## 2.4 Finitely Generated and Free Modules

Let A be a nonzero ring and M an A-module.

**Observation:** If  $M\cong A$  as A-modules then M may be regarded as a ring which is isomorphic to A .

**Proof:** Let  $\theta : M \to A$  be an *A*-module isomorphism.

Define multiplication  $\cdot$  on M by, for  $m_1, m_2 \in M$  :

The ring axioms are easily verified. For example, if  $x,y,z\in M$  then

$$(x \cdot y) \cdot z = \theta(\theta(x) y) z$$
$$= (\theta(x)\theta(y)) z$$

since  $\theta$  preserves scalar multiplication =  $\theta(x)(\theta(y) z)$ =  $x \cdot (y \cdot z)$ .

The ring identity element of M is  $\theta^{-1}(1)$  .

But  $\theta$  also becomes a  $\mathbf{ring}$  isomorphism, since for all  $m_1,m_2\in M$  ,

$$heta(m_1 \cdot m_2) \;\; = \;\; heta( heta(m_1) \; m_2)$$

$$= \theta(m_1)\theta(m_2) ,$$

since  $\theta$  preserves scalar multiplication.

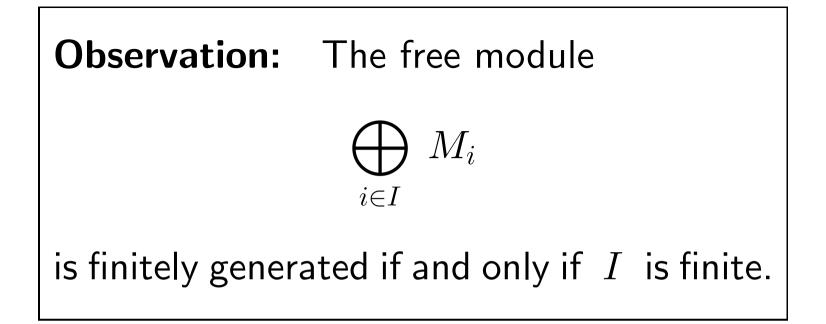
An *A*-module is called **free** if it is module isomorphic to

 $\rightarrow M_i$  $i \in I$ 

for some family  $\{ M_i \mid i \in I \}$  of A-modules, each  $M_i$  being module isomorphic to A.

Such a free module may also be denoted by

 $A^{(I)}$ .



## **Proof:** $(\Longrightarrow)$ is left as an **exercise**.

 $(\Longleftarrow)$  Suppose I is finite. For each  $i\in I$  , let  $1_i$  denote the identity element of  $M_i$  , regarded as ring isomorphic to A , so

$$M_i = A 1_i$$

and put

$$M'_i = \{ (x_j)_{j \in I} \mid x_j = 0 \quad \forall j \neq i \}$$

$$\mathbf{e}_i = (y_j)_{j \in I}$$

where

 $\quad \text{and} \quad$ 

$$y_j = \begin{cases} 0 & \text{if } j \neq i \\ 1_i & \text{if } j = i \end{cases}$$

## Then

$$\bigoplus_{i \in I} M_i = \sum_{i \in I} M'_i = \sum_{i \in I} A \mathbf{e}_i$$
$$= \langle \mathbf{e}_i \mid i \in I \rangle,$$

generated by the finite set  $\{\mathbf{e}_i \mid i \in I\}$ .

Thus a finitely generated free A-module is isomorphic to

$$A^n = A \oplus \ldots \oplus A$$

(with n summands), for some n.

Convention:  $A^0 = \{0\}$  , the zero module.

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The word "free" is justified by the following:

**Proof:**  $(\Longrightarrow)$  Suppose that M is generated by  $x_1, \ldots, x_n$ .

Define  $\phi: A^n \to M$  by

$$(a_1,\ldots,a_n) \mapsto a_1x_1 + \ldots + a_nx_n$$

Clearly  $\phi$  preserves addition and scalar multiplication, and  $\phi$  is onto because

$$M = \langle x_1, \ldots, x_n \rangle.$$

Therefore

$$M \cong A^n / \ker \phi ,$$

which is a quotient of  $A^n$ .

 $(\Leftarrow)$  Suppose  $\phi: A^n \to M$  is an onto module homomorphism.

But  $A^n = \langle x_1, \ldots, x_n \rangle$ , for some  $x_1, \ldots, x_n$ by the earlier Observation, so if  $m \in M$  then, for some  $\lambda_1, \ldots, \lambda_n \in A$ ,

$$m = \phi(\lambda_1 x_1 + \ldots + \lambda_n x_n)$$
  
=  $\lambda_1 \phi(x_1) + \ldots + \lambda_n \phi(x_n)$ ,

proving

$$M = \langle \phi(x_1), \ldots, \phi(x_n) \rangle,$$

so M is finitely generated.