Exercises: Let $I_1, I_2 \lhd A$ and $J_1, J_2 \lhd B$. Verify that

(i) $$(I_1 + I_2)^{\mathsf{e}} = I_1^{\mathsf{e}} + I_2^{\mathsf{e}} \ ,$$
$$(J_1 + J_2)^{\mathsf{c}} \supseteq J_1^{\mathsf{c}} + J_2^{\mathsf{c}} \ ;$$

(ii) $$(I_1 \cap I_2)^{\mathsf{e}} \subseteq I_1^{\mathsf{e}} \cap I_2^{\mathsf{e}} \ ,$$
$$(J_1 \cap J_2)^{\mathsf{c}} = J_1^{\mathsf{c}} \cap J_2^{\mathsf{c}} \ ;$$

(iii) $$(I_1 I_2)^{\mathsf{e}} = I_1^{\mathsf{e}} I_2^{\mathsf{e}} \ ,$$
$$(J_1 J_2)^{\mathsf{c}} \supseteq J_1^{\mathsf{c}} J_2^{\mathsf{c}} \ ;$$

Exercises continued: Let $I_1, I_2 \lhd A$ and $J_1, J_2 \lhd B$. Verify that

(iv) $$(I_1 : I_2)^{\mathsf{e}} \subseteq (I_1^{\mathsf{e}} : I_2^{\mathsf{e}}) \ ,$$
$$(J_1 : J_2)^{\mathsf{c}} \subseteq (J_1^{\mathsf{c}} : J_2^{\mathsf{c}}) \ ;$$

(v) $$r(I_1)^{\mathsf{e}} \subseteq r(I_1^{\mathsf{e}}) \ ,$$
$$r(J_1)^{\mathsf{c}} = r(J_1^{\mathsf{c}}) \ ;$$

and find examples which for which the set containments in (i) – (v) are proper.

Observation: Let $f : A \to B$ be a ring homomorphism, $I \lhd A$ and $J \lhd B$. Then

(i) $I \subseteq I^{\mathsf{ec}}$ and $J \supseteq J^{\mathsf{ce}}$;

(ii) $J^{\mathsf{c}} = J^{\mathsf{cec}}$ and $I^{\mathsf{e}} = I^{\mathsf{ece}}$.

Proof: (i) Observe that

$$I^{\mathsf{ec}} = f^{-1}\big(\langle f(I) \rangle \big)$$

$$\supseteq f^{-1}\big(f(I)\big) \supseteq I ,$$

and

$$
\begin{aligned}
J^{\mathsf{ce}} &= \langle f(f^{-1}(J))\rangle \\
&\subseteq \langle J\rangle = J\ .
\end{aligned}
$$

(ii) By (i) we see that

$$J^{\mathsf{c}} \subseteq (J^{\mathsf{c}})^{\mathsf{ec}} = (J^{\mathsf{ce}})^{\mathsf{c}} \subseteq J^{\mathsf{c}}\ ,$$

so $J^{\mathsf{c}} = J^{\mathsf{cec}}$. Similarly $I^{\mathsf{e}} = I^{\mathsf{ece}}$.

Call an ideal I of A **contracted** if $I = J^{\mathsf{c}}$ for some ideal J of B .

Call an ideal J of B **extended** if $J = I^{\mathsf{e}}$ for some ideal I of A .

Put

$$\mathcal{C} \;=\; \{\,\text{contracted ideals in } A\,\}\,,$$

$$\mathcal{E} \;=\; \{\,\text{extended ideals in } B\,\}\,.$$

Then

Proposition:

$$\mathcal{C} \;=\; \{\, K \lhd A \mid K^{\mathsf{ec}} = K \,\}\,,$$

$$\mathcal{E} \;=\; \{\, L \lhd B \mid L^{\mathsf{ce}} = L \,\}\,,$$

and

$$K \;\mapsto\; K^{\mathsf{e}} \qquad (K \in \mathcal{C})$$

defines a bijection from $\mathcal{C}$ to $\mathcal{E}$ whose inverse is

$$L \;\mapsto\; L^{\mathsf{c}} \qquad (L \in \mathcal{E})\,.$$

Proof: If $K \in \mathcal{C}$ then $K = L^{\mathsf{c}}$ for some $L \lhd B$, so

$$K^{\mathsf{ec}} \ = \ L^{\mathsf{cec}} \ = \ L^{\mathsf{c}} \ = \ K \ .$$

Thus

$$\mathcal{C} \ \subseteq \ \{ \, K \lhd \ A \ | \ K^{\mathsf{ec}} \ = \ K \, \} \, .$$

Reverse set containment is obvious, so the sets are equal. A similar observation applies to $\mathcal{E}$.

It is immediate then that extension and contraction are mutually inverse bijections from $\mathcal{C}$ to $\mathcal{E}$ and $\mathcal{E}$ to $\mathcal{C}$ respectively.

Exercise: Verify that

(i) $\mathcal{E}$ is closed under sum and product of ideals; and

(ii) $\mathcal{C}$ is closed under intersection, forming ideal quotients and taking radicals.