2.7 Tensor Products

Let A be a ring and M , N , P be A-modules.

Call a mapping $f : M \times N \rightarrow P$ A-**bilinear** (or simply **bilinear**) if

(i) for all $x \in M$ the mapping : $N \rightarrow P$ defined by

$$y \mapsto f(x, y) \qquad (y \in N)$$

is an A-module homomorphism; and

(ii) for all $y \in N$ the mapping : $M \to P$ defined by

$$x \mapsto f(x, y) \qquad (x \in M)$$

is an A-module homomorphism;

Thus $f : M \times N \to P$ is bilinear iff f is linear in each coordinate, that is,

$$(\forall x_1 \, , \; x_2 \, , \; x \;\in\; M \,) \, (\forall y_1 \, , \; y_2 \, , \; y \;\in\; N \,)$$

$$(\forall a \, , \; b \;\in\; A \,)$$

$$f(ax_1 + bx_2, y) \;\; = \;\; af(x_1, y) + bf(x_2, y)$$

and

$$f(x, ay_1 + by_2) \;\; = \;\; af(x, y_1) + bf(x, y_2) \; .$$

Example: Let $A = F$ be a field,

$$M = F[x] , \quad N = F[y] , \quad P = F[x,y] ,$$

polynomial rings regarded as vector spaces over F .

Easy to check:

$f : M \times N \to P$ where

$$f(\alpha, \beta) = \alpha\beta$$

is F-bilinear.

We will construct an A-module T, called the **tensor product** of M, N, denoted

$$T = M \otimes_A N = M \otimes N$$

which "contains" $M \times N$ (collapsing may occur) with the property that

A-bilinear mappings : $M \times N \to P$
"correspond" to
A-module homomorphisms : $T \to P$.

Theorem: Let M , N be A-modules. There exists a pair (T, g) where T is an A-module and $g : M \times N \to T$ is A-bilinear, such that

$$(\ \forall A\text{-bilinear } f : M \times N \to P\ )$$

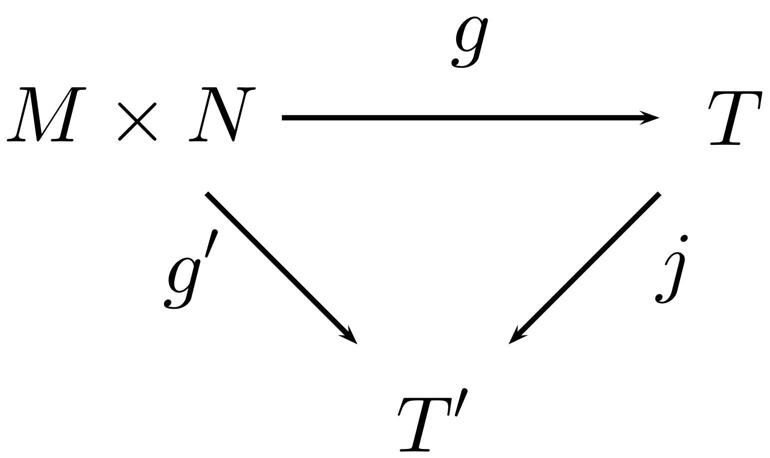
$$(\ \exists!\ A\text{-module homomorphism } f' : T \to P\ )$$

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad g \quad} & T \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f'} & \\ & P & \end{array}$$

commutes.

Theorem continued: Moreover, if (T, g) and (T', g') are two pairs with this property then

$$(\exists! \text{ isomorphism } j : T \rightarrow T')$$



commutes.

Proof: **Existence of (T, g)**

Put

$$C = A^{(M \times N)}$$

$$\equiv \{ \text{ formal linear combinations of elements of } M \times N \text{ with coefficients from } A \}$$

$$= \{ \sum_{i=1}^{n} a_i \cdot (x_i, y_i) \mid n \geq 0 , \quad a_i \in A , \; x_i \in M , \; y_i \in N \quad \forall i \} .$$

Let D be the submodule of C generated by elements of the following types, where

$$x\ ,\ x' \in M\ ,\quad y\ ,\ y' \in N\ ,\quad a \in A\ :$$

$$(x + x', y) - (x, y) - (x', y)\ ;$$

$$(x, y + y') - (x, y) - (x, y')\ ;$$

$$(ax, y) - a \cdot (x, y)\ ;$$

$$(x, ay) - a \cdot (x, y)\ .$$

Put

$$T \;=\; C/D\,.$$

Define $g : M \times N \to T$ by, for $x \in M$, $y \in N$,

$$g(x,y) \;=\; x \otimes y \;=\; (x,y) + D\,.$$

(Thus g is the restriction of the natural map from C to C/D.)

Need to check that g is A-bilinear.

If $x, x' \in M$, $y \in N$, $a, b \in A$ then

$$g(ax + bx', y) = (ax + bx') \otimes y$$

$$= (ax + bx', y) + D$$

$$= (ax + bx', y) - \left[(ax + bx', y) - (ax, y) - (bx', y)\right] + D$$

[since $(ax + bx', y) - (ax, y) - (bx', y) \in D$]

so that

$$
\begin{aligned}
g(ax + bx', y) &= (ax, y) + (bx', y) + D \\
&= (ax, y) - \big[(ax, y) - a \cdot (x, y)\big] \\
&\qquad + (bx', y) - \big[(bx', y) - b \cdot (x', y)\big] \\
&\qquad\qquad + D
\end{aligned}
$$

$\Big[$ since $(ax, y) - a \cdot (x, y) \in D$

and $(bx', y) - b \cdot (x', y) \in D \Big]$

yielding finally that

$$
\begin{aligned}
g(ax + bx', y) &= a \cdot (x, y) + b \cdot (x', y) + D \\
&= \Big(\, a \cdot (x, y) + D \,\Big) \;+\; \Big(\, b \cdot (x', y) + D \,\Big) \\
&= a \;\cdot\; \Big(\, (x, y) + D \,\Big) \;+\; b \;\cdot\; \Big(\, (x', y) + D \,\Big) \\
&= a \;\cdot\; (x \otimes y) \;+\; b \;\cdot\; (x' \otimes y) \\
&= a \;\cdot\; g(x, y) \;+\; b \;\cdot\; g(x', y) \;.
\end{aligned}
$$

Similarly one can show that g is linear in the second variable, which proves

g is A-bilinear.

Now let P be an A-module and $f : M \times N \to P$ be A-bilinear.

Define $\overline{f} : C \to P$ by

$$\overline{f}\Big(\sum_{i=1}^{n} a_i \cdot (x_i, y_i) \Big) = \sum_{i=1}^{n} a_i \, f(x_i, y_i) \; .$$

It is routine to check that $\overline{f}$ is an A-module homomorphism.

(We say that $\overline{f}$ **extends** f **by linearity**.)

We check that $\overline{f}$ vanishes on generators of D .

Let $x, x' \in M$, $y, y' \in N$, $a \in A$.

Then, using bilinearity of f ,

$$\overline{f}\Big(\ (x{+}x',\ y) - (x,y) - (x',y)\ \Big)$$

$$= \quad f(x+x',y) - f(x,y) - f(x',y)$$

$$= \quad f(x,y) + f(x',y) - f(x,y) - f(x',y) \quad = \quad 0\ ;$$

$$\overline{f}\Big(\ (ax,y) - a\cdot(x,y)\ \Big) \quad = \quad f(ax,y) - a\,f(x,y)$$

$$= \quad a\,f(x,y) - a\,f(x,y) \quad = \quad 0\ ;$$

and similarly (in the second variable)

$$\overline{f}\Big(\ (x, y + y') - (x, y) - (x, y')\ \Big) \quad = \quad 0\ ;$$

and

$$\overline{f}\Big(\ (x, ay) - a \cdot (x, y)\ \Big) \quad = \quad 0\ .$$

Thus

$\overline{f}$ vanishes on generators of D

so

$\overline{f}$ vanishes on D .

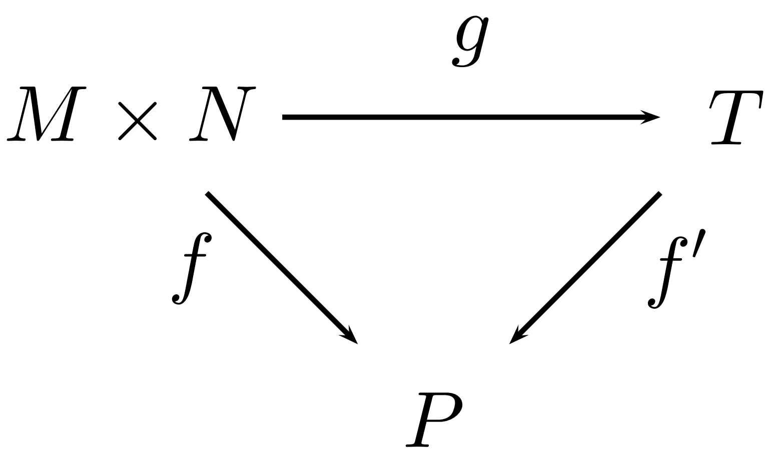
Thus $\overline{f}$ induces an A-module homomorphism

$$f' \; : \; T \; = \; C/D \; \rightarrow \; P$$

defined by

$$f'(\alpha + D) \;\; = \;\; \overline{f}(\alpha) \qquad\qquad (\alpha \in C) \; .$$

We have



This diagram commutes because, for $x \in M$, $y \in N$,

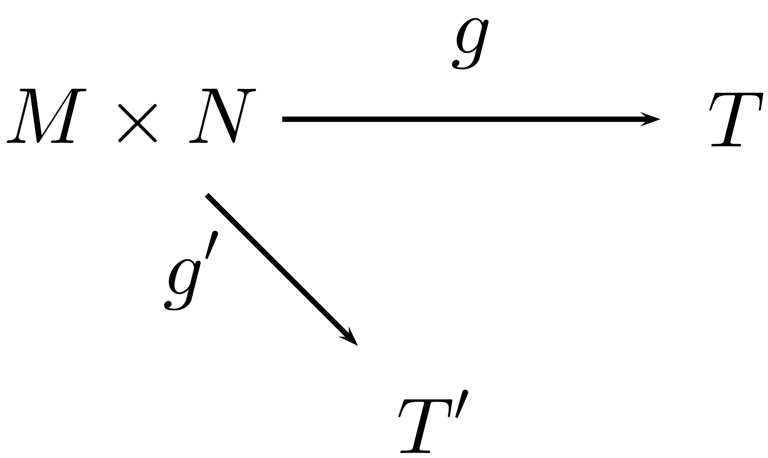
$$(f' \circ g)(x, y) \;=\; f'(g(x, y)) \;=\; f'(x \otimes y)$$

$$=\; f'\Big(\ (x, y) + D\ \Big) \;=\; \overline{f}(x, y) \;=\; f(x, y)\,.$$

Further, f' is unique with these properties because the images of the generators of T under f' are forced by the commutative diagram.

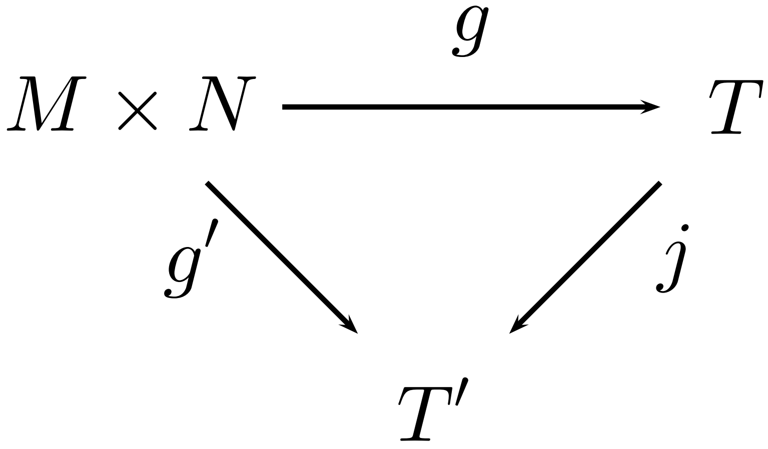
Uniqueness of (T, g)

Suppose (T', g') is another pair with the given property. We have

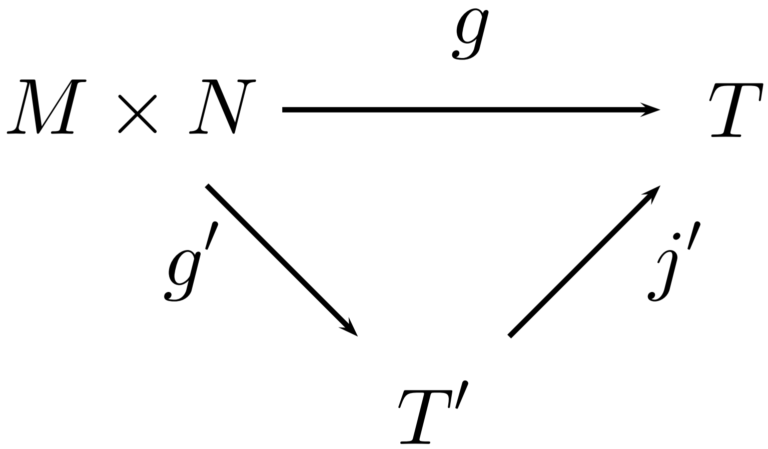


By the properties for (T, g) and (T', g') we have the following commuting diagrams:

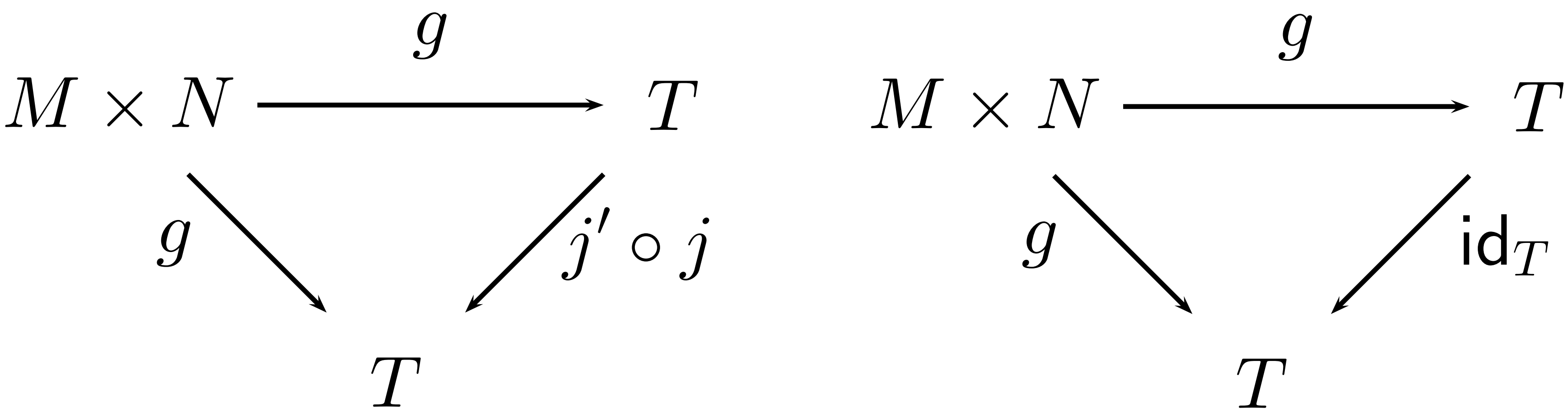
$(\exists!\ j)$



$(\exists!\ j')$



Hence $g \;=\; j' \circ g' \;=\; j' \circ (j \circ g) \;=\; (j' \circ j) \circ g$, so that the following diagrams commute:



By uniqueness, $j' \circ j = \mathsf{id}_T$.

Similarly, $j \circ j' = \mathsf{id}_{T'}$, and it follows that j is bijective, so j is an isomorphism.

The Theorem is proved.

Note the following rules for manipulating tensors:

$$(x + x') \otimes y \;=\; x \otimes y \;+\; x' \otimes y$$

$$x \otimes (y + y') \;=\; x \otimes y \;+\; x \otimes y'$$

$$ax \otimes y \;=\; x \otimes ay \;=\; a(x \otimes y)$$

In particular,

$$x \otimes 0 \ = \ x \otimes (0+0) \ = \ x \otimes 0 \ + \ x \otimes 0 \ ,$$

so

$$(\forall x \in M) \qquad x \otimes 0 \ = \ 0 \ ,$$

and similarly

$$(\forall y \in N) \qquad 0 \otimes y \ = \ 0 \ ,$$

the zero of $M \otimes N$.

Remarks:

(1) As an A-module

$$M \otimes N \;=\; \langle\, u \otimes v \;\mid\; u \in M\,,\;\; v \in N \,\rangle\ .$$

Suppose $M \;=\; \langle X \,\rangle$, $N \;=\; \langle Y \,\rangle$.

Consider $u \otimes v$ where $u \in M$, $v \in N$, say

$$u \;=\; \sum a_i x_i\ , \qquad v \;=\; \sum b_j y_j\ .$$

Then

$$
\begin{aligned}
u \otimes v &= \Big(\sum a_i x_i \Big) \otimes \Big(\sum b_j y_j \Big) \\
&= \sum_i a_i \Big(x_i \otimes \Big(\sum_j b_j y_j \Big) \Big) \\
&= \sum_{i,j} a_i b_j \left(x_i \otimes y_j \right) .
\end{aligned}
$$

This proves

$$M \otimes N \;=\; \langle\, x \otimes y \mid x \in X \,,\; y \in Y \,\rangle$$

Thus

if M and N are finitely generated, then so is $M \otimes N$.

(2) The notation $x \otimes y$ is ambiguous.

If M' , N' are submodules of M , N respectively and $x \in M'$, $y \in N'$ then $x \otimes y$ may have different properties

as an element of $M \otimes N$ or as an element of $M' \otimes N'$.

Example: Take $A = \mathbb{Z}$,

$$M = \mathbb{Z} , \quad M' = 2\mathbb{Z} , \quad N = N' = \mathbb{Z}_2 .$$

As an element of $M \otimes N$,

$$2 \otimes 1 \;=\; (1+1) \otimes 1 \;=\; (1 \otimes 1) \,+\, (1 \otimes 1)$$

$$=\; 1 \otimes (1+1) \;=\; 1 \otimes 0 \;=\; 0\,.$$

As an element of $M' \otimes N'$,

$$2 \otimes 1 \;\neq\; 0$$

because of the following:

Claim: $2\mathbb{Z} \otimes \mathbb{Z}_2$ is not the zero module.

Note that

$$2\mathbb{Z} \otimes \mathbb{Z}_2 \;=\; \langle\, 2 \otimes 1 \,\rangle$$

because $2\mathbb{Z} \;=\; \langle\, 2 \,\rangle$, $\mathbb{Z}_2 \;=\; \langle\, 1 \,\rangle$,

so that if $2 \otimes 1 \;=\; 0$ then $2\mathbb{Z} \otimes \mathbb{Z}_2$ would be the zero module, contradicting the Claim.

Proof of the Claim: To help distinguish integers from elements of $\mathbb{Z}_2$ write $\mathbb{Z}_2 \;=\; \{\, \overline{0} \,,\;\; \overline{1} \,\}$.

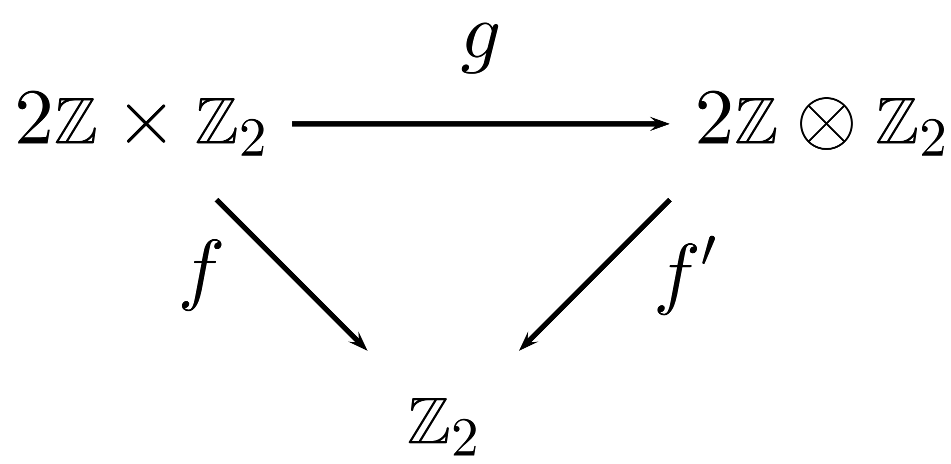
Define $f : 2\mathbb{Z} \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ by

$$f(x, \overline{y}) = \overline{(xy/2)}$$

for all $x \in 2\mathbb{Z}$, $y \in \{ 0, 1 \}$.

Then it is routine to check that f is $\mathbb{Z}$-bilinear (because overline preserves addition and multiplication).

Hence we have the following commutative diagram



for some g, f' where f' is a $\mathbb{Z}$-module homomorphism.

But f is onto and $\mathbb{Z}_2$ is not the zero module.

Hence $2\mathbb{Z} \otimes \mathbb{Z}_2$ cannot be the zero module, and the Claim is proved.