2.3 Operations on Submodules

Let M be an A-module. If $X \subseteq M$, put

$$\langle X \rangle \;=\; \text{submodule of } M \text{ generated by } X$$

$$= \;\bigcap \{ \text{ submodules of } M \text{ containing } X \ \} \, ,$$

with usual conventions such as

$$\langle x_1, \ldots, x_n \rangle \;=\; \langle \{ x_1, \ldots, x_n \} \rangle \; .$$

Define the **sum** of a family $\{ M_i \mid i \in I \}$ of submodules of M as for ideals of a ring:

$$\sum_{i \in I} M_i \;=\; \Big\{ \sum_{i \in I} x_i \;\Big|\; x_i \in M_i \quad (\forall i \in I), \text{ and only finitely many } x_i \text{ are nonzero} \Big\}$$

Easy to check:

$$\sum_{i \in I} M_i \;=\; \Big\langle \bigcup_{i \in I} M_i \Big\rangle \,.$$

As for ideals of a ring,

the set of submodules of M forms a complete lattice with respect to $\subseteq$ where

$$\text{g.l.b.} \quad = \quad \text{intersection} \ ,$$

$$\text{l.u.b.} \quad = \quad \text{sum} \ .$$

Isomorphism Theorems:

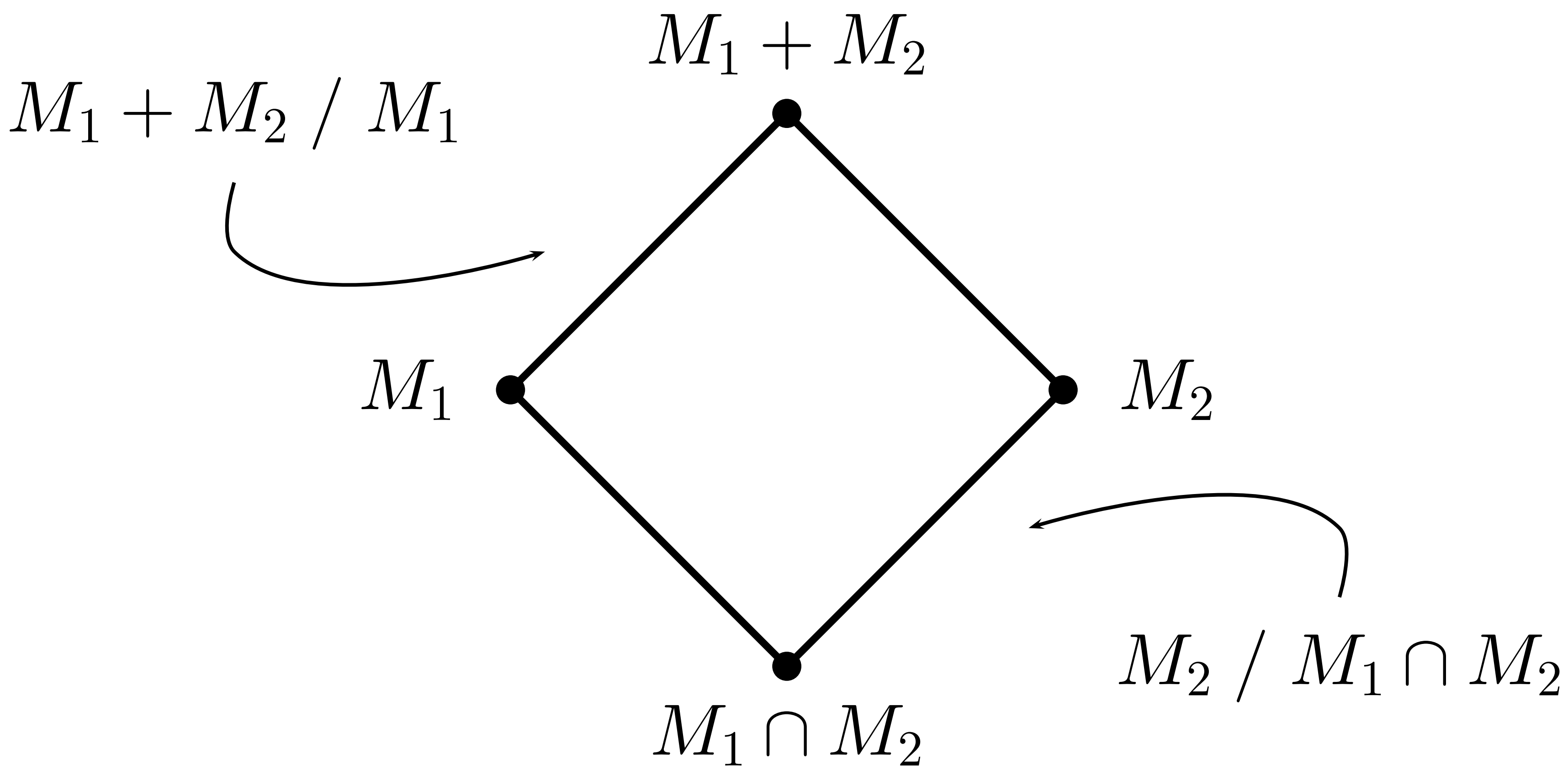
(i) If $L \supseteq M \supseteq N$ are a chain of A-submodules, then

$$(L/N)/(M/N) \cong L/M .$$

(ii) If M_1 , M_2 are submodules of an A-module M then

$$(M_1 + M_2)/M_1 \cong M_2 / M_1 \cap M_2 .$$

Visualize (ii) thus:



"opposite sides" represent isomorphic quotient modules

Proof of (i): Let $L \supseteq M \supseteq N$ be a chain of A-submodules. Consider the natural surjective map

$$\phi : L \rightarrow L/M .$$

Because $N \subseteq M = \ker \phi$ we get an induced surjective homomorphism

$$\overline{\phi} : L/N \rightarrow L/M$$

whose kernel is M/N .

By the Fundamental Homomorphism Theorem for modules,

$$(L/N)/(M/N) \;=\; (L/N)/\ \ker\overline{\phi} \;\cong\; L/M\ .$$

Proof of (ii): Let M_1 , M_2 be submodules of an A-module M . Define

$$\theta\ :\ M_2 \;\longrightarrow\; (M_1+M_2)/M_1 \quad \text{by} \quad x \;\mapsto\; x+M_1\ .$$

which is easily seen to be a surjective module homomorphism with kernel $M_1 \cap M_2$.

Hence, again by the Fundamental Homomorphism Theorem,

$$M_1 + M_2 \; / \; M_1 \;\; \cong \;\; M_2 \; / \; \ker \theta$$

$$= M_2 \; / \; M_1 \cap M_2 \; ,$$

and (ii) is proved.