/s)$ is a unit in B and

$$h'(1/s) \;=\; \big[h'(s/1)\big]^{-1}\,.$$

Hence

$$\begin{aligned} h'(a/s) \;&=\; h'(a/1)\,\big[h'(s/1)\big]^{-1} \\ &=\; h'(f(a))\,\big[h'(f(s))\big]^{-1} \;=\; g(a)g(s)^{-1} \;=\; h(a/s)\,. \end{aligned}$$

This proves $h' = h$, and so h is unique with the required properties.

Observe that $S^{-1}A$ and

$$f : A \longrightarrow S^{-1}A\,, \quad a \mapsto a/1$$

have the following properties:

(1) $s \in S$ implies $f(s)$ is a unit in $S^{-1}A$

(because $s/1$ has inverse $1/s$);

(2) $f(a) = 0$ implies $as = 0 \quad (\exists s \in S)$

(because the zero in $S^{-1}A$ is $0/1$);

(3) every element of $S^{-1}A$ has the form

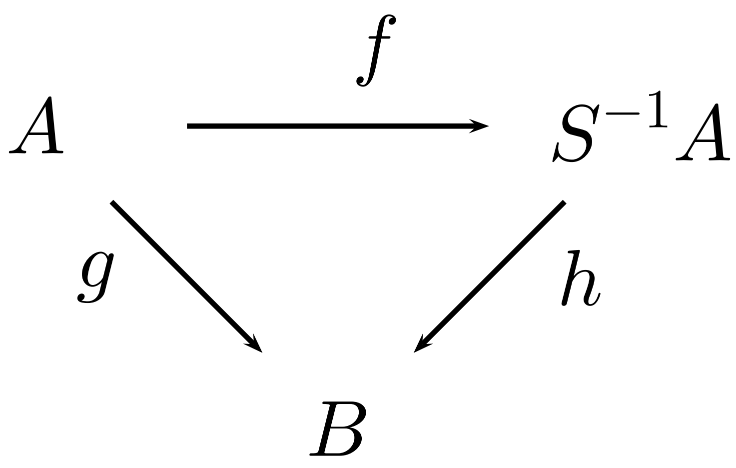
$$f(a)f(s)^{-1} \qquad (\exists a \in A)(\exists s \in S)$$

(because $a/s = a/1 \cdot 1/s$).

Conversely, properties (1), (2), (3) characterize $S^{-1}A$ up to isomorphism:

Corollary: Let $g : A \longrightarrow B$ be a ring homomorphism such that properties (1), (2) and (3) hold with g replacing f and B replacing $S^{-1}A$.

Then there is a unique **isomorphism** h such that the following diagram commutes:



Proof: By (1) and the previous Theorem, there is a unique homomorphism $h : S^{-1}A \to B$ such that

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & S^{-1}A \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & B & \end{array}$$

commutes.

Further, from the proof,

$$h(a/s) \;=\; g(a)\, g(s)^{-1} \qquad (a \in A\,,\;\; s \in S)\;.$$

By (3), h is onto.

If $a/s \in \ker h$ for some $a \in A$, $s \in S$, then

$$0 \;=\; h(a/s) \;=\; g(a)\, g(s)^{-1} \;,$$

so that $g(a) \;=\; 0\, g(s) \;=\; 0$, yielding, by (2),

$$at \;=\; 0 \qquad (\exists t \in S) \;,$$

whence $(a, s) \;\equiv\; (0, 1)$, that is, $a/s \;=\; 0$ in $S^{-1}A$.

Thus h is one-one, so h is an isomorphism.

Examples:

(1) Let P be a prime ideal of A, and put

$$S = A \backslash P ,$$

which is multiplicatively closed. Form

$$A_P = S^{-1} A ,$$

and put

$$M = \{ a/s \in A_p \mid a \in P \} .$$

Claim: A_P is a local ring with unique maximal ideal M .

The process of passing from A to A_P is called **localization at** P .

e.g. If $A = \mathbb{Z}$ and $P = p\mathbb{Z}$ where p is a prime integer, then localization at P produces

$$A_P = \{ a/b \mid a, b \in \mathbb{Z} , \; p \not| b \} .$$

Proof of Claim: We first prove

$$(*) \quad \boxed{\begin{array}{c} (\forall b \in A)\ (\forall t \in S) \\ b/t \in M \quad \Longrightarrow \quad b \in P \end{array}}$$

Suppose

$$b/t \;=\; a/s$$

where $b \in A$, $a \in P$ and $s, t \in S$. Then

$$(at - bs)u \;=\; 0 \qquad (\exists u \in S)$$

so

$$at - bs \in P$$

since P is prime, $0 \in P$ and $u \notin P$.

Hence

$$bs = at - (at - bs) \in P .$$

But $s \notin P$, so $b \in P$, and $(*)$ is proved.

By $(*)$, certainly $1 = 1/1 \notin M$ (since $1 \notin P$)

so $M \neq A_P$.

It is easy to check that $M \lhd A_P$.

Further, if $b \in A$, $t \in S$ and $b/t \notin M$,

then, by definition of M , $b \notin P$, so $b \in S$, yielding

$$t/b \in A_P ,$$

whence b/t is a unit of A_P .

By (i) of an early Proposition (on page 105), A_P is local with unique maximal ideal M .

Examples (continued):

(2) $S^{-1}A$ is the zero ring iff $0 \in S$.

Proof: $(\Longleftarrow)$ If $0 \in S$ then, for all $a, b \in A$, $s, t \in S$,

$$a/s \;=\; b/t$$

since

$$(at - bs)0 \;=\; 0\;,$$

so that all elements of $S^{-1}A$ are equal.

($\Longrightarrow$) If $S^{-1}A$ contains only one element then

$$(0,1) \equiv (1,1)$$

so that

$$0 = (0\cdot 1 - 1\cdot 1)t = -t \qquad (\exists t \in S)$$

so that $0 = t \in S$.

(3) Let $x \in A$ and put

$$S = \{ x^n \mid n \geq 0 \} \qquad \text{(where } x^0 = 1) .$$

Then S is multiplicatively closed, so we may form

$$A_x = S^{-1}A .$$

e.g. If $A = \mathbb{Z}$ and $0 \neq x \in \mathbb{Z}$ then

$A_x = \{$ rational numbers in reduced form whose denominators divide a power of $x \}$.

(4) Let I be an ideal of a ring A and put

$$S \;=\; 1+I \;=\; \{\, 1+x \;\mid\; x \in I \,\}\ .$$

Then S is easily seen to be multiplicatively closed, so we may form $S^{-1}A$.

e.g. If $A = \mathbb{Z}$ and $I \;=\; 6\mathbb{Z}$ then

$S^{-1}A \;=\; \{$ rational numbers in reduced form whose denominators divide some integer congruent to 1 mod 6 $\}$.