Commutative Algebra   Lecture 24   Wed 29/4/09

Tensor products continued

M, N are A-modules

> **Theorem**: $\exists!\ (T, g)$ where $M \times N \xrightarrow[\text{bilinear}]{g} T$
>
> such that
>
> $\forall$ bilinear $f : M \times N \longrightarrow P$
>
> $\exists!$ module hom $f' : T \longrightarrow P$
>
> such that
>
> $$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "f"'] & T \arrow[d, "f'"] \\ & P \end{tikzcd}$$ commutes

Uniqueness of $T \cong M \otimes N$ up to isomorphism:

Suppose also (T', g') satisfies the properties, so

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "g'"'] & T \arrow[d, bend left, "j"] \\ & T' \arrow[u, bend left, "j'"] \end{tikzcd}$$

$\exists!\ j, j'$

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so have

by uniqueness

$$\begin{tikzcd} M \times N \arrow[r, "g"] \arrow[dr, "g"'] & T \arrow[d, shift right, "1"'] \arrow[d, shift left, "j' \circ j = 1_T"] \\ & T \end{tikzcd}$$

and

$$\begin{tikzcd} M \times N \arrow[r, "g'"] \arrow[dr, "g'"'] & T' \arrow[d, shift right, "1"'] \arrow[d, shift left, "j \circ j' = 1_{T'}"] \\ & T' \end{tikzcd}$$

so j and j' are mutually inverse isos. $\square$

Claim: $\mathbb{Z}_+ \otimes_{\mathbb{Z}} \mathbb{Z}_+ \cong \mathbb{Z}_+$.

Proof: let $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$

$(m, n) \longmapsto mn$

easily checked to be bilinear

The main Theorem yields a commuting diagram:

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$$\begin{tikzcd} \mathbb{Z}_4 \times \mathbb{Z}_4 \arrow[rr, "g"] \arrow[dr, "f"'] & & \mathbb{Z}_4 \otimes \mathbb{Z}_4 \arrow[dl, bend left, "f'"] \\ & \mathbb{Z}_4 \arrow[ur, bend left, "h"] & \end{tikzcd}$$

Note that f' is onto because f is onto,

~~Define~~ $h : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \otimes \mathbb{Z}_4$

$n \longmapsto 1 \otimes n$

easily checked to be a module hom.

Then $(f' \circ h)(n) = f'(1 \otimes n) = f(1, n) = \underline{1} n = n$

$\forall n \in \mathbb{Z}_4$

so $f' \circ h = \underline{1}_{\mathbb{Z}_4}$

and $(h \circ f')(m \otimes n) = h(f'(m \otimes n)) = h(f(m,n))$

$= h(mn) = \underline{1} \otimes mn = m \otimes n$

so $h \circ f'$ fixes generators of $\mathbb{Z}_4 \otimes \mathbb{Z}_4$, so

$h \circ f' = \underline{1}_{\mathbb{Z}_4 \otimes \mathbb{Z}_4}$,

so f' and h are mutually inverse isos. ✓

Exercise: modify this argument to prove
$$A \otimes_A M \cong M \qquad \forall \text{ modules } M \text{ over } A.$$

Claim: $F[x] \otimes_F F[x] \cong F[x,y]$.

Proof: Let $f : F[x] \times F[x] \to F[x,y]$

$$(p(x), q(x)) \longmapsto p(x)\,q(y)$$

easily checked to be bilinear.

Main Theorem yields commuting diagram

$$\begin{tikzcd}
F[x] \times F[x] \arrow[rr, "g"] \arrow[dr, "f"'] & & F[x] \otimes F[x] \arrow[dl, "f'"] \\
& F[x,y] \arrow[ur, bend left, "h"] &
\end{tikzcd}$$

Define $h : F[x,y] \to F[x] \otimes F[x]$

$$\sum a_{ij} x^i y^j \longmapsto \sum a_{ij} (x^i \otimes x^j)$$

easily checked to be a module hom.

Then $f' \circ h \left(\sum a_{ij} x^i y^j \right)$

$$= f' \left(\sum a_{ij} (x^i \otimes x^j) \right)$$

$$= \sum a_{ij} f'(x^i \otimes x^j)$$

$$= \sum a_{ij} f(x^i, x^j)$$

$$= \sum a_{ij} x^i y^j$$

So $$f' \circ h = \mathbb{1}_{F[x,y]} ;$$

and $h \circ f'(x^i \otimes x^j) = h(f'(x^i \otimes x^j))$

$$= h(f(x^i, x^j)) = h(x^i y^j)$$

$$= x^i \otimes y^j ,$$

so $h \circ f'$ fixes generators of $F[x] \otimes F[x]$,

so $$h \circ f' = \mathbb{1}_{F[x] \otimes F[x]} .$$

Hence f' and h are mutually inverse

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Exercise: Prove

$$\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_d$$

where $d = \gcd(m,n)$.

Tensoring homomorphisms

Suppose $M \xrightarrow{f} M'$, $N \xrightarrow{g} N'$ are A-module homomorphisms. Then define

$$f \times g : M \times N \longrightarrow M' \times N'$$
$$(m,n) \longmapsto (f(m), g(n))$$

and we get commuting diagram:

$$\begin{tikzcd}
(m,n) \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & M \times N \arrow[r, "\text{bilinear}"] \arrow[d, "f \times g"'] \arrow[dr, "\text{bilinear}"] & M \otimes N \arrow[d, "\exists!\, f \otimes g"] & m \otimes n \arrow[d, mapsto] \\
(f(m), g(n)) \arrow[rrr, mapsto, bend right] & M' \times N' \arrow[r, "\text{bilinear}"'] & M' \otimes N' & f(m) \otimes g(n)
\end{tikzcd}$$

so $\boxed{f \otimes g : m \otimes n \longmapsto f(m) \otimes g(n)}$

<u>Example</u>:

$$F[x] \xrightarrow{f} F, \qquad F[x] \xrightarrow{g} F$$

$$p(x) \longmapsto p(\alpha) \qquad q(x) \longmapsto q(\beta)$$

evaluation maps at α, β respectively, where F is a field and $\alpha, \beta \in F$.

Then

$$F[x] \otimes F[x] \xrightarrow{f \otimes g} F \otimes F$$

$$p(x) \otimes q(x) \longmapsto p(\alpha) \otimes q(\beta)$$

<u>Interpretation?</u>

$$\sum a_{ij}\, x^i \otimes x^j \longmapsto \sum a_{ij}\, \alpha^i \otimes \beta^j$$

$$\begin{tikzcd}
F[x] \otimes F[x] \arrow[r, "f \otimes g"] & F \otimes F \arrow[d, "\text{iso } \psi"] \\
F[x,y] \arrow[u, "\text{iso } \varphi"] \arrow[r] & F
\end{tikzcd}$$

$1 \otimes xy = x \otimes y \longmapsto xy$

$$p(x,y) = \sum a_{ij}\, x^i y^j \longmapsto \sum a_{ij}\, \alpha^i \beta^j = p(\alpha, \beta)$$

evaluation map