$(\Longleftarrow)$ Suppose conversely, for all A-modules N, that

$$0 \longrightarrow \text{Hom}\,(M'', N) \xrightarrow{\overline{v}} \text{Hom}\,(M, N) \xrightarrow{\overline{u}} \text{Hom}\,(M', N)$$

is exact.

(i) $\boxed{\text{We show } v \text{ is surjective:}}$

Put $N = M''/\mathrm{im}\, v$ and let $f : M'' \to N$ be the natural map.

Observe that $\overline{v}(f) = f \circ v = 0$, the zero map, by definition of f, so $f = 0$, since $\overline{v}$ is injective.

But this means $M'' = \operatorname{im} v$, that is, v is surjective.

(ii) $\boxed{\text{We show } \operatorname{im} u \subseteq \ker v :}$

Put $N = M''$ and let $f : M'' \to N$ be the identity mapping. Then

$$0 = (\overline{u} \circ \overline{v})(f) = f \circ v \circ u = v \circ u$$

(since $\operatorname{im} \overline{v} = \ker \overline{u}$), which proves $\operatorname{im} u \subseteq \ker v$.

(iii) $\boxed{\text{We show } \ker v \subseteq \operatorname{im} u :}$

Put $N = M/\operatorname{im} u$ and let $f : M \to N$ be the natural map.

Certainly $\overline{u}(f) = f \circ u = 0$ (by definition of f).

so $f \in \ker \overline{u} = \operatorname{im} \overline{v}$, yielding

$$f = \overline{v}(g) = g \circ v$$

for some $g \in \mathrm{Hom}\,(M'', N)$.

But $\ker(g \circ v) \supseteq \ker v$, so

$$\operatorname{im} u = \ker f = \ker(g \circ v) \supseteq \ker v .$$

Facts (i), (ii), (iii) establish that

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact, and **(1)** of the Theorem is proved.

Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & N' & \xrightarrow[u']{} & N & \xrightarrow[v']{} & N'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of A-modules and homomorphisms, with exact rows.

Theorem: With the above, there exists an exact sequence

$$\begin{tikzcd}
 & 0 \arrow[r] & \ker f' \arrow[r, "\overline{u}"] & \ker f \arrow[r, "\overline{v}"] & \ker f'' \arrow[dlll, "d"'] \\
\operatorname{coker} f' \arrow[r, "\overline{u}'"'] & \operatorname{coker} f \arrow[r, "\overline{v}'"'] & \operatorname{coker} f'' \arrow[r] & 0 &
\end{tikzcd}$$

for some homomorphism d .

In the above diagram

$\overline{u}$, $\overline{v}$ denote the restrictions of u , v respectively, and

$\overline{u}'$, $\overline{v}'$ are induced by composites of u' , v' respectively with natural maps.

Proof: Define

$$d : \ \ker f'' \ \to \ \operatorname{coker} f' \ = \ N/\mathrm{im}\ f'$$

as follows:

Let $x'' \in \ker f''$. Then, since v is onto,

$$x'' = v(x) \qquad \exists\, x \in M$$

so

$$v'(f(x)) = f''(v(x)) = f''(x'') = 0 ,$$

yielding

$$f(x) \in \ker v' = \operatorname{im} u' ,$$

whence

$$f(x) \; = \; u'(y') \qquad \exists \, y' \in N' \, .$$

Now put

$$\boxed{\; d(x'') \; = \; y' \; + \; f'(M') \; \in \; N'/\mathrm{im}\, f' \, . \;}$$

(i) $\boxed{\text{Check that } \; d \; \text{ is well-defined:}}$

This is a simple **exercise**, using exactness at M, commutativity of the first square and the fact that u' is injective.

(ii) $\boxed{\text{Check that } d \text{ is a module homomorphism:}}$

This follows easily, tracing through the definition of d and using the fact that each of v, f and u' are homomorphisms.

(iii) $\boxed{\text{Check exactness at } \ker f' \text{ and } \operatorname{coker} f'' :}$

This is immediate because $\overline{u}$ is injective

(restriction of an injective map)

and $\overline{v}'$ is surjective (induced by a surjective map).

(iv) $\boxed{\text{Check exactness at } \ker f :}$

If $x \in \ker \overline{v}$ then $x \in \ker v = \operatorname{im} v$, so

$$x = u(x') \qquad \exists\, x' \in M'$$

and

$$u'(f'(x')) \;=\; f(u(x')) \;=\; f(x) \;=\; 0\,,$$

so

$$f'(x') \;=\; 0 \qquad \text{(since } u' \text{ is injective)}$$

yielding $x' \in \ker f'$, whence

$$x \;=\; u(x') \;=\; \overline{u}(x') \;\in\; \mathrm{im}\,\overline{u}\,.$$

Thus $\ker \overline{v} \;\subseteq\; \mathrm{im}\,\overline{u}$.

Conversely, if $x \in \operatorname{im} \overline{u}$ then

$$x \;=\; \overline{u}(x') \;=\; u(x') \qquad\qquad \exists\, x' \in \ker f'$$

so

$$f(x) \;=\; f(u(x')) \;=\; u'(f'(x')) \;=\; u'(0) \;=\; 0\ ,$$

so

$$x \;\in\; \ker f \;\cap\; \operatorname{im} u \;\;=\;\; \ker f \;\cap\; \ker v$$

so $x \;\in\; \ker \overline{v}$. Thus $\operatorname{im} \overline{u} \;=\; \ker \overline{v}$, and equality holds.

(v) $\boxed{\text{Check exactness at } \operatorname{coker} f :}$

This is left as an **exercise**.

(vi) $\boxed{\text{Check exactness at } \ker f'' :}$

Suppose $x'' \in \ker d$, so

$$x'' = v(x) \quad \exists\, x \in M \quad , \quad f(x) = u'(y') \quad \exists\, y' \in N' ,$$

and

$$f'(M') \;=\; d(x'') \;=\; y' + f'(M') \; .$$

Thus $y' \in f'(M')$, so

$$y' \;=\; f'(x') \qquad \exists\, x' \in M'$$

yielding

$$f(x) \;=\; u'(y') \;=\; u'(f'(x')) \;=\; f(u(x')) \; ,$$

so $x - u(x') \in \ker f$. Observe now that

$$\overline{v}(x - u(x')) = v(x) - v(u(x'')) = v(x) = x'' ,$$

proving $\ker d \subseteq \mathrm{im}\, \overline{v}$.

Conversely, if $x'' \in \mathrm{im}\, \overline{v}$ then

$$x'' = \overline{v}(x) = v(x) \qquad \exists\, x \in \ker f$$

so $f(x) = 0 = u'(0)$, so (by definition)

$$d(x'') = 0 + f'(M') = f'(M') ,$$

proving $x'' \in \ker d$, whence $\operatorname{im} \overline{v} = \ker d$.

(vii) $\boxed{\text{Check exactness at } \operatorname{coker} f' :}$

This is left as an **exercise**.

The Theorem is proved.

Exercise: In the earlier diagram with commuting squares and exact rows, find an example in which each of

$$\ker f' \;,\; \ker f \;,\; \ker f'' \;,$$

$$\operatorname{coker} f' \;,\; \operatorname{coker} f \;,\; \operatorname{coker} f'' \;,$$

is not a zero module, and each of

$$\overline{u} \;,\; \overline{v} \;,\; d \;,\; \overline{u}' \;,\; \overline{v}'$$

is not a zero homomorphism.

Let $\mathcal{C}$ be a class of A-modules containing the zero module.

Call $\lambda : \mathcal{C} \to \mathbb{Z}$ **additive** if, for each short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where $M', M, M'' \in \mathcal{C}$ we have

$$\boxed{\lambda(M) = \lambda(M') + \lambda(M'') .}$$

Note that

$$0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0$$

is exact, so $\lambda(0) = \lambda(0) + \lambda(0)$, yielding

$$\boxed{\lambda(0) = 0 .}$$

Example: Let $A = F$ be a field and $\mathcal{C}$ the class of all finite dimensional vector spaces over F .

If

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact, then

$$M'' \;\cong\; M/\ker g \;=\; M/f(M')$$

so (by the Rank-Nullity Theorem)

$$\begin{aligned} \dim M'' \;&=\; \dim(M/f(M')) \;=\; \dim M \;-\; \dim f(M') \\ &=\; \dim M \;-\; \dim M' \,, \end{aligned}$$

which proves $\dim : \mathcal{C} \to \mathbb{Z}$ is additive.

Example: Let $\mathcal{C}$ denote the class of all finite abelian groups, regarded as $\mathbb{Z}$-modules.

Let $\mathcal{P}$ be some given set of primes (possibly all primes). If $A \in \mathcal{C}$ then

$$|A| = \Big(\prod_{p \in \mathcal{P}} p^{\alpha_p} \Big) \, q$$

where q is coprime to all elements of $\mathcal{P}$.

Define $\lambda(A) = \sum_{p \in \mathcal{P}} \alpha_p$.

Clearly λ is additive.

Example: Let $\mathcal{C}$ denote the class of all finitely generated abelian groups, regarded as $\mathbb{Z}$-modules.

If $A \in \mathcal{C}$ then $A \cong \mathbb{Z}^n \oplus B$ for some $n \geq 0$ and finite abelian group B .

Define $\lambda(A) = n =$ **torsion free rank** .

Exercise: Prove λ is additive.

Proposition: Let

$$0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} M_n \longrightarrow 0$$

be exact where all modules and kernels belong to $\mathcal{C}$, and let λ be additive. Then

$$\sum_{i=0}^{n} (-1)^i \, \lambda(M_i) \;=\; 0 \, .$$

Proof: We have that

$$
\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[dr] & & \cdots & & 0 \arrow[dr] & & 0 \arrow[dr] & & 0 \\
& 0 \arrow[ur] \arrow[dr] & & N_1 \arrow[ur] \arrow[dr] & & N_2 \arrow[ur] \arrow[dr] & & \cdots \arrow[ur] \arrow[dr] & & N_n \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
0 \arrow[rr] & & M_0 \arrow[rr] \arrow[ur] & & M_1 \arrow[rr] \arrow[ur] & & \cdots & & M_{n-1} \arrow[rr] \arrow[ur] & & M_n \arrow[rr] \arrow[ur] & & 0
\end{tikzcd}
$$

is a commutative diagram where

$$N_i \;=\; \mathrm{im}\, f_{i-1} \;=\; \ker f_i$$

and $0 \longrightarrow N_i \longrightarrow M_i \longrightarrow N_{i+1} \longrightarrow 0$

is exact for $i = 1 , \ldots , n-1$.

Then, noting $\lambda(0) = 0$,

$$\lambda(M_0) - \lambda(M_1) + \ldots + (-1)^n \lambda(M_n)$$

$$\begin{aligned} = & - \big(\lambda(0) - \lambda(M_0) + \lambda(N_1) \big) \\ & + \big(\lambda(N_1) - \lambda(M_1) + \lambda(N_2) \big) \\ & - \cdots \\ & + (-1)^{n-1} \big(\lambda(N_n) - \lambda(M_n) + \lambda(0) \big) \end{aligned}$$

$$= 0 .$$