

Tensor products continued

M, N are A -modules

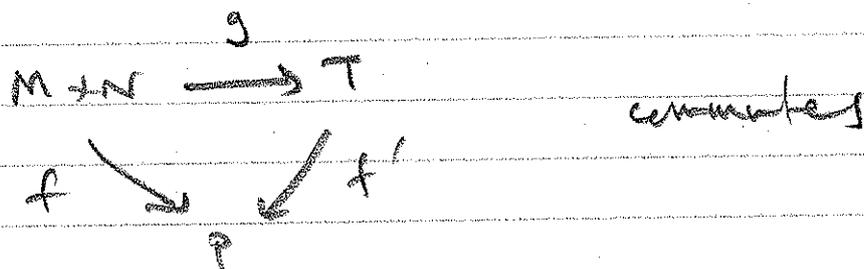
Theorem: $\exists!$ (T, g) where $M \times N \xrightarrow{g} T$
bilinear

such that

\forall bilinear $f: M \times N \rightarrow P$

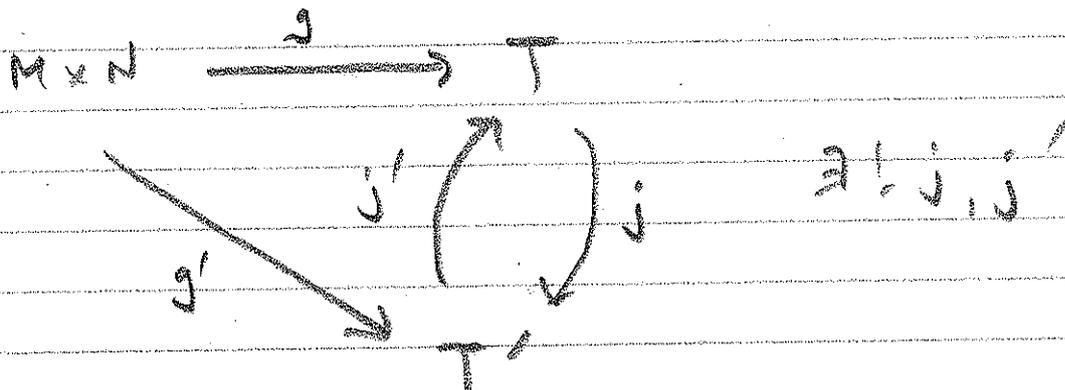
$\exists!$ module hom $f': T \rightarrow P$

such that



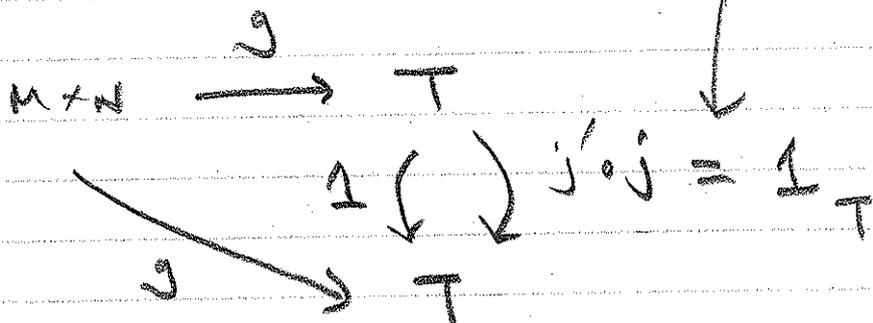
Uniqueness of $T \cong M \otimes N$ up to isomorphism:

Suppose also (T', g') satisfies the properties, so

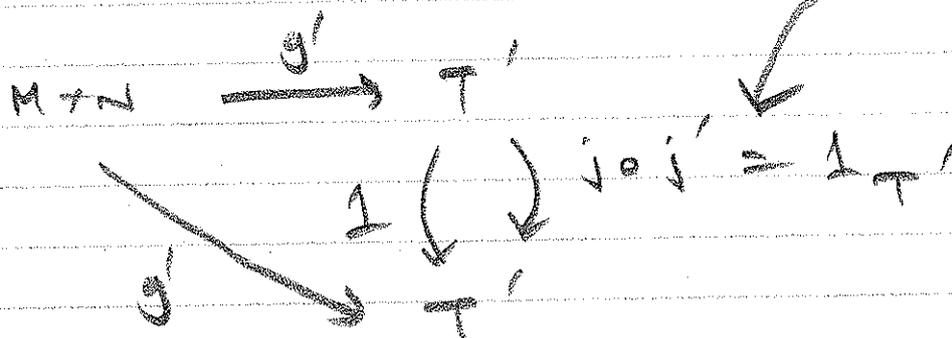


(B)

so have



and



so j and j' are mutually inverse isos. □

Claim: $\mathbb{Z}_+ \oplus_{\mathbb{Z}} \mathbb{Z}_+ \cong \mathbb{Z}_+$

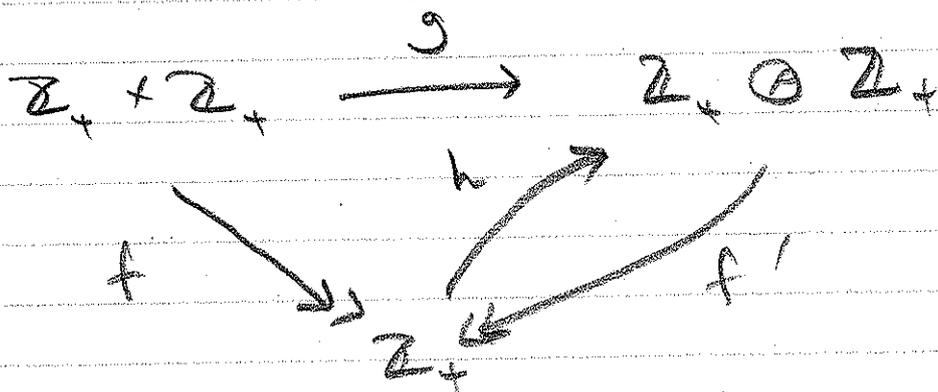
Proof: Let $f: \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$
 $(m, n) \mapsto mn$

easily checked to be bilinear

The main theorem yields a commutative

Diagram:

(c)



Note that f' is onto because f is onto.

Define $h: \mathbb{Z}_4 \rightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_4$
 $n \mapsto 1 \oplus n$

easily checked to be a module hom.

Then

$$(f' \circ h)(n) = f'(1 \oplus n) = f(1, n) = 1n = n$$

$$\forall n \in \mathbb{Z}_4$$

so

$$f' \circ h = 1_{\mathbb{Z}_4}$$

and

$$\begin{aligned} (h \circ f')(m \oplus n) &= h(f'(m \oplus n)) = h(f(m, n)) \\ &= h(mn) = 1 \oplus mn = m \oplus n \end{aligned}$$

so $h \circ f'$ fixes generators of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, so

$$h \circ f' = 1_{\mathbb{Z}_4 \oplus \mathbb{Z}_4}$$

so f' and h are mutually inverse isom. ✓

(3)

Exercise: modify this argument to

prove

$$A \otimes_A M \cong M$$

\forall modules M
over A .

Claim: $F[x] \otimes_F F[x] \cong F[x, y]$.

Proof: Let $f: F[x] \times F[x] \rightarrow F[x, y]$

$$(p(x), q(x)) \mapsto p(x)q(y)$$

easily checked to be bilinear.

Main Theorem yields commutative diagram

$$\begin{array}{ccc} F[x] \times F[x] & \xrightarrow{g} & F[x] \otimes F[x] \\ & \searrow f & \nearrow h \\ & & F[x, y] \end{array}$$

Define $h: F[x, y] \rightarrow F[x] \otimes F[x]$

$$\sum a_{ij} x^i y^j \mapsto \sum a_{ij} (x^i \otimes x^j)$$

easily checked to be a module hom.

(F)

$$\begin{aligned} \text{Then } f' \circ h \left(\sum a_{ij} x^i y^j \right) &= f' \left(\sum a_{ij} (x^i \otimes x^j) \right) \\ &= \sum a_{ij} f'(x^i \otimes x^j) \\ &= \sum a_{ij} f(x^i, x^j) \\ &= \sum a_{ij} x^i y^j \end{aligned}$$

$$\Rightarrow f' \circ h = \mathbb{1}_{F[x, y]},$$

$$\begin{aligned} \text{and } h \circ f' (x^i \otimes x^j) &= h(f'(x^i \otimes x^j)) \\ &= h(f(x^i, x^j)) = h(x^i y^j) \\ &= x^i \otimes y^j, \end{aligned}$$

so $h \circ f'$ fixes generators of $F[x] \otimes F[x]$,

$$\Rightarrow h \circ f' = \mathbb{1}_{F[x] \otimes F[x]}.$$

Hence f' and h are mutually inverse maps.



(F)

Exercise : Prove

$$\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_d$$

where $d = \text{g.c.d.}(m, n)$

Tensoring homomorphisms

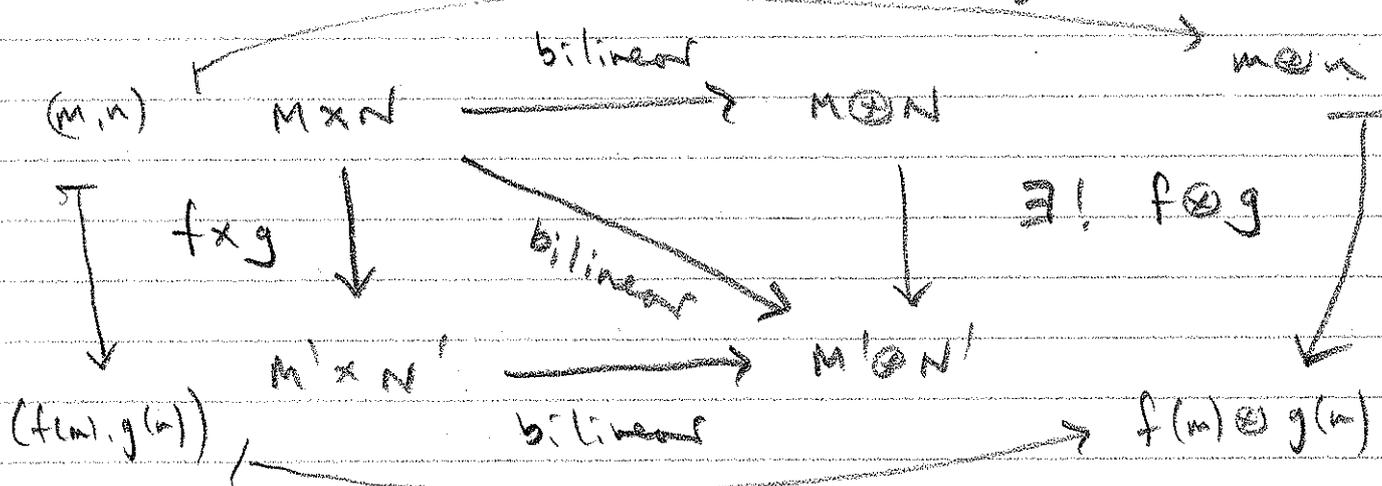
Suppose $M \xrightarrow{f} M'$, $N \xrightarrow{g} N'$ are

A -module homomorphisms. Then define

$$f \times g : M \times N \longrightarrow M' \times N'$$

$$(m, n) \longmapsto (f(m), g(n))$$

and we get commutative diagram:



so $f \otimes g : m \otimes n \longmapsto f(m) \otimes g(n)$

(a)

Example:

$$F[x] \xrightarrow{f} F, \quad F[x] \xrightarrow{g} F$$

$$p(x) \mapsto p(\alpha), \quad q(x) \mapsto q(\beta)$$

evaluation maps at α, β respectively, where F is a field and $\alpha, \beta \in F$.

Then

$$F[x] \otimes F[x] \xrightarrow{f \otimes g} F \otimes F$$

$$p(x) \otimes q(x) \mapsto p(\alpha) \otimes q(\beta)$$

Interpretation?

