Module Homomorphisms

A mapping $f : M \to N$ between A-modules M , N is called an

A-**module homomorphism**

or

A-**linear**

if it respects addition and scalar multiplication, that is, for $x, y \in M$ and $a \in A$

$$f(x+y) = f(x) + f(y) \quad , \quad f(ax) = af(x)$$

(so f is an abelian group homomorphism which respects the action of the ring).

If A is a field then an A-module homomorphism is just a linear transformation.

Put

$$\text{Hom}_A(M, N) = \{ A\text{-module homomorphisms } : M \to N \}$$

also written Hom (M, N) if A is clear from context.

Define **pointwise** addition and scalar multiplication on Hom (M, N) :

$$\forall f, g \in \text{Hom } (M, N) \qquad \forall a \in A$$

$$(f + g)(x) \ = \ f(x) + g(x) \ \ , \ \ (af)(x) \ = \ af(x) \ .$$

Routine to check:

under these operations Hom $_A(M, N)$ becomes an A-module.

Induced homomorphisms:

Suppose M' , M , N , N' are A-modules and

$$u : M' \to M \quad , \quad v : N \to N'$$

are A-module homomorphisms.

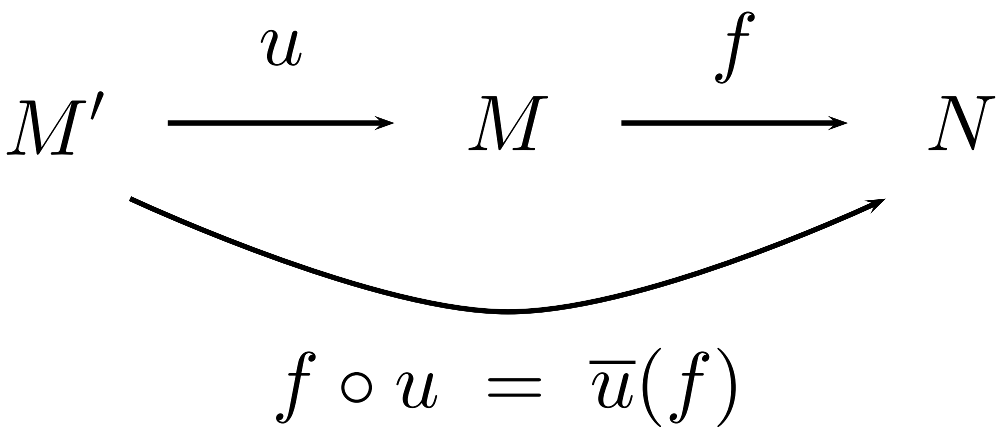
Composition of mappings induces homomorphisms between appropriate Hom modules.

Define

$$\overline{u} : \text{Hom}\ (M, N) \to \text{Hom}\ (M', N)$$

by

$$\overline{u}(f) \;=\; f \circ u\ .$$

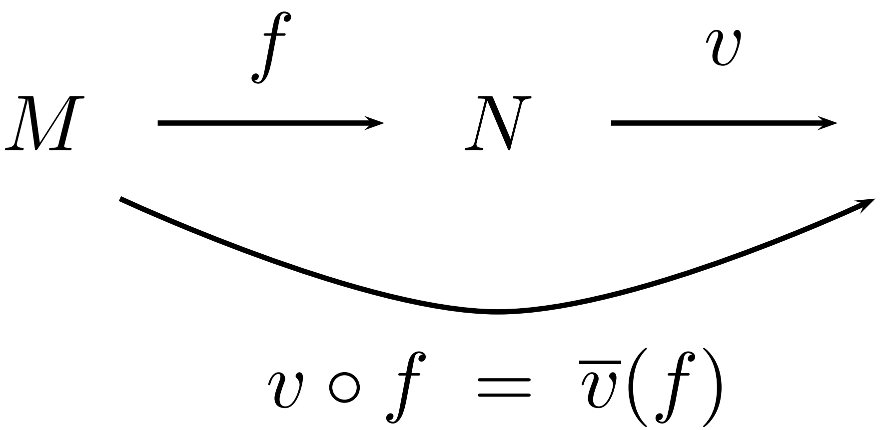


Define

$$\overline{v} : \text{Hom}\ (M, N) \to \text{Hom}\ (M, N')$$

by

$$\overline{v}(f) \;=\; v \circ f\ .$$



Easy to check:

$\overline{u}$, $\overline{v}$ are themselves A-module homomorphisms

and we say that $\overline{u}$, $\overline{v}$ are **induced** from u , v respectively.

Example: Suppose that A is a field and M' , M , N , N' are finite dimensional vector spaces of dimension m' , m , n , n' respectively.

Linear transformations may be identified with matrices

so

$$\begin{aligned}\text{Hom}\,(M,N) &\equiv \text{Mat}\,(n,m) \\ &= \{\, n \times m \text{ matrices over } A \,\}\ ,\end{aligned}$$

$$\text{Hom}\,(M',N) \equiv \text{Mat}\,(n,m')\ ,$$

$$\text{Hom}\,(M,N') \equiv \text{Mat}\,(n',m)\ .$$

Let $u : M' \to M$, $v : N \to N'$ be linear transformations. Regard

u as an $m \times m'$ matrix,

and

v as an $n' \times n$ matrix.

Then the induced homomorphism become simply **pre** and **post**-multiplication respectively by matrices:

$$\overline{u} : \mathsf{Mat}\,(n,m) \;\longrightarrow\; \mathsf{Mat}\,(n,m')$$

$$x \;\longmapsto\; x\,u\;;$$

$$\overline{v} : \mathsf{Mat}\,(n,m) \;\longrightarrow\; \mathsf{Mat}\,(n',m)$$

$$x \;\longmapsto\; v\,x\;.$$

Observation: For any A-module M ,

$$\text{Hom}_A(A, M) \cong M .$$

Proof: It is routine to check that

$$f \mapsto f(1) \qquad \forall f \in \text{Hom}_A(A, M)$$

is a bijective A-module homomorphism.