We will use the notion of coprimeness to write down a criterion for a ring to be isomorphic to a direct sum of a finite collection of its own quotients.

Let $A_1 , \ldots , A_n$ be rings. Call

$$A = \{ (x_1, \ldots, x_n) \mid x_i \in A_i \quad \forall i \}$$

the **direct sum** of $A_1 , \ldots , A_n$, written

$$A = A_1 \oplus \ldots \oplus A_n$$

or

$$A = \oplus_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i$$

which is a ring with coordinatewise operations, and identity element $(1, \ldots, 1)$.

The **projection mapping**, for each i ,

$$p_i : A \longrightarrow A_i \,, \quad (x_1, \ldots, x_n) \mapsto x_i$$

is an onto ring homomorphism.

Now let A be any ring, $n \geq 2$ and $J_1 , \ \ldots \ , \ J_n \lhd A$. Define

$$\phi : A \longrightarrow \bigoplus_{i=1}^{n} A/J_i$$

by

$$x \mapsto (J_1 + x \ , \ \ldots \ , \ J_n + x) \qquad (x \in A) \ .$$

Clearly ϕ is a ring homomorphism with kernel

$$\ker \phi \ = \ \bigcap_{i=1}^{n} J_i \ .$$

Proposition:

(i) ϕ is injective iff $\bigcap_{i=1}^{n} J_i = \{0\}$.

(ii) ϕ is surjective iff J_i and J_k are coprime whenever $i \neq k$.

(iii) If J_i , J_k are coprime whenever $i \neq k$ then

$$\prod_{i=1}^{n} J_i = \bigcap_{i=1}^{n} J_i .$$

Corollary:

The ring A is isomorphic to the direct sum of A/J_1 , … , A/J_n by the “natural” map ϕ iff the ideals intersect trivially and are pairwise coprime.

Proof of Proposition: (i) is clear.

(ii) $(\Longrightarrow)$ Suppose ϕ is surjective.

Then, for some $x \in A$,

$$(J_1 + 1\ ,\ \ J_2\ ,\ \ldots\ ,\ \ J_n) \ = \ x\phi$$

$$= \ (J_1 + x\ ,\ \ J_2 + x\ ,\ \ldots\ ,\ \ J_n + x)\ .$$

In particular $x \ \in \ (J_1 + 1) \cap J_2$, so

$$1 \ = \ (1 - x) + x \ \in \ J_1 + J_2\ ,$$

proving J_1 and J_2 are coprime. Similarly J_i and J_k are coprime whenever $i \neq k$.

($\Longleftarrow$) Suppose conversely that J_i and J_k are coprime whenever $i \neq j$. Then

$$(\forall k \geq 2)(\exists u_k \in J_1)(\exists v_k \in J_k) \qquad u_k + v_k \;=\; 1 \,.$$

Let $a \in A$ and put

$$x \;=\; a\, v_2 \;\ldots\; v_n \,.$$

Then

$$x \;\in\; J_k \qquad \text{for all } k \geq 2 \,,$$

and

$$x \quad = \quad a\,(1-u_2)\ \ldots\ (1-u_n) \quad \in \quad J_1 + a\ ,$$

so

$$(J_1 + a\ ,\ \ J_2\ ,\ \ldots\ ,\ \ J_n)\ \in\ \text{im}\ \phi\ .$$

Similarly, for $i \geq 2$,

$$(J_1\ ,\ \ldots\ ,\ J_{i-1}\ ,\ \ J_i{+}a\ ,\ \ J_{i+1}\ ,\ \ldots\ ,\ \ J_n)\ \in\ \text{im}\ \phi\ .$$

Thus, for all $a_1 , \ \ldots , \ a_n \in A$,

$$(J_1{+}a_1 , \ \ldots , \ J_n + a_n)$$

$$= \sum_{i=1}^{n} (J_1 , \ \ldots , J_{i-1} , \ J_i + a_i , \ J_{i+1} , \ \ldots , \ J_n) ,$$

$\in$ im ϕ ,

proving ϕ is surjective.

(iii) Suppose J_i and J_k are coprime whenever $i \neq k$.

If $n = 2$ then, by an earlier Observation,

$$J_1 \cap J_2 \;=\; J_1\, J_2 \;,$$

which starts an induction. Suppose $n > 2$ and (as inductive hypothesis)

$$\prod_{i=1}^{n-1} J_i \;=\; \bigcap_{i=1}^{n-1} J_i \;.$$

Put

$$K = \prod_{i=1}^{n-1} J_i \ .$$

But

$$(\forall i = 1, \ldots n-1)(\exists x_i \in J_i \ , \quad y_i \in J_n)$$

$$x_i + y_i \quad = \ 1$$

so that

$$
\begin{aligned}
1 &= 1-(x_1\ldots x_{n-1})+(x_1\ldots x_{n-1}) \\
&= 1-\big[(1-y_1)\ldots(1-y_{n-1})\big]+(x_1\ldots x_{n-1}) \\
&= 1-\big[1+\underbrace{\ldots}_{\in J_n}\big]+\underbrace{(x_1\ldots x_{n-1})}_{\in K}
\end{aligned}
$$

yielding $1 \in J_n + K$.

Thus

$$\prod_{i=1}^{n} J_i = \left(\prod_{i=1}^{n-1} J_i\right) J_n$$

$$= K\, J_n = K \cap J_n$$

$$= \left(\bigcap_{i=1}^{n-1} J_i\right) \cap J_n = \bigcap_{i=1}^{n} J_i \ ,$$

completing the proof by induction.

The next result gives useful connections between **prime** ideals, unions and intersections:

Theorem: Let A be a ring.

(i) Let $J_1 , \ldots , J_n \lhd A$ and P a prime ideal of A such that

$$P \supseteq \bigcap_{i=1}^{n} J_i .$$

Then $P \supseteq J_k$ for some k .

If $P = \bigcap_{i=1}^{n} J_i$ then $P = J_k$ for some k .

Theorem (continued):

(ii) Let $P_1 , \ldots , P_n$ be prime ideals of A and $J \lhd A$ such that

$$J \subseteq \bigcup_{i=1}^{n} P_i .$$

Then $J \subseteq P_k$ for some k .

Proof: (i) Suppose $P \not\supseteq J_i$ for all i . Then

$$(\forall i) \qquad \exists\, x_i \in J_i \backslash P .$$

Put

$$y = x_1 \ldots x_n .$$

Then

$$y \in \bigcap_{i=1}^{n} J_i \subseteq P ,$$

so, since P is prime,

$$(\exists k) \qquad x_k \in P ,$$

contradicting that $x_k \in J_k \backslash P$.

Hence $P \supseteq J_k$ for some k .

If $P = \bigcap_{i=1}^{n} J_i$ then

$$J_k \subseteq P \subseteq J_k ,$$

so $P = J_k$, and (i) is proved.

(ii) If $n = 1$ then $J \subseteq P_1$, which starts an induction. We will show

(∗) $$\boxed{J \subseteq \bigcup_{i \neq j} P_i \qquad \exists j \in \{1, \dots n\} .}$$

Suppose that $(*)$ fails, so

$$\big(\forall j \in \{1, \ldots, n\}\big) \qquad \exists\, x_j \;\in\; J \setminus \bigcup_{i \neq j} P_i \;.$$

But

$$J \;\subseteq\; \bigcup_{i=1}^{n} P_i \;,$$

so

$$\big(\forall j \in \{1, \ldots, n\}\big) \qquad x_j \in P_j \;.$$

Put

$$y = \sum_{j=1}^{n} x_1 \ldots x_{j-1}\, x_{j+1} \ldots x_n \,.$$

Then

$$y \in J \subseteq \bigcup_{i=1}^{n} P_i$$

so

$$y \in P_k \qquad \big(\exists k \in \{1, \ldots, n\}\big) \,.$$

Hence

$$x_1 \ \ldots \ x_{k-1}\, x_{k+1} \ \ldots \ x_n$$

$$= \quad y \; - \; \Big(\sum_{j \neq k} \; x_1 \ \ldots \ x_{j-1}\, x_{j+1} \ \ldots \ x_n \Big) \quad \in \quad P_k$$

since $y \in P_k$ and $x_k \in P_k$ is a factor of

$$x_1 \ \ldots \ x_{j-1}\, x_{j+1} \ \ldots \ x_n \qquad \text{for } \ j \neq k \ .$$

But P_k is prime, so $x_j \in P_k$ for some $j \neq k$.

This contradicts that

$$x_j \quad \notin \quad \bigcup_{i \neq j} P_i \quad \supseteq \quad P_k \ .$$

Hence $(*)$ holds.

By an inductive hypothesis, $J \ \subseteq \ P_i$ for some i , and (ii) is proved.

Ideal quotients:

Let $I\,,\ J \ \triangleleft\ A$.

The **ideal quotient** of I by J is

$$(I : J) \ = \ \{\, x \in A \ | \ Jx \subseteq I \,\}\,.$$

It is routine to verify that

$$\boxed{(I : J) \ \triangleleft \ A\,.}$$

We write

$$\mathsf{Ann}(J) \;=\; (0:J) \;=\; (\{0\}:J)$$

$$= \; \{\, x \in A \;\mid\; Jx \;=\; \{0\} \,\} \, ,$$

called the **annihilator** of J .

If $y \in A$ then we write

$$(I:y) \;=\; (I:Ay) \quad \text{and} \quad \mathsf{Ann}(y) \;=\; \mathsf{Ann}(Ay) \, .$$

Easy to see:

$$\bigcup_{x \neq 0} \text{Ann}(x) \;=\; \{ \text{ zero-divisors in } A \} \,.$$

Example: Put $A = \mathbb{Z}$, and let m , $n \in \mathbb{Z}^+$. Then

$$m \;=\; p_1^{\alpha_1} \ldots p_k^{\alpha_k} \,, \qquad n \;=\; p_1^{\beta_1} \ldots p_k^{\beta_k}$$

for some prime numbers $p_1, \ldots, p_k$ and nonnegative integers $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$.

Then

$$
\begin{aligned}
(m\mathbb{Z} : n) &= (m\mathbb{Z} : n\mathbb{Z}) \\
&= \{ \, z \in \mathbb{Z} \;\mid\; zn \in m\mathbb{Z} \, \} \\
&= q\mathbb{Z}
\end{aligned}
$$

where

$$
q = p_1^{\gamma_1} \, \ldots \, p_k^{\gamma_k}
$$

such that, for each i ,

$$\gamma_i \;=\; \max\{\alpha_i - \beta_i, 0\} \;=\; \alpha_i - \min\{\alpha_i, \beta_i\} \; .$$

Thus

$$(m\mathbb{Z} : n\mathbb{Z}) \;=\; q\mathbb{Z}$$

where

$$q \;=\; \frac{m}{\text{g.c.d.}\,\{m, n\}} \; .$$

Exercises: Let I , J , $K \lhd A$. Verify the following:

(1) $I \subseteq (I:J)$;

(2) $(I:J)\,J \subseteq I$;

(3) $\big((I:J):K\big) = (I:JK) = \big((I:K):J\big)$;

(4) Verify that if $I_\ell \lhd A$ for all $\ell \in X$, where X is some indexing set, and $J \lhd A$, then

$$\left(\bigcap_{\ell \in X} I_\ell \ : \ J \right) \ = \ \bigcap_{\ell \in X} (I_\ell : J) \ .$$

(5) Verify that if $J_\ell \lhd A$ for all $\ell \in X$, where X is some indexing set, and $I \lhd A$, then

$$\left(I \ : \ \sum_{\ell \in X} J_\ell \right) \ = \ \bigcap_{\ell \in X} (I : J_\ell) \ .$$