(A)

Commutative Algebra   Lecture 30   Wed 13/5/09

Universal property of $S^{-1}A$:

$g : A \to B$ ring hom with $g(s)$ unit $\forall s \in S$

$\Rightarrow \exists!$ hom h yielding commuting diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & S^{-1}A \arrow[d, "h"] \\ & B \end{tikzcd} \qquad a/s \mapsto g(a)\,g(s)^{-1}$$

can be an iso

Theorem: $B \cong S^{-1}A$ under h if $g : A \to B$ satisfies

(1) $(\forall s \in S)$ $g(s)$ is a unit in B

(2) $\ker g = \{a \in A \mid as = 0 \ \exists s \in S\}$

(3) $(\forall b \in B)$ $b = g(a)\,g(s)^{-1}$ $\exists a \in A, s \in S$

injectivity of h surjectivity of h

These hold for $g = f : A \to S^{-1}A$, $a \mapsto a/1$:

✓ (1) $f(s) = s/1 = (1/s)^{-1}$ in $S^{-1}A$ $\forall s \in S$

✓ (2) $\ker f = \{a \in A \mid a/1 = 0/1\} = \{a \in A \mid as = 0 \ \exists s \in S\}$

✓ (3) $(\forall\, a/s \in S^{-1}A)$ $a/s = a/1 \cdot 1/s = a/1 \,(s/1)^{-1}$,

(B)

<u>Examples</u> : (i) $S^{-1}\mathbb{Z} \cong \mathbb{Q}$ where $S = \mathbb{Z} \setminus \{0\}$

under inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$

(ii) $S^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_3$ where $S = \{1, 2\}$

under $g : \mathbb{Z}_6 \to \mathbb{Z}_3$ mod 3 hom

$0, 3 \mapsto 0, \quad 1, 4 \mapsto 1, \quad 2, 5 \mapsto 2$

✓ (1) $g(1) = 1, g(2) = 2$ units of $\mathbb{Z}_3$

✓ (2) $\ker g = \{0, 3\} = \{a \in \mathbb{Z}_6 \mid 2a = 0\}$

✓ (3) holds trivially since g onto

(iii) $A = F[x]$, $S = \{1, x, x^2, \ldots\}$

$S^{-1}A \cong F[x, x^{-1}]$, ring of Laurent polys

$\sum_{i \in \mathbb{Z}} a_i x^i$ of finite support

under inclusion $F[x] \hookrightarrow F[x, x^{-1}]$

✓ (1) all powers of x are invertible

✓ (2) kernel $= \{0\} = \{p(x) \mid x^i p(x) = 0 \ \exists i\}$

✓ (3) all Laurent polys have form

$x^{-N} p(x) = p(x) (x^N)^{-1}$

$\exists N \geq 0$, $p(x) \in F[x]$.

(c)

Modules of fractions

Let M be an A-module and $S \subseteq A$ be multiplicatively closed. Form

$$\underbrace{M}_{\text{numerators}} \times \underbrace{S}_{\text{denominators}} = \{(m,s) \mid m \in M,\ s \in S\}$$

and define a relation on $M \times S$:

$$\boxed{(m,s) \equiv (m',s') \iff t(sm' - s'm) = 0 \quad \exists\, t \in S}$$

easily checked to be an equivalence relation.

Write
$$m/s = \text{equivalence class of } (m,s)$$

and put
$$S^{-1}M = \{ m/s \mid m \in M,\ s \in S \}$$

with operations

$$\boxed{\begin{aligned} m/s + m'/s' &= sm' + s'm / ss' \\ \underbrace{a/t}_{\text{element } \in S^{-1}A}\ m/s &= am/ts \end{aligned}}$$

These operations are well-defined and $S^{-1}M$ forms an $S^{-1}A$-module, called the <u>module of fractions with respect to S</u>. (quite a lot of checking)

Recall $A \xrightarrow{f} S^{-1}A$, $a \mapsto a/1$

is a homomorphism, so, by restriction of scalars (with respect to f),

$S^{-1}M$ becomes an A-module

where $a \cdot \frac{m}{s} = \frac{a}{1}\frac{m}{s} = \frac{am}{s}$

$\in A$ $\in M$ $f(a)$

The mapping

$$M \xrightarrow{F} S^{-1}M, \quad m \mapsto m/1$$

then becomes an A-module homomorphism, and

F is injective

$\iff (\forall x \in M, x \neq 0) \ \mathrm{Ann}(x) \cap S = \emptyset$

Examples:

(1) Let $A = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$, so $S^{-1}A = \mathbb{Q}$,

and $M = \mathbb{Z} \oplus \mathbb{Z} = \{(a,b) \mid a, b \in \mathbb{Z}\}$.

In $S^{-1}M$, $\frac{(a,b)}{s} = \frac{(c,d)}{t}$

$\Longleftrightarrow$ $u\big(t(a,b) - s(c,d)\big) = 0$ $\exists u \neq 0$

$\Longleftrightarrow$ $(ta, tb) = (sc, sd)$

$\Longleftrightarrow$ $\left(\frac{a}{s}, \frac{b}{s}\right) = \left(\frac{c}{t}, \frac{d}{t}\right)$

setting up an isomorphism: $S^{-1}M \cong \mathbb{Q} \oplus \mathbb{Q}$

need common denominators to get onto:

$\left(\frac{a}{s}, \frac{b}{t}\right) = \left(\frac{at}{st}, \frac{bs}{st}\right)$

both as $\mathbb{Z}$ and $\mathbb{Q}$-modules

More generally,

$$\boxed{\begin{array}{l}\text{If } A \text{ is an integral domain with field of fractions } F \\ \text{and } M \text{ is a free } A\text{-module of rank } n, \text{ then} \\ \qquad S^{-1}M \cong F^n \quad (n\text{-dimensional vector space})\end{array}}$$

(F)

(2) Let $A = \mathbb{Z}$, $S = \{1, 3, 3^2, 3^3, \ldots\}$,

so $S^{-1}A = \left\{ a/3^i \mid a \in \mathbb{Z},\ i \in \mathbb{N} \right\}$

Put $M = \mathbb{Z}_6 \oplus \mathbb{Z}_6$.

Easy to check from the definition, in $S^{-1}M$

$$\frac{(a,b)}{3^i} = \begin{cases} \frac{(0,0)}{1} & \text{if } a,b \text{ even} \\ \frac{(0,1)}{1} & \text{if } a \text{ even}, b \text{ odd} \\ \frac{(1,0)}{1} & \text{if } a \text{ odd}, b \text{ even} \\ \frac{(1,1)}{1} & \text{if } a,b \text{ odd} \end{cases}$$

so that

$$S^{-1}M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

as $\mathbb{Z}$, $S^{-1}\mathbb{Z}$ and $\mathbb{Z}_2$-modules

noting $\mathrm{Ann}(S^{-1}M) = \{ a/3^i \mid a \in 2\mathbb{Z},\ i \in \mathbb{N} \} \underset{max}{\triangleleft} S^{-1}\mathbb{Z}$

and $S^{-1}\mathbb{Z} / \mathrm{Ann}(S^{-1}M) \cong \mathbb{Z}_2$

Note also $\mathrm{Ann}(3) = 3\mathbb{Z}$ and $3\mathbb{Z} \cap S \neq \emptyset$,

and $F: \mathbb{Z}_6 \oplus \mathbb{Z}_6 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the mod 2 map which is not injective.

(G)

Connection with tensor products:

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}: \text{ Always} \\ S^{-1}M \cong S^{-1}A \otimes_A M = \{ \tfrac{1}{s} \otimes m \mid s \in S, m \in M \}\end{array}}$$

unique iso with property
$am/s \longmapsto \frac{a}{s} \otimes m$

note ordinary brackets!!

because of common denominators:

$$\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^{n} \frac{a_i t_i}{s} \otimes m_i$$

where $s = s_1 \cdots s_n$

$t_i = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n$

$$= \sum \left(\frac{a_i t_i}{1} \frac{1}{s} \right) \otimes m_i$$

$$= \sum \frac{a_i t_i}{1} \left(\frac{1}{s} \otimes m_i \right)$$

"buffer" scalar multiplication

$$= \sum (a_i t_i) \cdot \left({}^1\!/_s \otimes m_i \right)$$

restriction of scalars

$$= \sum {}^1\!/_s \otimes (a_i t_i m_i)$$

$$= \frac{1}{s} \otimes \left(\sum a_i t_i m_i \right)$$

✓