1.5 The Nil and Jacobson Radicals

The idea of a "radical" of a ring A is an ideal I comprising some "nasty" piece of A such that

$$\boxed{A/I \text{ is well-behaved or tractable.}}$$

Two types considered here are the **nil** and **Jacobson radicals**, which are intimately connected with prime and maximal ideals respectively.

Factoring out by the nil and Jacobson radicals yields quotients closely related to integral domains and fields respectively (in a sense which will be made precise).

Consider first all nilpotent elements of a ring:

Theorem: Let A be a ring and put

$$N = \{ x \in A \mid x \quad \text{is nilpotent} \} .$$

Then $N \lhd A$ and A/N has no nonzero nilpotent elements.

Proof: Let $x, y \in N$, $a \in A$. Then

$$x^m \;=\; y^n \;=\; 0 \qquad (\exists m, n \in \mathbb{Z}^+) \;.$$

Clearly

$$(ax)^m \;=\; (-x)^m \;=\; 0 \;,$$

so

$$ax \;,\; -x \;\in\; N \;.$$

Also

$$(x+y)^{m+n-1} \quad = \quad \sum_{i=0}^{m+n-1} \binom{m+n-1}{i} x^i y^{m+n-1-i}$$

where

$$\binom{m+n-1}{i}$$

denotes the usual binomial coefficient, and we interpret integer multiples in the usual way.

If $i \geq m$ then $x^i = 0$.

If $i < m$ then

$$m + n - 1 - i \;=\; n + (m - i - 1) \;\geq n$$

so that

$$y^{m+n-1-i} \;=\; 0\ .$$

Thus

$$(x+y)^{m+n-1} \;=\; 0\ ,$$

so $x + y \in N$, completing the proof that $N \lhd A$.

Further, if $N + a$ is a nilpotent element of A/N then

$$(N + a)^k \;=\; N + a^k \;=\; N \qquad (\exists k \in \mathbb{Z}^+)$$

so $a^k \in N$, so

$$a^{kl} \;=\; (a^k)^l \;=\; 0 \qquad (\exists l \in \mathbb{Z}^+)\,,$$

which shows $a \in N$ and $N + a = N$.

Thus all nonzero elements of A/N are not nilpotent, and the theorem is proved.

The ideal N of the previous theorem is called the **nilradical** of A, and the last part of the previous theorem says that

> the nilradical of what is left, after factoring out by the nilradical, is itself trivial.

An alternative description is provided by:

> **Theorem:** The nilradical of a ring is the intersection of all of the prime ideals of the ring.

Before proving this, we observe that its importance derives from the following concept:

Suppose $\{ A_i \mid i \in I \}$ is a family of rings and consider the ring

$$A = \prod_{i \in I} A_i = \{ (x_i)_{i \in I} \mid x_i \in A_i \ \forall i \}$$

with coordinatewise operations, called the **direct product** of the family.

Call a ring B a **subdirect product** of this family if there exists a ring homomorphism

$$\psi : B \rightarrow A$$

such that

(i) ψ is injective; and

(ii) $\rho_j \circ \psi : B \rightarrow A_j$ is surjective for all j

where $\rho_j : A \rightarrow A_j$ is the **projection mapping** $(x_i)_{i \in I} \mapsto x_j$.

Think of subdirect products as being "close" to direct products.

Observation: Let B be a ring and $\{ J_i \mid i \in I \}$ be a family of ideals of B. Put

$$J = \bigcap_{i \in I} J_i \ .$$

Then $J \lhd B$ and B/J is a subdirect product of the family $\{ B/J_i \mid i \in I \}$.

Proof of the Observation: The map

$$\psi : B \to \prod_{i \in I} B/J_i \ , \quad x \mapsto (J_i + x)_{i \in I}$$

is easily seen to be a ring homomorphism with kernel

$$\ker \psi \ = \ \bigcap_{i \in I} J_i \ = \ J \ .$$

Thus

$$B/J \ \cong \ \text{im} \ \psi \ .$$

(by the Fundamental Homomorphism Theorem).

Further, clearly the composite

$$B \xrightarrow{\psi} \prod_{i \in I} B/J_i \xrightarrow{\rho_j} B/J_j$$

is surjective for each j , so

im ψ , and hence B/J ,

is a subdirect product of the family

$$\{ B/J_i \mid i \in I \} ,$$

and the Observation is proved.

Corollary to the Theorem: If A is any ring and N its nilradical, then A/N is a subdirect product of integral domains.

Proof: Let $\{ J_i \mid i \in I \}$ be the family of all prime ideals of a ring A. By the Theorem,

$$N = \bigcap_{i \in I} J_i$$

is the nilradical of A, and by the Observation, A/N is a subdirect product of the family $\{ A/J_i \mid i \in I \}$, each member of which is an integral domain.

Corollary: A ring with trivial nilradical is a subdirect product of integral domains.

Proof of the Theorem: Let N denote the nilradical and N' the intersection of all prime ideals of the ring A .

We have to show $N = N'$.

Consider a prime ideal P of A and let $x \in N$.

Then

$$x(x^{k-1}) \;=\; x^k \;=\; 0 \;\in\; P \qquad (\exists k \in \mathbb{Z}^+)\ ,$$

so, since P is prime,

$$x \in P \qquad \text{or} \qquad x^{k-1} \in P\ .$$

By a simple induction, $x \in P$. Thus $N \subseteq P$. Hence

$$\boxed{N \;\subseteq\; N'\ .}$$

Suppose $x \in A \backslash N$, so x is **not** nilpotent. Put

$$\Sigma \;=\; \{\, I \,\lhd\, A \mid (\forall k \in \mathbb{Z}^+) \quad x^k \notin I \,\}\,,$$

which is a poset with respect to $\subseteq$.

Certainly $\Sigma \neq \emptyset$, since $\{0\} \in \Sigma$ (because x is not nilpotent).

> It is easy to verify that Zorn's Lemma applies, guaranteeing the existence of a maximal element P of Σ.

$\boxed{\text{We will show } P \text{ is prime.}}$

Suppose $a, b \in A$ and $ab \in P$.

Suppose $a, b \notin P$, so

$$P \quad \subset \quad P + aA \qquad \text{and} \qquad P \subset \quad P + bA \ .$$

But

$$P + aA \ , \quad P + bA \quad \lhd \quad A$$

and P is maximal in Σ . Hence

$$P + aA \ , \quad P + bA \ \notin \ \Sigma \ ,$$

so

$$x^k \in P + aA\ , \quad x^l \in P + bA \qquad\qquad (\exists k, l \in \mathbb{Z}^+)\ .$$

Then

$$x^{k+l} \;=\; x^k\, x^l \;\in P + abA \;\lhd\; A\ ,$$

so

$$P + abA \;\not\in\; \Sigma\ .$$

But $ab \in P$, so $P + abA = P \in \Sigma$, a contradiction.

Hence $a \in P$ or $b \in P$, proving P is prime.

In particular $x \notin P$, so $x \notin N'$.

This proves

$$\boxed{\quad N' \;\subseteq\; N\ , \quad}$$

so we conclude

$$\boxed{\quad N \;=\; N'\ , \quad}$$

and the Theorem is proved.

Define the **Jacobson radical** R of a ring A to be the intersection of all of the maximal ideals of A . Thus

$$\boxed{A/R \text{ is a subdirect product of fields.}}$$

Clearly, the maximal ideals of A/R have the form M/R where M is a maximal ideal of A , so the Jacobson radical of A/R is just $R/R = \{R\}$, the trivial ideal.

Since all maximal ideals are prime, we get immediately that

$$\boxed{\text{nilradical} \quad \subseteq \quad \text{Jacobson radical.}}$$

The nilradical was defined in terms of a membership test for elements (that they be nilpotent). A membership test exists also for the Jacobson radical.

Theorem: Let $x \in A$. Then

$$x \in R \quad \Longleftrightarrow \quad (\forall y \in A) \quad 1 - xy \quad \text{is a unit.}$$

Proof: $(\Longrightarrow)$ Suppose $1 - xy$ is **not** a unit for some $y \in A$.

By Zorn's Lemma (an earlier Corollary)

$$1 - xy \ \in \ M$$

for some maximal ideal M .

If $x \in R$ then $x \in M$

(since R is the intersection of all maximal ideals) and so

$$1 \;=\; \underbrace{(1 - xy)} \;+\; \underbrace{xy} \;\in\; M \;,$$

elements of M

which is impossible since $M \neq A$.

Hence $x \notin R$.

$(\Longleftarrow)$ Suppose $x \notin R$.

Then $x \notin M$ for some maximal ideal M

(since R is the intersection of all maximal ideals).

By maximality,

$$A = \langle M \cup \{x\} \rangle$$

$$= \{ m + xy \mid m \in M , \ y \in A \} .$$

In particular,

$$1 \;=\; m + xy \qquad (\exists m \in M)(\exists y \in A)$$

so

$$1 - xy \;=\; m \;\in\; M\ .$$

If $1 - xy$ is a unit then $M = A$.

But $M \neq A$, so $1 - xy$ is **not** a unit.

The Theorem is proved.

Examples: (1) In any integral domain the nilradical is trivial, and in any local ring the Jacobson radical is the unique maximal ideal.

Since there are local rings which are integral domains but not fields, the nil and Jacobson radicals need not be equal.

(2) Let $A = \mathbb{Z}$.

$$\boxed{\textbf{Claim:} \quad R = N = \{0\} .}$$

Proof: Consider $0 \neq a \in \mathbb{Z}$.

If $1-3a=1$ then $a=0$,

whilst if $1-3a=-1$ then $a=2/3$,

both of which are impossible.

Hence $1-3a$ is not a unit, so $a \notin R$, so $R=\{0\}$.

Alternatively (since maximal ideals in $\mathbb{Z}$, being a PID, are precisely nonzero prime ideals)

$$R \;=\; \bigcap_{p \text{ prime}} p\mathbb{Z} \;=\; \{0\} \, .$$

(3) Let $A = F[x]$ where F is a field.

$$\boxed{\textbf{Claim:} \quad R = N = \{0\}\ .}$$

Proof: Consider $0 \neq \alpha \in F[x]$.

Then $1 - x\alpha$ is a polynomial of degree ≥ 1 , so cannot be a unit

(units in $F[x]$ being just the nonzero constants).

Hence $\alpha \notin R$, so again $R = \{0\}$.

(4) Let $A = F[G]$, a group ring, where F is a field of prime characteristic p, and G is an abelian p-group.

Claim: In this example

$$R = N = \{ \sum \alpha_g g \mid \sum \alpha_g = 0 \} .$$

Proof: Consider the onto ring homomomorphism

$$\phi : A \to F \qquad \text{where} \qquad \sum \alpha_g g \mapsto \sum \alpha_g ,$$

whose kernel is

$$\ker\phi \;=\; \{\, \sum \alpha_g\, g \;\mid\; \sum \alpha_g \;=\; 0 \,\} \,.$$

Thus

$$A/\ker\phi \;\cong\; F$$

(by the Fundamental Homomorphism Theorem),

so $\ker\phi$ is a maximal ideal of A . Hence

$$R \;\subseteq\; \ker\phi \,.$$

Consider

$$x \;=\; \alpha_1 g_1 + \ldots + \alpha_n g_n \in \ker\phi .$$

Then

$$\alpha_1 + \ldots + \alpha_n \;=\; 0 ,$$

and, for some sufficiently high power $m = p^k$,

$$g_i^m \;=\; 1 \qquad (\forall i = 1, \ldots, n) .$$

Hence

$$
\begin{aligned}
x^m &= (\alpha_1 g_1 + \ldots + \alpha_n g_n)^m \\
&= (\alpha_1 g_1)^m + \ldots + (\alpha_n g_n)^m
\end{aligned}
$$

[by the "Freshman's Dream"

$$(a+b)^p = a^p + b^p$$

in any field of characteristic p],

so

$$\begin{aligned} x^m &= \alpha_1^m g_1^m + \ldots + \alpha_n^m g_n^m \\ &= \alpha_1^m + \ldots + \alpha_n^m \\ &= (\alpha_1 + \ldots + \alpha_n)^m \\ &= 0^m = 0 . \end{aligned}$$

Thus $x \in N$. Hence $R \subseteq \ker\phi \subseteq N \subseteq R$, so

$$R = N = \ker\phi \, ,$$

and the Claim is proved.

Exercise: Let A be a finite ring.

(1) Prove that if $x \in A$ then some power of x is idempotent.

(2) Verify that if $0 \neq e = e^2 \in A$ then $1 - e$ is idempotent so cannot be a unit.

(3) Deduce that $N = R$.