2.8 Multilinear Mappings and Tensor Products

Let $M_1, \ldots, M_r$, P be A-modules.

Call a mapping $f : M_1 \times \ldots \times M_r \rightarrow P$ **multilinear** if it is linear in each variable, that is,

$$f(\, m_1 \,,\, \ldots \,,\, m_{i-1} \,,\, ax_1 + bx_2y \,,\, m_i \,,\, \ldots \,,\, m_r \,)$$
$$= a\, f(\, m_1 \,,\, \ldots \,,\, m_{i-1} \,,\, x_1 \,,\, m_i \,,\, \ldots \,,\, m_r \,)$$
$$+ \; b\, f(\, m_1 \,,\, \ldots \,,\, m_{i-1} \,,\, x_2 \,,\, m_i \,,\, \ldots \,,\, m_r \,)$$

Following through the proof of the main Theorem on tensor products, using C and D with r-tuples instead of ordered pairs,

we obtain a **tensor product**

$$T = M_1 \otimes \ldots \otimes M_r ,$$

generated by products

$$m_1 \otimes \ldots \otimes m_r = (m_1, \ldots, m_r) + D$$

with the following properties:

Theorem: There exists a pair (T, g) where T is an A-module and $g : M_1 \times \ldots \times M_r \to T$ is multilinear, such that

$$(\forall \text{ multilinear } f \ : \ M_1 \times \ldots M_r \ \to \ P \ )$$

$$(\exists! \text{ module homomorphism } f' \ : \ T \ \to \ P \ )$$

$$\begin{tikzcd} M_1 \times \ldots \times M_r \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, "f'"] \\ & P \end{tikzcd}$$

commutes.

Theorem continued: Moreover, if (T, g) and (T', g') are two pairs with this property then

$$(\exists! \text{ isomorphism } j : T \to T')$$

$$\begin{tikzcd}
M_1 \times \ldots \times M_r \arrow[rr, "g"] \arrow[dr, "g'"'] & & T \arrow[dl, "j"] \\
& T' &
\end{tikzcd}$$

commutes.

We list a couple of model "canonical isomorphisms":

Theorem: Let M, N, P be A-modules. Then there exist unique isomorphisms extending the following mappings on generators:

(1) $$(M \otimes N) \otimes P \to M \otimes (N \otimes P) \to M \otimes N \otimes P$$

$$(x \otimes y) \otimes z \ \mapsto \ x \otimes (y \otimes z) \mapsto \ x \otimes y \otimes z \ ;$$

(2) $$(M \oplus N) \otimes P \ \to \ (M \otimes P) \oplus (N \otimes P)$$

$$(x, y) \otimes z \ \mapsto \ (x \otimes z, y \otimes z) \ ;$$

Proof: We prove **(1)** and leave

$$\boxed{\text{the proof of } \textbf{(2)} \text{ as an } \textbf{exercise}.}$$

Fix $z \in P$ and define

$$F_z \ : \ M \times N \ \longrightarrow \ M \otimes N \otimes P$$

by

$$F_z(x, y) \ = \ x \otimes y \otimes z \ .$$

Then F_z is linear in the first variable, because of properties of “triple” tensors:

$$
\begin{aligned}
F_z(ax_1 + bx_2, y) &= (ax_1 + bx_2) \otimes y \otimes z \\
&= a(x_1 \otimes y \otimes z) \; + \; b(x_2 \otimes y \otimes z) \\
&= aF_z(x_1, y) \; + \; bF_z(x_2, y) \; .
\end{aligned}
$$

Similarly in the second variable, so

$$\boxed{F_z \text{ is bilinear.}}$$

Hence there is a unique homomorphism f_z such that the following diagram commutes:

$$\begin{tikzcd}
M \times N \arrow[rr] \arrow[dr, "F_z"'] & & M \otimes N \arrow[dl, "f_z"] \\
& M \otimes N \otimes P &
\end{tikzcd}$$

Now consider $a, b \in A$ and $z_1, z_2 \in P$. We verify that

$$\boxed{f_{az_1+bz_2} \ = \ af_{z_1} \ + \ bf_{z_2} \ .}$$

Then, for $x \in M$, $y \in N$,

$$f_{az_1+bz_2}(x \otimes y) \quad = \quad F_{az_1+bz_2}(x, y)$$

$$= \quad x \otimes y \otimes (az_1 + bz_2) \quad = \quad a\big(x \otimes y \otimes z_1\big) \; + \; b\big(x \otimes y \otimes z_2\big)$$

$$= \quad aF_{z_1}(x, y) \; + \; bF_{z_2}(x, y) \quad = \quad af_{z_1}(x \otimes y) \; + \; bf_{z_2}(x \otimes y)$$

$$= \quad \Big(af_{z_1} \; + \; bf_{z_2}\Big)(x \otimes y) \; .$$

Hence $f_{az_1+bz_2}$ and $af_{z_1} + bf_{z_2}$ agree on generators, so

$$\boxed{f_{az_1+bz_2} \;=\; af_{z_1} + bf_{z_2}\;.}$$

Now define

$$H \;:\; \big(M \otimes N\big) \times P \;\to\; M \otimes N \otimes P$$

by, for $t \in M \otimes N$ and $z \in P$,

$$H(t, z) \;=\; f_z(t)\;.$$

Then, since f_z is a module homomorphism,

$$\begin{aligned} H(at_1 + bt_2, z) &= f_z(at_1 + bt_2) \;=\; af_z(t_1) + bf_z(t_2) \\ &= aH(t_1, z) \;+\; bH(t_2, z)\ , \end{aligned}$$

and, from the previous fact we proved,

$$\begin{aligned} H(t, az_1 + bz_2) &= f_{az_1+bz_2}(t) \;=\; \big(af_{z_1} \;+\; bf_{z_2}\big)(t) \\ &= af_{z_1}(t) \;+\; bf_{z_2}(t) \\ &= aH(t, z_1) \;+\; bH(t, z_2)\ . \end{aligned}$$

Hence

$$\boxed{H \text{ is bilinear,}}$$

so there is a unique homomorphism h such that the following diagram commutes:

$$\begin{tikzcd}
(M \otimes N) \times P \arrow[rr] \arrow[dr, "H"'] & & (M \otimes N) \otimes P \arrow[dl, "h"] \\
& M \otimes N \otimes P &
\end{tikzcd}$$

yielding

$$\begin{aligned} h\big((x \otimes y) \otimes z\big) &= H(x \otimes y, z) = f_z(x \otimes y) \\ &= F_z(x, y) \\ &= x \otimes y \otimes z \ . \end{aligned}$$

Now define

$$F' \ : \ M \times N \times P \ \to \ (M \otimes N) \otimes P$$

by

$$F'(x, y, z) \ = \ (x \otimes y) \otimes z \ .$$

It is easy to verify that F' is multilinear, so there is a unique homomorphism f' such that the following diagram commutes:

$$\begin{tikzcd}
M \times N \times P \arrow[rr] \arrow[dr, "F'"'] & & M \otimes N \otimes P \arrow[dl, "f'"] \\
& (M \otimes N) \otimes P &
\end{tikzcd}$$

whence

$$f'(x \otimes y \otimes z) \;=\; F'(x, y, z) \;=\; (x \otimes y) \otimes z \;.$$

But h and f' undo each other on generators of $M \otimes N \otimes P$ and $(M \otimes N) \otimes P$ respectively, so

$$h \circ f' = \mathsf{id}_{M \otimes N \otimes P} \qquad \text{and} \qquad f' \circ h = \mathsf{id}_{(M \otimes N) \otimes P} \,.$$

Hence h and f are isomorphims, and clearly h is unique with the given property.

By a similar argument there is a unique isomorphism:

$$M \otimes (N \otimes P) \to M \otimes N \otimes P \qquad x \otimes (y \otimes z) \to x \otimes y \otimes z$$

and **(1)** of the Theorem is proved.