2.10 Exactness Properties of the Tensor Product

We are able to tensor modules and module homomorphisms,

so the question arises whether we can use tensors to build new exact sequences from old ones.

First we prove a close relationship between tensor products and modules of homomorphisms:

Theorem: Let A be a ring and M , N , P be A-modules. Then

$$\text{Hom}\,(M \otimes N, P) \;\cong\; \text{Hom}\,(M, \text{Hom}\,(N, P))$$

as A-modules.

Proof: Define

$$\Phi : \text{Hom}\,(M \otimes N, P) \;\rightarrow\; \text{Hom}\,(M, \text{Hom}\,(N, P))$$

as follows:

$$\text{for} \;\; f \in \mathrm{Hom}\,(M \otimes N, P)\ ,$$

define

$$\Phi(f) \;\in\; \mathrm{Hom}\,(M, \mathrm{Hom}\,(N, P))$$

by,

$$\text{for } x \in M\ ,\ y \in N\ ,$$

$$\big[\Phi(f)(x)\big](y) \;=\; f(x \otimes y)\ .$$

Check $\Phi(f)(x) \in$ Hom (N, P) :

$$
\begin{aligned}
\big[\Phi(f)(x)\big](ay_1 + by_2) \quad &= \quad f\Big(x \otimes (ay_1 + by_2)\Big) \\
&= \quad f\Big(a(x \otimes y_1) + b(x \otimes y_2)\Big) \\
&= \quad a\, f(x \otimes y_1) + b\, f(x \otimes y_2) \\
&= \quad a\,\big[\Phi(f)(x)\big](y_1) \; + \; b\,\big[\Phi(f)(x)\big](y_2)\ .
\end{aligned}
$$

Check $\Phi(f) \in \mathsf{Hom}\,(M, \mathsf{Hom}\,(N, P))$:

$$\big[\Phi(f)(ax_1 + bx_2)\big](y) \quad = \quad f\Big((ax_1 + bx_2) \otimes y\Big)$$

$$= \quad f\Big(a(x_1 \otimes y) + b(x_2 \otimes y)\Big)$$

$$= \quad a\,f(x_1 \otimes y) + b\,f(x_2 \otimes y)$$

$$= \quad \Bigg(a\left[\Phi(f)(x_1)\right] \; + \; b\left[\Phi(f)(x_2)\right]\Bigg)(y)\ .$$

Check Φ is a module homomorphism:

$$\Big(\Phi(af_1+bf_2)(x)\Big)(y) \;=\; \big(af_1 + bf_2\big)(x \otimes y)$$

$$= \; a\, f_1(x \otimes y) + b\, f_2(x \otimes y)$$

$$= \; a\,[\Phi(f_1)(x)](y) \; + \; b\,[\Phi(f_2)(x)](y)$$

$$= \; \Big(\big[a\,\Phi(f_1) \; + \; b\,\Phi(f_2)\big](x)\Big)(y)\;.$$

Check Φ is one-one:

Suppose $f_1, f_2 \in$ Hom $(M \otimes N, P)$ and $\Phi(f_1) = \Phi(f_2)$.

If $x \in M$, $y \in N$ then

$$\begin{aligned} f_1(x \otimes y) &= \big[\Phi(f_1)(x)\big](y) \\ &= \big[\Phi(f_2)(x)\big](y) = f_2(x \otimes y) , \end{aligned}$$

which shows f_1 and f_2 agree on generators, so $f_1 = f_2$.

Check Φ is onto:

Suppose $g \in \text{Hom}\,(M, \text{Hom}\,(N, P))$.

We want to find a module homomorphism

$$f' : M \otimes N \to P$$

such that $\Phi(f') \;=\; g$.

Define $f : M \times N \;\to\; P$ by, for $x \in M$, $y \in N$,

$$f(x, y) \;\;=\;\; g(x)(y) \; .$$

Check f is linear in the first variable (because g is linear):

$$
\begin{aligned}
f(ax_1 + bx_2, y) &= g(ax_1 + bx_2)(y) \\
&= \big[ag(x_1) + bg(x_2)\big](y) \\
&= a\big(g(x_1)(y)\big) \ + \ b\big(g(x_2)(y)\big) \\
&= af(x_1, y) \ + \ bf(x_2, y) \ .
\end{aligned}
$$

Check f is linear in the second variable (because each $g(x)$ is linear):

$$\begin{aligned} f(x, ay_1 + by_2) &= g(x)(ay_1 + by_2) \\ &= a[g(x)(y_1)] \ + \ b[g(x)(y_2)] \\ &= af(x, y_1) \ + \ bf(x, y_2) \ . \end{aligned}$$

$$\boxed{\text{Thus } f \text{ is bilinear.}}$$

Hence there is a unique module homomorphism f' making the following diagram commute:

$$\begin{tikzcd} M \times N \arrow[rr] \arrow[dr, "f"'] & & M \otimes N \arrow[dl, "f'"] \\ & P & \end{tikzcd}$$

Then

$$\big[\Phi(f')(x)\big](y) \;=\; f'(x \otimes y) \;=\; f(x, y) \;=\; g(x)(y)\ ,$$

proving $\Phi(f') \;=\; g$, verifying that Φ is onto.

This completes the proof of the Theorem.

Theorem: Suppose that

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \qquad (*)$$

is exact and let N be any A-module. Then

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \longrightarrow 0 \qquad (**)$$

is exact where $1 = \mathrm{id}_N$.

Note:

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

exact does **not** imply

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N$$

exact.

Example: Take $A = \mathbb{Z}$. Let $f : \mathbb{Z} \to \mathbb{Z}$ where $f(z) = 2z$.

Then

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;f\;} \mathbb{Z}$$

is exact, but

$$0 \longrightarrow \mathbb{Z} \bigotimes \mathbb{Z}_2 \xrightarrow{\;f \bigotimes 1\;} \mathbb{Z} \bigotimes \mathbb{Z}_2$$

is not exact, because

$f \bigotimes 1$ is not injective.

To see this note that, for all $x \in \mathbb{Z}$, $y \in \mathbb{Z}_2$,

$$(f \otimes 1)(x \otimes y) \;=\; 2x \;\otimes\; y \;=\; x \;\otimes\; 2y$$

$$=\; x \otimes 0 \;=\; 0 \;,$$

so $f \otimes 1$ is the zero homomorphism, yet

$$\mathbb{Z} \otimes \mathbb{Z}_2 \;\cong\; \mathbb{Z}_2$$

is not the trivial module.

Proof of the Theorem: Let P be any A-module. Then Hom (N, P) is also an A-module, so by $(*)$,

$$0 \longrightarrow \text{Hom}\,(M'', \text{Hom}\,(N, P))$$

$$\xrightarrow{\overline{g}} \text{Hom}\,(M, \text{Hom}\,(N, P))$$

$$\xrightarrow{\overline{f}} \text{Hom}\,(M', \text{Hom}\,(N, P))$$

$$(***)$$

is exact (by the Theorem on page 356).

Let

$$\Phi'' \quad : \quad \text{Hom}\,(M'' \otimes N, P) \longrightarrow \text{Hom}\,(M'', \text{Hom}\,(N, P))$$

$$\Phi \quad : \quad \text{Hom}\,(M \otimes N, P) \longrightarrow \text{Hom}\,(M, \text{Hom}\,(N, P))$$

$$\Phi' \quad : \quad \text{Hom}\,(M' \otimes N, P) \longrightarrow \text{Hom}\,(M', \text{Hom}\,(N, P))$$

be the isomorphisms in the proof of the Theorem on page 473.

Claim: The following squares commute:

$$\begin{array}{ccc}
\text{Hom}\,(M \otimes N, P) & \xrightarrow{\overline{f \otimes 1}} & \text{Hom}\,(M' \otimes N, P) \\
\Phi \Big\downarrow & & \Big\downarrow \Phi' \\
\text{Hom}\,(M, \text{Hom}\,(N, P)) & \xrightarrow[\overline{f}]{} & \text{Hom}\,(M', \text{Hom}\,(N, P))
\end{array}$$

Claim continued:

$$\begin{array}{ccc}
\text{Hom}\,(M''\otimes N,P) & \xrightarrow{\overline{g\otimes 1}} & \text{Hom}\,(M\otimes N,P) \\
\Big\downarrow{\scriptstyle \Phi''} & & \Big\downarrow{\scriptstyle \Phi} \\
\text{Hom}\,(M'',\text{Hom}\,(N,P)) & \xrightarrow[\overline{g}]{} & \text{Hom}\,(M,\text{Hom}\,(N,P))
\end{array}$$

We verify commutativity of the first square, the other square being similar:

for $\alpha \in \mathrm{Hom}\,(M \otimes N, P)$, $x' \in M'$, $y \in N$,

$$\left[\Phi'\big(\,\overline{f \otimes 1}\,(\alpha)\,\big)(x')\right](y) \;=\; \left(\overline{f \otimes 1}\,(\alpha)\,\right)(x' \otimes y)$$

$$=\; \alpha\big(\,(f \otimes 1)\,(x' \otimes y)\,\big)$$

$$=\; \alpha\big(\,(f(x') \otimes y)\,\big)$$

$$= \left[\Phi(\alpha)\ (f(x'))\right](y)$$

$$= \left[\overline{f}\left(\Phi(\alpha)\,(x')\right)\right](y)\ ,$$

so that $\phi \circ (\overline{f \otimes 1}) \;=\; \overline{f} \circ \Phi\ .$

We now verify exactness of

$$0 \longrightarrow \mathsf{Hom}\,(M'' \otimes N, P) \xrightarrow{\overline{g \otimes 1}} \mathsf{Hom}\,(M \otimes N, P) \xrightarrow{\overline{f \otimes 1}} \mathsf{Hom}\,(M' \otimes N, P) \qquad (****)$$

(i) By commutativity of the second square in the above Claim,

$$\overline{g \otimes 1} \;=\; \Phi^{-1} \circ \overline{g} \circ \Phi'' \,.$$

But, from the exactness of $(***)$, $\overline{g}$ is injective, so

$$\overline{g \otimes 1} \text{ is injective,}$$

being the composite of injective functions.

(ii) Observe that, using commutativity of both squares in the above Claim, and the exactness of $(***)$:

$$\begin{aligned}
\ker(\,\overline{f \otimes 1}\,) \;&=\; \ker\left(\,\Phi'^{\,-1} \circ \overline{f} \circ \Phi\,\right) \\
&=\; \ker\left(\,\overline{f} \circ \Phi\,\right) \\
&=\; \Phi^{-1}\,(\,\ker \overline{f}\,)
\end{aligned}$$

$$= \quad \Phi^{-1}\left(\operatorname{im} \overline{g} \right) \quad = \quad \operatorname{im}\left(\Phi^{-1} \circ \overline{g} \right)$$

$$= \quad \operatorname{im}\left(\Phi^{-1} \circ \overline{g} \circ \Phi'' \right) \quad = \quad \operatorname{im}\left(\overline{g \otimes 1} \right) .$$

(i) and (ii) verify that $(* * **)$ is exact for all P , so by the Theorem on page 356,

$$(**) \text{ is exact,}$$

and our proof is complete.