Proof: (1. implies 2.)

If A is a field and $\{0\} \neq I \lhd A$ then

$$a \in I \qquad \text{for some} \qquad a \neq 0 \ ,$$

so a is a unit, so $I \supseteq aA = A$, yielding

$$I = A \ .$$

(2. implies 3.)

Suppose the only ideals of A are $\{0\}$ and A, and let $f : A \to B$ be a ring homomorphism **onto** a nonzero ring B.

Certainly $\ker f \neq A$. But

$$\ker f \ \triangleleft \ A$$

so $\ker f = \{0\}$, from which it follows that f is injective.

Exercise: A ring homomorphism is injective iff its kernel is trivial.

(3. implies 1.)

Suppose that 3. holds and let $x \in A$ where x is **not** a unit.

Then $xA \neq A$, and $xA \lhd A$ so A/xA is not the zero ring.

By 3., the natural map : $A \to A/xA$ is injective, so its kernel is trivial, that is,

$$xA = \{0\} .$$

In particular,

$$x = x \cdot 1 = 0 .$$

This proves all nonzero elements are units, so A is a field, and the proof of the Proposition is complete.

Prime and maximal ideals:

An ideal P of A is called **prime** if $P \neq A$ and

$$(\forall x, y \in A) \qquad xy \in P \quad \Longrightarrow \quad x \in P \text{ or } y \in P\ .$$

Observation: Let $P \lhd A \neq \{0\}$.

Then P is prime iff A/P is an integral domain.

Proof: ($\Longrightarrow$) Suppose P is prime, and let $P + x$ be a zero-divisor in A/P , that is,

$$P \;=\; (P+x)(P+y) \;=\; P + xy$$

for some $y \notin P$. Then $xy \in P$, so

$$x \in P \qquad \text{(because } P \text{ is prime and } y \notin P \text{)},$$

yielding $P + x \;=\; P$, the zero of A/P .

($\Longleftarrow$) Proof is left as an exercise.

Corollary: If A is a nonzero ring then $\{0\}$ is a prime ideal of A iff A is an integral domain.

Example: The prime ideals of $\mathbb{Z}$ are $\{0\}$ and $p\mathbb{Z}$ where p is a prime number.

Exercise: Verify that if p is prime and $p\mathbb{Z} \subseteq I \lhd \mathbb{Z}$ then $I = p\mathbb{Z}$ or $I = \mathbb{Z}$.

An ideal M of A is called **maximal** if $M \neq A$ and there is no ideal I of A such that

$$M \subset I \subset A$$

(proper set containment).

Observation: Let $M \lhd A \neq \{0\}$.

Then M is maximal iff A/M is a field.

Proof: Observe that M is maximal

$$\iff$$

the only ideals of A/M are A/M and M/M (by the Proposition on page 38)

$$\iff$$

A/M is a field (by the Proposition on page 51).

$\boxed{\textbf{Corollary:} \quad \text{All maximal ideals are prime.}}$

However, prime ideals need not be maximal,

e.g. $\{0\}$ is a prime ideal of $\mathbb{Z}$ which is not maximal.

The property of being prime is preserved under taking preimages with respect to a homomorphism:

Observation: Let $f : A \to B$ be a ring homomorphism and Q be a prime ideal of B . Then

$$f^{-1}(Q) \;=\; \{\, x \in A \mid f(x) \in Q \,\}$$

is a prime ideal of A .

Proof: $$A \xrightarrow{f} B \xrightarrow{\phi} B/Q$$

where ϕ is the natural map. Then

$$\ker\,(\phi \circ f) \;=\; \{\, x \in A \mid f(x) \in Q \,\} \;=\; f^{-1}(Q)\;,$$

so

$$\begin{aligned} A/f^{-1}(Q) \;&=\; A/\ker\,(\phi \circ f) \\ &\cong\; \text{subring of}\;\; B/Q \end{aligned}$$

(by the Fundamental Homomorphism Theorem).

But B/Q is an integral domain (since Q is prime),

so also $A/f^{-1}(Q)$ is an integral domain

(since subrings of integral domains are clearly integral domains).

Hence $f^{-1}(Q)$ is a prime ideal of A, and the proof is complete.

Note: Preimages of maximal ideals need not be maximal.

e.g. Let $f : \mathbb{Z} \longrightarrow \mathbb{Q}$ be the identity injection. Then $\{0\} = f^{-1}(\{0\})$ is not maximal in $\mathbb{Z}$, yet $\{0\}$ is maximal in $\mathbb{Q}$.

We will discuss the existence of maximal ideals, after first digressing briefly on **Zorn's Lemma**.