Consider an ideal J of A and an A-module M .

Define the **product**

$$JM \;=\; \Big\{ \sum_{i=1}^{n} a_i\, x_i \;\Big|\; n \in \mathbb{Z}^+ ,\ a_i \in J ,\ x_i \in M \quad (\forall i) \Big\} .$$

Easy to check:

$$\boxed{JM \text{ is an } A\text{-submodule of } M .}$$

Consider now two submodules N and P of M .

Define

$$(N : P) \ = \ \{ \, a \in A \ | \ aP \ \subseteq N \, \}$$

(analogous to an ideal quotient).

Easy to check:

$$\boxed{(N : P) \ \lhd \ A \, .}$$

Write

$$\begin{aligned}\text{Ann}\,(M) &= (\{0\} : M) \\ &= \{\, a \in A \mid aM = \{0\}\,\} \\ &= \{\, a \in A \mid ax = 0 \quad (\forall x \in M)\,\}\,,\end{aligned}$$

called the **annihilator** of M .

Thus

$$\boxed{\text{Ann}\,(M) \ \lhd \ A\,.}$$

Consider an ideal J of A such that

$$J \subseteq \mathrm{Ann}\,(M)\,.$$

Then M becomes an A/J-module by defining

$$(J + a)\,x \;=\; a\,x \qquad (\forall a \in A, x \in M)\,.$$

This is well-defined because if

$$J + a \;=\; J + a_0$$

then $a_0 - a \in \mathsf{Ann}\,(M)$,

so that, for $x \in M$,

$$a\,x \;=\; a\,x + 0 \;=\; a\,x + (a_0 - a)\,x \;=\; a_0\,x\ .$$

It is routine to verify the module axioms.

Call an A-module M **faithful** if

$$\text{Ann}\,(M) \;=\; \{0\}\,.$$

Always then,

$$\boxed{\text{M is faithful as an A/J-module when we put $J = \text{Ann}\,(M)$,}}$$

because

$$\{\ J + a \ \mid\ a \in A \quad \text{and} \quad (J + a)\, M \ = \ \{0\}\ \}$$

$$= \ \{\ J + a \ \mid\ a \in A \quad \text{and} \quad a\, M \ = \ \{0\}\ \}$$

$$= \ \{\ J + a \ \mid\ a \ \in\ \mathsf{Ann}\,(M)\ \}$$

$$= \ \mathsf{Ann}\,(M)/J \ = \ J/J\ .$$

Reason for terminology:

if M is a faithful A-module then

$$\begin{aligned} \theta \ : \ A \ &\longrightarrow \ \text{End}\,(M) \\ a \ &\mapsto \ \theta(a) \ : \ x \ \mapsto \ ax \end{aligned}$$

is a **faithful** (that is, one-one) ring homomorphism.

because $\ker\theta \ = \ \text{Ann}\,(M) \ = \{0\}$.

Easy Exercises: Let N , P be submodules of an A-module M . Verify that

(i) $\text{Ann}\,(N+P) \;=\; \text{Ann}\,(N) \cap \text{Ann}\,(P)$;

(ii) $(N:P) \;=\; \text{Ann}\,\big(\,(N+P)/N\,\big)$,

where $(N+P)/N$ is regarded as an A-submodule of M/N .

If $x \in M$ then write

$$Ax = \{ \, ax \mid a \in A \, \} = \langle x \rangle \, .$$

If $X \subseteq M$ then call X a **set of generators** for M if

$$M = \langle X \rangle = \Big\langle \bigcup_{x \in X} Ax \Big\rangle = \sum_{x \in X} Ax \, ,$$

so that every element of M can be expressed as a **linear combination**

$$a_1 \, x_1 + \ldots + a_n \, x_n$$

for some $n \geq 1$ and some $a_1, \ldots, a_n \in A$, $x_1, \ldots, x_n \in X$.

If $M = \langle X \rangle$ for some **finite** set X then we say that M is **finitely generated**.

Note that expressions of module elements as linear combinations of generators need not be unique, even when the generating set is minimal.

Example: Regarded as a $\mathbb{Z}$-module (over itself)

$$\mathbb{Z} \;=\; \langle\, 2\;,\; 3\,\rangle$$

and $\{2,3\}$ is a minimal generating set, yet

$$1 \;=\; (-1)2+(1)3 \;=\; (2)2+(-1)3$$

so uniqueness of expressions fails.

Direct sum and product

Consider a family $\{ M_i \mid i \in I \}$ of A-modules.

Define the **direct product**

$$\prod_{i \in I} M_i \;=\; \{ (x_i)_{i \in I} \mid x_i \in M_i \quad \forall i \}$$

which is easily seen to be an A-module with respect to coordinatewise operations.

Define the **direct sum**

$$\bigoplus_{i \in I} M_i \;=\; \{\, \alpha \,\in\, \prod M_i \;|\; \alpha \text{ has finite support } \} \,,$$

which is a submodule of $\prod M_i$, with equality (in the case that each M_i is nonzero) iff I is finite.

We give a criterion for deciding when a given A-module is isomorphic to the direct sum of some of its submodules.

Suppose now that $\{ M_i \mid i \in I \}$ is a family of submodules of an A-module M .

Call M the **internal direct sum** of the family if

(i) $M = \sum_{i \in I} M_i$; and

(ii) For all $j \in I$,

$$M_j \cap \Big(\sum_{i \neq j} M_i \Big) = \{0\} .$$

Exercise: Verify that TFAE:

(i) M is the internal direct sum of the family $\{M_i \mid i \in I\}$.

(ii) Each $m \in M$ can be expressed **uniquely** as

$$m = \sum_{i \in I} m_i$$

where $m_i \in M_i$ for each i only finitely many of which are nonzero.

It is common then to write

$$\text{“}\ M \;=\; \bigoplus_{i \in I} M_i\ \text{”}$$

because of the following:

Corollary: An internal direct sum is isomorphic to the external direct sum.

Proof: If M is the internal direct sum of $\{\ M_i \mid i \in I\ \}$ then

$$m \;\mapsto\; (m_i)_{i \in I} \qquad (m \in M)$$

where $m = \sum_{i \in I} m_i$ (for $m_i \in M_i$ for each i)

is a mapping: $M \to \bigoplus_{i \in I} M_i$,

which is well-defined by part (ii) of the Exercise,

and clearly one-one, onto and homomorphic.

Connection between module and ring direct sums:

Let $A_1, \ldots, A_n$ be rings and form the **ring** direct sum

$$A = A_1 \oplus \ldots \oplus A_n .$$

For $i = 1 , \ldots , n$ put

$$B_i = \{ (a_1, \ldots, a_n) \in A \mid a_j = 0 \quad \text{if} \quad j \neq i \} .$$

For each i it is easy to see that

$$\boxed{B_i \ \lhd \ A}$$

so that B_i may be regarded as an A-module, and

$$\boxed{B_i \ \cong \ A_i}$$

ring isomorphic

Also easy to see:

$$A \;=\; \sum_{i=1}^{n} B_i \;,$$

and, for $j \in \{1, \ldots, n\}$,

$$B_j \;\cap\; \sum_{i \neq j} B_i \;=\; \{\, (0, \ldots, 0) \,\} \;.$$

Thus

$$\boxed{A \;=\; \bigoplus_{i=1}^{n} B_i}$$

internal direct sum of modules

This process can be reversed.

Suppose now that A is the internal direct sum of ideals B_1 , ... , B_n regarded as A-modules. Then

$$1 \;=\; e_1 \,+\, \ldots \,+\, e_n$$

for unique $e_i \in B_i$.

If $b \in B_i$ then

$$b \;=\; 1\,b \;=\; (e_1 \,+\, \ldots \,+\, e_n)\,b$$

$$=\; e_1\,b \,+\, \ldots \,+\, e_i\,b \,+\, \ldots \,+ e_n\,b\;,$$

so, by uniqueness of linear combinations, $e_i\,b \;=\; b$.

Thus

$$\boxed{\text{For each } i\text{, } B_i \text{ is a ring with identity } e_i\text{.}}$$

(Note: B_i is not a subring of A unless it is the only nontrivial ideal in the list.)

Then

$$A \cong \bigoplus_{i=1}^{n} B_i$$

both as a direct sum of A-modules **and** as a direct sum of rings.

The very last assertion follows because

$\theta \;:\; a \;\mapsto\; (b_1, \ldots, b_n)$ where $a \;=\; b_1 + \ldots + b_n$

preserves **ring** multiplication:

if $a \;=\; b_1 + \ldots + b_n$, $a' \;=\; b'_1 + \ldots + b'_n$ then

$$a\,a' \;=\; b_1\,b'_1 \;+\; \ldots \;+\; b_n\,b'_n$$

since $b_i b'_j \;\in\; B_i \cap B_j \;=\; \{0\}$ if $i \neq j$, so that

$$
\begin{aligned}
\theta(a\,a') &= (\, b_1 b_1' \,,\ \ldots\ ,\ b_n b_n' \,) \\
&= (b_1, \ldots, b_n)(b_1', \ldots, b_n') \\
&= \theta(a)\theta(a')\ .
\end{aligned}
$$