2.9 Restriction and Extension of Scalars

Let $f : A \to B$ be a ring homomorphism and let N be a B-module.

> We want to exploit f to regard N as an A-module.

Define scalar multiplication by elements of A by, for $a \in A$, $x \in N$,

$$a\,x \;=\; f(a)\,x\;.$$

Because f is a ring homomorphism, it is routine to check that

N becomes an A-module, said to be obtained by **restriction of scalars**.

In particular, since B is a module over itself,

f defines A-module operations on B .

Proposition: Suppose N is finitely generated as a B-module, and B is finitely generated as an A-module (via f).
Then N is finitely generated as an A-module (via f).

Proof: Let

$$N = \langle y_1, \ldots, y_n \rangle_{B\text{-module}}$$

and

$$B = \langle x_1, \ldots, x_m \rangle_{A\text{-module}}$$

Let $z \in N$. Then

$$z = \sum_{i=1}^{n} b_i \, y_i \qquad \exists b_1, \ldots, b_n \in B \ .$$

But, for each i ,

$$b_i = \sum_{j=1}^{m} f(a_{ij}) \, x_j \qquad \exists a_{i1}, \ldots, a_{im} \in A \ .$$

Hence

$$
\begin{aligned}
z &= \sum_i \Big(\sum_j f(a_{ij})\, x_j \Big)\, y_i \\
&= \sum_{i,j} f(a_{ij})\, (x_j y_i) \;=\; \sum_{i,j} a_{ij}\, (x_j y_i)
\end{aligned}
$$

where scalar multiplication in the last summation is as an A-module. Thus

$$
N = \langle\, x_j y_i \;|\; 1 \le j \le m\,,\;\; 1 \le i \le n \,\rangle_{A\text{-module}}\,.
$$

Suppose now that $f : A \to B$ is a ring homomorphism and

$$M \text{ is an } A\text{-module.}$$

By restriction of scalars, B is also an A-module, so we may form

$$M_B = B \otimes_A M .$$

But M_B may also be regarded as a B-module.

Define, for $b, b' \in B$, $x \in M$,

$$b'\,(b \otimes m) \;\;=\;\; (b'\,b) \,\otimes\, x$$

being defined multiplication in B

and extending by linearity.

It is routine to check the module axioms. We call M_B the B-module obtained from M by

extension of scalars.

Check that this action is well-defined:

Fix $b' \in B$ and define

$$h : B \times M \;\longrightarrow\; B \otimes_A M$$

by

$$h(b, x) \;=\; (b'b) \otimes x \;.$$

Then, for $b_1, b_2 \in B$, $a_1, a_2 \in A$, $x \in M$,

$$h(a_1 \cdot b_1 + a_2 \cdot b_2 \;,\; x) \;=\; [b'(a_1 \cdot b_1 + a_2 \cdot b_2)] \otimes x$$

$$=\; [b'(f(a_1)b_1 + f(a_2)b_2)] \otimes x \;;$$

$$
\begin{aligned}
h(a_1 \cdot b_1 + a_2 \cdot b_2\ ,\ x) &= [f(a_1)b'b_1 + f(a_2)b'b_2] \otimes x \\
&= [a_1 \cdot b'b_1 + a_2 \cdot b'b_2)] \otimes x \\
&= a_1\big((b'b_1) \otimes x\big) + a_2\big((b'b_2) \otimes x\big) \\
&= a_1 h(b_1, x) + a_2 h(b_2, x)\ .
\end{aligned}
$$

Similarly in the second variable, which verifies that h is bilinear.

Hence we have a commutative diagram for some unique h' :

$$\begin{tikzcd}
B \times M \arrow[rr] \arrow[dr, "h"'] & & B \otimes M \arrow[dl, "h'"] \\
& B \otimes M &
\end{tikzcd}$$

If $b \otimes x$ is a generator of $B \otimes M$ then

$$h'(b \otimes x) \;=\; h(b, x) \;=\; b'b \otimes x \;,$$

so the action given earlier is sensibly defined.

Proposition: Let M be finitely generated as an A-module.

Then $M_B = B \otimes_A M$ is finitely generated when regarded as a B-module.

Proof: Let $M = \langle x_1, \ldots, x_n \rangle_{A\text{-module}}$.

Elements of M_B are sums of elements of the form

$$b \otimes m \ \text{ where } \ b \in B \ , \ m \in M \ ,$$

and

$$m \;=\; \sum_{i=1}^{n} c_i\, x_i \qquad \exists c_i \in A \;,$$

so

$$b \otimes m \;=\; b \otimes \big(\sum c_i\, x_i\big) \;=\; \sum c_i\,(b \otimes x_i)$$

$$=\; \sum (c_i \cdot b) \otimes x_i \;=\; \sum \big(f(c_i)\, b\big) \otimes x_i$$

$$=\; \sum \,[f(c_i)\, b]\; (1 \otimes x_i) \;,$$

so

$$b \otimes m \ \in \ \langle\, 1 \otimes x_1 \, , \, \ldots \, , \, 1 \otimes x_n \,\rangle_{\, B\text{-module}} \, .$$

Hence

$$M_B \ = \ \langle\, 1 \otimes x_1 \, , \, \ldots \, , \, 1 \otimes x_n \,\rangle_{\, B\text{-module}} \, ,$$

so M_B is finitely generated, and the Proposition proved.