1.6 Operations on Ideals

Let A be a ring, and consider $X \subseteq A$.

Put

$$\begin{aligned} \langle X \rangle &= \text{ideal generated by } X \\ &= \bigcap \{ J \mid X \subseteq J \lhd A \} . \end{aligned}$$

Write $\langle x_1, \ldots, x_n \rangle = \langle \{ x_1, \ldots, x_n \} \rangle$, and note that

$$\langle \emptyset \rangle = \{0\} \, .$$

Put

$$\mathcal{L}(A) = \{ J \mid J \lhd A \}$$

which is a poset with respect to $\subseteq$. Then

$\mathcal{L}(A)$ is a complete lattice.

A **lattice** is a poset in which the greatest lower bound (g.l.b.) and least upper bound (l.u.b.) of a pair of elements exist.

A lattice is **complete** if the g.l.b. and l.u.b. of any subset (not necessarily finite) exist.

If $\mathcal{S} \subseteq \mathcal{L}(A)$ then

$$\text{g.l.b.}\, \mathcal{S} \;=\; \cap \{ J \mid J \in \mathcal{S} \} \,;$$

$$\text{l.u.b.}\, \mathcal{S} \;=\; \langle \cup \{ J \mid J \in \mathcal{S} \} \rangle \,.$$

Sum of sets:

If $X, Y \subseteq A$ then put

$$X + Y \;=\; \{\, x + y \mid x \in X\,,\;\; y \in Y \,\}$$

and write

$$X + y \;=\; X + \{\, y \,\}$$

(so if $X \lhd A$ then, consistent with earlier notation, $X + y$ is the coset of X containing y).

Easy to see:

$$J\,,\ K \lhd A \quad \Longrightarrow \quad J+K \ = \ \langle\, J \ \cup \ K \,\rangle\,.$$

(Need to check: (i) $J \cup K \ \subseteq J + K$;

(ii) $J + K$ is an ideal

(iii) $J \cup K \ \subseteq L \lhd \ A \quad \Longrightarrow \quad J + K \ \subseteq \ L$.)

Consider a family of ideals

$$\{\ J_i \ \lhd\ A \ \mid\ i \in I\ \}$$

where I is some indexing set. Define the **sum**

$$\sum_{i\in I} J_i \;=\; \Big\{\ \sum_{i\in I} x_i \ \Big|\ x_i \in J_i \ \ (\forall i \in I)$$

and only finitely many x_i are nonzero $\Big\}$.

Exercise: Verify that

$$\sum_{i \in I} J_i \;=\; \langle \cup_{i \in I} J_i \rangle \, .$$

Thus

finding least upper bounds in $\mathcal{L}(A)$ amounts to taking sums of families of ideals.

Product of an ideal with a set:

Let $J \lhd A$, $X \subseteq A$.

Define the **product**

$$JX = \Big\{ \sum_{i=1}^{n} a_i x_i \;\Big|\; n \geq 1 \text{ and } (\forall i) \quad a_i \in J \,,\; x_i \in X \Big\} .$$

Easy to check:

$$\boxed{JX = \langle\, ax \mid a \in J \,,\; x \in X \,\rangle \lhd A \,.}$$

Write

$$Jx \;=\; J\{x\} \;.$$

Easy to see

$$Jx \;=\; \{\, ax \mid a \in J \,\} \;.$$

Special case:

the principal ideal generated by x is

$$\langle x \rangle \;=\; Ax \;.$$

Other routine facts:

$$\boxed{\text{if } \; J, K, L \lhd A \; \text{ then } \; (JK)L = J(KL) \; ,}$$

so brackets may be ignored, and

$$\text{if } J_1 \, , \; \ldots \, , \; J_k \lhd A \; \text{ then}$$

$$J_1 \ldots J_k \; = \; \langle \, x_1 \ldots x_k \; | \; x_j \in J_j \quad \forall j \, \rangle$$

$$= \; \Big\{ \sum_{i=1}^{n} x_{i1} \ldots x_{ik} \; \Big| \; n \geq 1 \, , \; x_{ij} \in J_j \quad \forall i, j \, \Big\} \, .$$

Powers of an ideal J are defined in a natural way:

$$\begin{aligned} J^0 &= A \\ J^1 &= J \\ J^k &= J\,J\;\dots\;J \qquad (k \text{ times}) \\ &= \langle\, x_1 \dots x_k \mid x_1\,,\,\dots\,,\,x_k \in J \,\rangle\,. \end{aligned}$$

It is immediate that

$$J^k = \{0\}$$

iff all products of k elements of J are zero.

Note: Always we have

$$J\,,\ K \lhd A \qquad \Longrightarrow \qquad JK \subseteq J \cap K\,.$$

Examples:

(1) Put $A = \mathbb{Z}$, $J = \langle m \rangle$, $K = \langle n \rangle$ where $m, n \in \mathbb{Z}$.

Easy to see:

$$J \cap K \quad = \quad \langle \text{ l.c.m.}\{m, n\} \rangle$$

and

$$J + K \quad = \quad \langle \text{ g.c.d.}\{m, n\} \rangle$$

Hence the lattice of ideals of $\mathbb{Z}$ can be identified with the lattice

$$(\mathbb{Z}^+ \cup \{0\} \, , \ \ \leq)$$

where $a \leq b$ means $b \mid a$.

Further, clearly, $\boxed{JK \ = \ \langle \, mn \, \rangle}$

so

$$\boxed{JK \ = \ J \cap K \ \text{ iff } m \text{ and } n \text{ are coprime.}}$$

(2) Put $A = F[x_1, \ldots, x_n]$ where F is a field and $x_1, \ldots, x_n$ are n commuting indeterminates.

Put

$$J = \langle\, x_1, \ldots, x_n \,\rangle$$

$$= \{\, p \in A \;\mid\; p \text{ has constant term } 0 \,\}\ .$$

Clearly, if $m \geq 1$ then

$$J^m = \{ p \in A \mid \text{each term of } p \text{ has degree } \geq m \}$$

where the **degree** of a term

$$\lambda \, x_1^{d_1} \, x_2^{d_2} \, \ldots \, x_n^{d_n} \qquad (\lambda \in F)$$

is

$$d_1 + d_2 + \ldots + d_n \, .$$

Easy Exercises: Let $J, K, L \lhd A$.

(1) $J(K+L) = JK + JL$;

(2) $J \cap (K+L) \supseteq (J \cap K) + (J \cap L)$;

(3) $J \supseteq K \implies$

$$J \cap (K+L) = (J \cap K) + (J \cap L)$$

(the **modular law**);

(4) $(J+K)(J \cap K) \subseteq JK$.

Exercise: Find a ring A and ideals J , K , L such that

$$(J \cap K) + (J \cap L) \subset J \cap (K + L) .$$

Exercise: Find a ring A and ideals J , K such that

$$(J + K)(J \cap K) \subset J K .$$

Example: Put $A = \mathbb{Z}$ and let $a, b, c \in \mathbb{Z}^+ \cup \{0\}$.

Exercise: Verify that

$$\text{l.c.m.}\{\, a \,,\ \ \text{g.c.d.}\{b, c\}\, \}$$

$$= \ \text{g.c.d.}\{\, \text{l.c.m.}\{a, b\} \,,\ \ \text{l.c.m.}\{a, c\}\, \}$$

Put $J = a\mathbb{Z}$, $K = b\mathbb{Z}$, $L = c\mathbb{Z}$. Then

$$J \cap (K + L) = a\mathbb{Z} \cap (\text{g.c.d.}\{b, c\}\, \mathbb{Z})$$

$$= \text{l.c.m.}\{\, a \,,\ \text{g.c.d.}\{b, c\}\ \}\, \mathbb{Z}$$

$$= \text{g.c.d.}\{\ \text{l.c.m.}\{a, b\} \,,\ \text{l.c.m.}\{a, c\}\ \}\, \mathbb{Z}$$

$$= \text{l.c.m.}\{a, b\}\, \mathbb{Z} \ + \ \text{l.c.m.}\{a, c\}\ \}\, \mathbb{Z}$$

$$= (J \cap K) + (J \cap L)\ .$$

Thus, since $\mathbb{Z}$ is a PID,

For all ideals J, K, L of $\mathbb{Z}$,

$$J \cap (K + L) \;=\; (J \cap K) + (J \cap L) \;.$$

Further, if $a, b \in \mathbb{Z}^+ \cup \{0\}$ then clearly

$$ab \;=\; (\text{g.c.d.}\{a, b\})(\text{l.c.m.}\{a, b\})$$

so that

$$(a\mathbb{Z})(b\mathbb{Z}) \;=\; ab\,\mathbb{Z} \;=\; (\text{g.c.d.}\{a,b\}\,\mathbb{Z})(\text{l.c.m.}\{a,b\}\,\mathbb{Z})$$

$$= \;(a\mathbb{Z} + b\mathbb{Z})(a\mathbb{Z} \cap b\mathbb{Z})\ .$$

Again, since $\mathbb{Z}$ is a PID,

For all ideals J, K of $\mathbb{Z}$,

$$(J + K)(J \cap K) \;=\; JK\ .$$

Call ideals J, K of a ring A **coprime** or **comaximal** if

$$J + K \;=\; A \,,$$

equivalently

$$(\exists x \in J \,,\; y \in K) \qquad 1 \;=\; x + y \;.$$

e.g. If $A = \mathbb{Z}$ then ideals are coprime iff their respective generators are coprime as integers.

Observation: If J, K are coprime then

$$J \cap K \;=\; JK \; .$$

Proof: If $J + K = A$ then

$$J \cap K \;=\; A(J \cap K) \;=\; (J+K)(J \cap K)$$

$$\subseteq \; JK \; \subseteq \; J \cap K \; ,$$

whence equality holds.