2.4 Finitely Generated and Free Modules

Let A be a nonzero ring and M an A-module.

> **Observation:** If $M \cong A$ as A-modules then M may be regarded as a ring which is isomorphic to A .

Proof: Let $\theta : M \rightarrow A$ be an A-module isomorphism.

Define multiplication $\cdot$ on M by, for $m_1, m_2 \in M$:

$$m_1 \cdot m_2 \;=\; \theta(m_1)\, m_2 \,.$$

↑ scalar multiplication

The ring axioms are easily verified. For example, if $x, y, z \in M$ then

$$
\begin{aligned}
(x \cdot y) \cdot z &= \theta\big(\theta(x)\ y\big)\ z \\
&= \big(\theta(x)\theta(y)\big)\ z
\end{aligned}
$$

since θ preserves scalar multiplication

$$
\begin{aligned}
&= \theta(x)\big(\theta(y)\ z\big) \\
&= x \cdot (y \cdot z)\ .
\end{aligned}
$$

The ring identity element of M is $\theta^{-1}(1)$.

But θ also becomes a **ring** isomorphism, since for all $m_1, m_2 \in M$,

$$\begin{aligned} \theta(m_1 \cdot m_2) &= \theta(\theta(m_1)\, m_2) \\ &= \theta(m_1)\theta(m_2) \; , \end{aligned}$$

since θ preserves scalar multiplication.

An A-module is called **free** if it is module isomorphic to

$$\bigoplus_{i \in I} M_i$$

for some family $\{ M_i \mid i \in I \}$ of A-modules, each M_i being module isomorphic to A.

Such a free module may also be denoted by

$$A^{(I)} .$$

Observation: The free module

$$\bigoplus_{i \in I} M_i$$

is finitely generated if and only if I is finite.

Proof: ($\Longrightarrow$) is left as an **exercise**.

($\Longleftarrow$) Suppose I is finite. For each $i \in I$, let 1_i denote the identity element of M_i, regarded as ring isomorphic to A, so

$$M_i \;=\; A\,1_i$$

and put

$$M_i' \;=\; \{\,(x_j)_{j\in I} \mid x_j \;=\; 0 \quad \forall j \neq i\,\}$$

and

$$\mathbf{e}_i \;=\; (y_j)_{j\in I}$$

where

$$y_j \;=\; \begin{cases} 0 & \text{if } j \neq i \\ 1_i & \text{if } j = i\ . \end{cases}$$

Then

$$\bigoplus_{i \in I} M_i = \sum_{i \in I} M_i' = \sum_{i \in I} A\mathbf{e}_i$$

$$= \langle \mathbf{e}_i \mid i \in I \rangle ,$$

generated by the finite set $\{\mathbf{e}_i \mid i \in I\}$.

Thus a finitely generated free A-module is isomorphic to

$$A^n \;=\; A \oplus \ldots \oplus A$$

(with n summands), for some n .

Convention: $A^0 = \{0\}$, the **zero** module.

The word “free” is justified by the following:

> **Proposition:** M is a finitely generated A-module iff M is isomorphic to a quotient of A^n for some $n \geq 0$.

Proof: $(\Longrightarrow)$ Suppose that M is generated by $x_1, \ldots, x_n$.

Define $\phi : A^n \to M$ by

$$(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n .$$

Clearly ϕ preserves addition and scalar multiplication, and ϕ is onto because

$$M = \langle x_1 , \ldots , x_n \rangle .$$

Therefore

$$M \cong A^n / \ker \phi ,$$

which is a quotient of A^n .

($\Longleftarrow$) Suppose $\phi : A^n \to M$ is an onto module homomorphism.

But $A^n = \langle x_1, \ldots, x_n \rangle$, for some $x_1, \ldots, x_n$ by the earlier Observation, so if $m \in M$ then, for some $\lambda_1, \ldots, \lambda_n \in A$,

$$\begin{aligned} m &= \phi(\lambda_1 x_1 + \ldots + \lambda_n x_n) \\ &= \lambda_1 \phi(x_1) + \ldots + \lambda_n \phi(x_n) , \end{aligned}$$

proving

$$M = \langle \phi(x_1) , \ldots , \phi(x_n) \rangle ,$$

so M is finitely generated.