2.6 Exact Sequences

Let A be a ring. A sequence (possibly infinite in one or both directions)

$$\cdots \longrightarrow M_{i-1} \overset{f_{i-1}}{\longrightarrow} M_i \overset{f_i}{\longrightarrow} M_{i+1} \longrightarrow \cdots$$

of A-modules and A-module homomorphisms is called **exact at** M_i if

$$\operatorname{im} f_{i-1} \;=\; \ker f_i \;,$$

and **exact** if it is exact at M_j for all j .

Notation: Write $0 = \{0\}$.

Then for any module M there are unique module homomorphisms

$$0 \longrightarrow M , \quad 0 \mapsto 0$$

and

$$\boxed{M \longrightarrow 0\,, \quad m \mapsto 0 \qquad \forall m \in M}$$

in each case called the **zero homomorphism**, also denoted by 0 .

Special cases:

(1)

$$0 \longrightarrow M' \xrightarrow{f} M$$

is exact iff $0 = \ker f$ iff f is injective.

(2)

$$M' \xrightarrow{g} M \longrightarrow 0$$

is exact iff $\operatorname{im} g = \ker 0 = M$ iff g is surjective.

(3)

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact iff f is injective, g is surjective and

$$\operatorname{im} f = \ker g \; ,$$

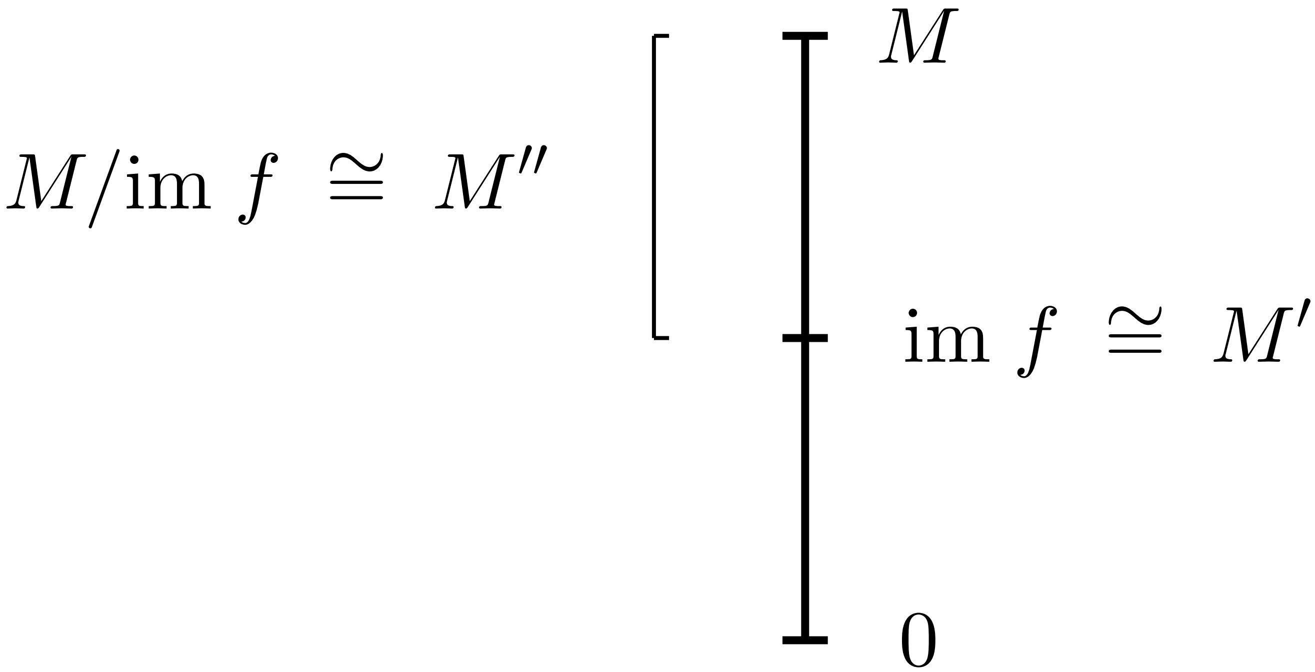
so that, by the Fundamental Homomorphism Theorem,

$$\operatorname{coker} f \;=\; M/\operatorname{im} f \;=\; M/\ker g \;\cong\; M''$$

so that we can think of

M as an **extension** of

$$M' \;\cong\; \operatorname{im} f \qquad \text{by} \qquad M'' \,.$$



Type (3) is called a **short exact sequence**.

Exact sequences stretching infinitely in both directions are called **long**.

Suppose

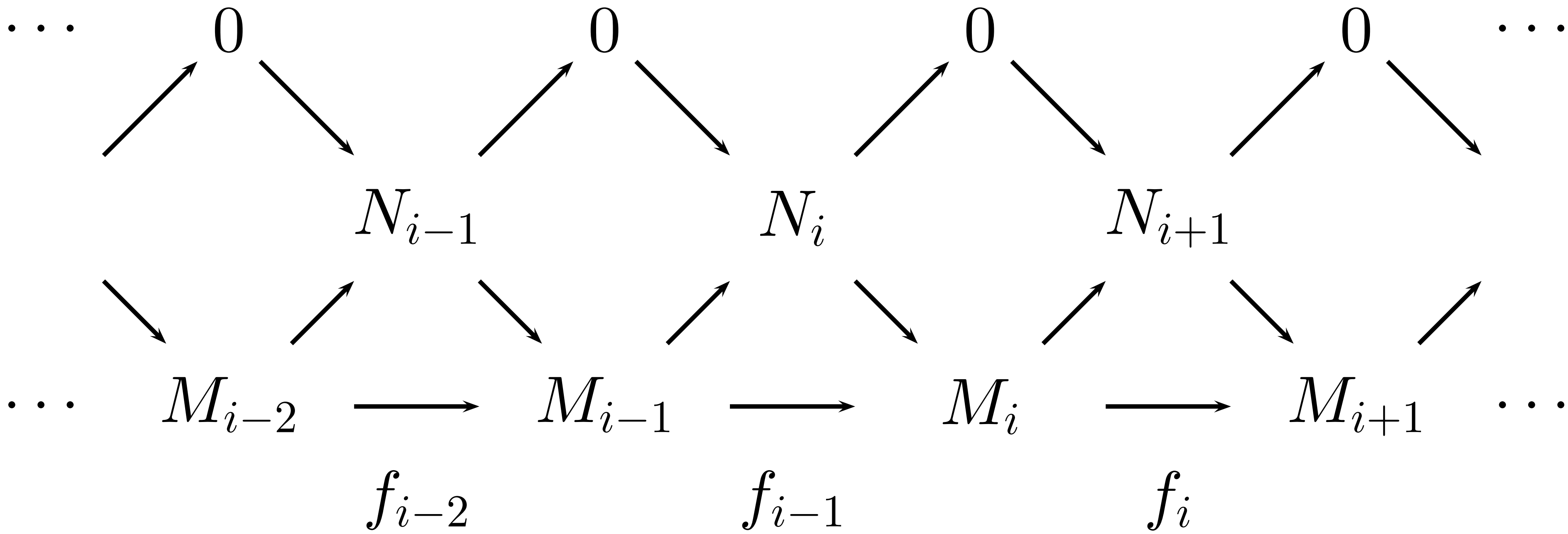
$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

is long exact.

Put

$$N_{j+1} = \operatorname{im} f_j = \ker f_{j+1} \qquad (\forall j) .$$

Then we have the following diagram of mappings:



where every square or triangle of mappings **commutes**, and

$$0 \longrightarrow N_i \longrightarrow M_i \longrightarrow N_{i+1} \longrightarrow 0$$

is short exact for each i .

Conversely, if we have a diagram of mappings with the above property then the lower chain of module homomorphisms is long exact.

Recall that if M , N are A-modules then

$$\text{Hom}_A(M,N) = \text{Hom}(M,N)$$

$$= \{ A\text{-module homomorphisms } : M \to N \},$$

becomes an A-module with respect to pointwise addition and scalar multiplication.

We prove a result linking exactness to the operators

$$\text{Hom}\,(M,-)\;:\;X\;\mapsto\;\text{Hom}\,(M,X)$$

and

$$\text{Hom}\,(-,N)\;:\;X\;\mapsto\;\text{Hom}\,(X,N)\;.$$

Theorem:

(1) The sequence

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact iff, for all A-modules N, the sequence

$$0 \longrightarrow \text{Hom}\,(M'', N) \xrightarrow{\overline{v}} \text{Hom}\,(M, N) \xrightarrow{\overline{u}} \text{Hom}\,(M', N)$$

is exact.

Theorem:

(2) The sequence

$$0 \longrightarrow N' \stackrel{u}{\longrightarrow} N \stackrel{v}{\longrightarrow} N''$$

is exact iff, for all A-modules M , the sequence

$$0 \longrightarrow \text{Hom}\,(M, N') \stackrel{\overline{u}}{\longrightarrow} \text{Hom}\,(M, N) \stackrel{\overline{v}}{\longrightarrow} \text{Hom}\,(M, N'')$$

is exact.

Note that, for all M , N ,

$$\text{Hom}\,(M,0) \;\cong\; \text{Hom}\,(0,N) \;\cong\; 0\,.$$

Proof: We will prove **(1)** and leave the proof of **(2)** as an **exercise**.

$(\Longrightarrow)$ Suppose that

$$M' \overset{u}{\longrightarrow} M \overset{v}{\longrightarrow} M'' \longrightarrow 0$$

is exact and let N be any A-module.

(i) **We show $\overline{v}$ is injective:**

Suppose $f, g \in$ Hom (M'', N) and

$$f \circ v = \overline{v}(f) = \overline{v}(g) = g \circ v \ .$$

If $x \in M''$ then $x = v(y)$ for some $y \in M$ (since v is surjective)

so

$$f(x) = f(v(y)) = (f \circ v)(y) = (g \circ v)(y) = g(x) \ .$$

Thus $f = g$, which proves $\overline{v}$ is injective.

(ii) We show $\ker \overline{u} \subseteq \operatorname{im} \overline{v}$:

Suppose $f \in \ker \overline{u}$, so $f \circ u = \overline{u}(f) = 0$.

Define $g : M'' \to N$ by, for $x \in M''$,

$$g(x) = f(y)$$

where $x = v(y)$ for some $y \in M$ (again since v is surjective).

Then g is sensibly defined, because if $v(y) = v(y_0)$ then

$$0 = v(y) - v(y_0) = v(y - y_0)$$

so $y - y_0 \in \ker v = \operatorname{im} u$,

so that

$$y - y_0 = u(z)$$

for some $z \in M'$, yielding

$$f(y) = f(y-y_0)+f(y_0) = f(u(z))+f(y_0) = f(y_0)$$

(since $f \circ u = 0$).

It is immediate that g is a module homomorphism

(since f is),

so $g \in$ Hom (M'', N) .

But, if $y \in M$ then

$$[\overline{v}(g)](y) \;=\; (g \circ v)(y) \;=\; g(v(y)) \;=\; f(y)$$

(by definition of g). Hence $\overline{v}(g) = f$, so $f \in \mathrm{im}\,\overline{v}$,

proving that $\ker \overline{u} \subseteq \mathrm{im}\,\overline{v}$.

(iii) We show $\operatorname{im} \overline{v} \subseteq \ker \overline{u}$:

Suppose $f = \overline{v}(g) \in \operatorname{im} \overline{v}$.

If $z \in M'$ then

$$[\overline{u}(f)](z) = [\overline{u}(\overline{v}(g)](z) = (g \circ v \circ u)(z)$$
$$= g(v(u(z))) = g(0) = 0$$

(since $\operatorname{im} u = \ker v$).

Thus $\overline{u}(f) = 0$, so $f \in \ker \overline{u}$, so $\operatorname{im} \overline{v} \subseteq \ker \overline{u}$.

Facts (i), (ii), (iii) together establish that

$$0 \longrightarrow \text{Hom}\ (M'', N) \xrightarrow{\overline{v}} \text{Hom}\ (M, N) \xrightarrow{\overline{u}} \text{Hom}\ (M', N)$$

is exact.