A

Commutative Algebra Lecture 34 Mon 25/5/09

Primary decompositions

– motivated by Fundamental Theorem of Arithmetic:

$$z = \underline{p_1^{\alpha_1}} \cdots \underline{p_n^{\alpha_n}}$$ $\exists$ primes $p_1, \dots, p_n$

$\exists\ \alpha_1, \dots, \alpha_n \geq 1$

prime powers

$$z\mathbb{Z} = (p_1^{\alpha_1}\mathbb{Z}) \cdots (p_n^{\alpha_n}\mathbb{Z})$$

$$= (p_1^{\alpha_1}\mathbb{Z}) \cap \cdots \cap (p_n^{\alpha_n}\mathbb{Z})$$

called **primary** ideals

intersection provides simple algorithmic tests for membership

Note

$$r(p_i^{\alpha_i}\mathbb{Z}) = p_i\mathbb{Z} \underset{\text{prime}}{\triangleleft} \mathbb{Z}$$

and

$$p_i^{\alpha_i}\mathbb{Z} = (p_i\mathbb{Z})^{\alpha_i}$$

$\boxed{\text{Do any of these features generalise?}}$

(B)

$\boxed{\text{Lasker-Noether Theorem: Primary decompositions of proper ideals always exist in Noetherian rings.}}$

ascending chain condition (a.c.c.)

intersections of primary ideals

finding the right generalisation of prime ??

Recall $P \underset{\text{prime}}{\lhd} A$ if $P \neq A$ and

$$xy \in P \Rightarrow x \in P \text{ or } y \in P.$$

If $p \in \mathbb{Z}$, p prime and $n \geq 1$ then

$$xy \in p^n\mathbb{Z} \implies p^n \mid xy$$

inherently lopsided

$$\Rightarrow p^n \mid x \text{ or } p \mid y$$

$$\left(\not\Rightarrow p^n \mid x \text{ or } p^n \mid y \quad \text{for } n \geq 2 \right)$$

$$\Rightarrow x \in p^n\mathbb{Z} \text{ or } y \in p\mathbb{Z}$$

$$\Rightarrow x \in p^n\mathbb{Z} \text{ or } y^n \in p^n\mathbb{Z}$$

$$\boxed{\text{Define } Q \underset{\text{primary}}{\lhd} A \text{ if } Q \neq A \text{ and } \quad xy \in Q \implies x \in Q \text{ or } y^n \in Q \quad \exists n \geq 1.}$$

e.g. $p^n\mathbb{Z} \underset{\text{primary}}{\lhd} \mathbb{Z}$ for p prime, $n \geq 1$.

Exercise: Find a ring A and $I \lhd A$ such that I is not primary, yet

$$xy \in I \implies x^n \in I \text{ or } y^n \in I \quad (\exists n \geq 1)$$

Claim: $Q \underset{\text{primary}}{\lhd} \mathbb{Z} \iff Q = p^n\mathbb{Z}$

$\exists$ prime p, $\exists n \geq 1$.

Proof: ($\Leftarrow$) done above

($\Rightarrow$) Suppose $Q \underset{\text{primary}}{\lhd} \mathbb{Z}$.

But $Q = m\mathbb{Z}$ $\quad \exists m \geq 2$ (since $\mathbb{Z}$ is a PID)

and $m = p^n q$ $\quad \exists$ prime p, $\exists n \geq 1$ and $\exists q$ coprime to p.

(D)

Then $p^n q \in Q$ so $p^n \in Q$ or $q^k \in Q$ $\exists k \geq 1$

But if $q^k \in Q$ then

$$p^n q \mid q^k \text{, so } p \mid q \quad \times .$$

Hence $p^n \in Q$, so $p^n q \mid p^n$, so $q = 1$

and $Q = p^n \mathbb{Z}$. $\square$

Exactly same argument shows

If A is a UFD and $I \triangleleft A$ is principal then

$$I \text{ is primary} \iff I = x^n A = (xA)^n$$

$\exists$ irreducible $x \in A$, $\exists n \geq 1$.

— no simple description for ~~non~~ principal ideals

~~look for examples in a ring~~ which is not a PID

Example : $A = F[x,y]$

$$Q = \langle x, y^2 \rangle = Ax + Ay^2 \trianglelefteq A$$

Claims :

(i) $Q \underset{\text{primary}}{\trianglelefteq} A$

(ii) $r(Q) = \langle x, y \rangle = Ax + Ay \underset{\text{prime}}{\trianglelefteq} A$

(iii) $r(Q)^2 \subsetneqq Q \subsetneqq r(Q)$

(iv) Q is not a power of a prime ideal.

Proofs : (i) $F[x,y] \xrightarrow{\varphi} F[y]/\langle y^2 \rangle$

$$p(x,y) \longmapsto p(0,y) + \langle y^2 \rangle$$

has kernel Q (easy to check), so

$$\alpha\beta \in Q \Rightarrow \varphi(\alpha)\varphi(\beta) = \varphi(\alpha\beta) = \langle y^2 \rangle = \langle y \rangle^2$$

$$\Rightarrow \alpha(0,y) \in \langle y^2 \rangle \text{ or } \beta(0,y) \in \langle y \rangle$$

$$\Rightarrow \alpha \in \ker\varphi \text{ or } \beta^2(0,y) \in \langle y \rangle^2 = \langle y^2 \rangle$$

$$\Rightarrow \alpha \in Q \text{ or } \beta^2 \in Q$$

Hence $Q \underset{\text{primary}}{\trianglelefteq} A$.

P

(ii) $r(Q) = r\langle x^2, y\rangle = \langle x, y\rangle \lhd F[x,y] = A$

clear

and $A/r(Q) \cong F$, so $r(Q) \underset{max}{\lhd} A$, so prime.

(iii) $r(Q)^2 = (Ax + Ay)^2 = Ax^2 + Ay^2 + Axy$

so $r(Q)^2 \subsetneq Q \subsetneq r(Q)$

$x \in Q \setminus r(Q)^2$ $\quad$ $y \in r(Q) \setminus Q$

(iv) If $Q = P^n$ $\exists\, P \underset{prime}{\lhd} A$ $\exists\, n \geq 1$

then $r(Q) = r(P^n) = P$

first assignment

so $P^2 = r(Q)^2 \subsetneq Q = P^n \subsetneq r(Q) = P$

$P^2 \subsetneq P^2$ $\longrightarrow$ contradiction if $n \geq 2$ $\quad$ contradiction if $n = 1$