2.2 Submodules and Quotient Modules

Call a subgroup M' of an A-module M a **submodule** if M' is closed under scalar multiplication, that is,

$$(\forall a \in A)(\forall x \in M') \qquad ax \in M' \, .$$

In this case we can form the quotient group

$$M/M' \ = \ \{ \, x + M' \ | \ x \in M \, \} \, ,$$

which becomes an A-module by defining

$$a(x + M') \;=\; ax \,+\, M' \qquad (\forall a \in A\,,\; x \in M)\,.$$

$\boxed{\text{It is routine to check that this is well-defined and that the module axioms are satisfied.}}$

Call M/M' the **quotient of** M **by** M' .

Easy to see:

The **natural map**:

$$M \to M/M' , \quad x \mapsto x + M'$$

is a surjective module homomorphism with kernel M',

which induces a one-one correspondence between submodules of M/M' and submodules of M which contain M'.

Let $f : M \to N$ be a module homomorphism.

Terminology and notation: write

$$\begin{aligned} \ker f &= \{ x \in M \mid f(x) = 0 \}, \\ \text{im } f &= f(M) = \{ f(x) \mid x \in M \}, \\ \text{coker } f &= N/\text{im } f \end{aligned}$$

for the **kernel**, **image** and **cokernel** of f respectively.

Easy to check:

$$\ker f \text{ is a submodule of } M \,,$$

and

$$\text{im } f \text{ is a submodule of } N$$

(so coker f makes sense).

Consider a submodule M' of M such that

$$M' \subseteq \ker f \,.$$

Define

$$\overline{f} \; : \; M/M' \; \longrightarrow \; N$$

by

$$x + M' \; \mapsto \; f(x) \qquad\qquad (\forall x \in M) \; .$$

This is well-defined because if

$$x + M' \;\; = \;\; x_0 + M'$$

then $x - x_0 \; \in \; M' \; \subseteq \ker f$, so that

$$f(x) \;=\; f(x - x_0 + x_0) \;=\; f(x - x_0) + f(x_0)$$

$$=\; 0 + f(x_0) \;=\; f(x_0)\ .$$

It is routine to verify that $\overline{f}$ is a module homomorphism, and

$$\ker \overline{f} \;=\; \ker f / M'\ .$$

Call $\overline{f}$ the homomorphism **induced** by f .

If $M' = \ker f$ then $\overline{f}$ becomes one-one, which proves another version of the

Fundamental Homomorphism Theorem:

$$M/\ker f \;\cong\; \text{im } f \; .$$