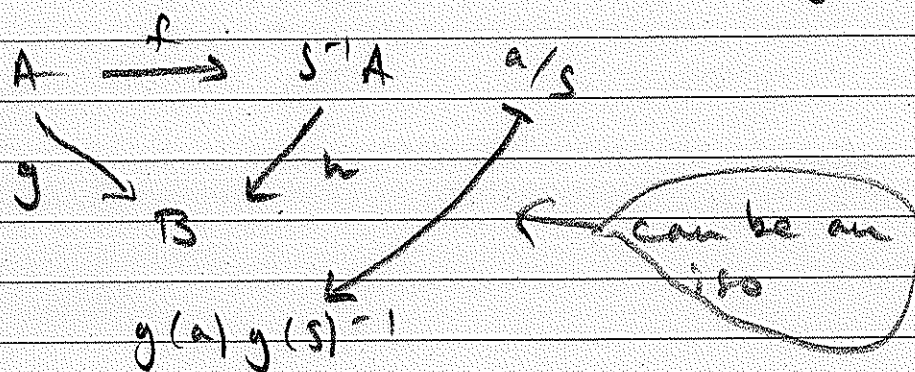


Universal property of $S^{-1}A$:

$g: A \rightarrow B$ ring hom with $g(s)$ unit $\forall s \in S$

$\Rightarrow \exists!$ hom h yielding commuting diagram



Theorem: $B \cong S^{-1}A$ under h if $g: A \rightarrow B$ satisfies

(1) $(\forall s \in S) \quad g(s)$ is a unit in B

→ (2) $\ker g = \{a \in A \mid as = 0 \exists s \in S\}$

(3) $(\forall b \in B) \quad b = g(a)g(s)^{-1} \exists a \in A, s \in S$

(injectivity of h)

(surjectivity of h)

These hold for $g = f: A \rightarrow S^{-1}A, a \mapsto a/1$:

✓ (1) $f(s) = s/1 = (1/s)^{-1}$ in $S^{-1}A \quad \forall s \in S$

✓ (2) $\ker f = \{a \in A \mid a/1 = 0/1\} = \{a \in A \mid as = 0 \exists s \in S\}$

✓ (3) $(\forall a/s \in S^{-1}A) \quad a/s = a/1 \cdot 1/s = a/1 \cdot (s/1)^{-1}$

(B)

Examples: (i) $S^{-1}\mathbb{Z} \cong \mathbb{Q}$ where $S = \mathbb{Z} \setminus \{0\}$
under inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$

(ii) $S^{-1}\mathbb{Z}_6 \cong \mathbb{Z}_3$ where $S = \{1, 2\}$

under $g: \mathbb{Z}_6 \rightarrow \mathbb{Z}_3 \pmod{3}$ hom

$$0, 3 \mapsto 0, \quad 1, 4 \mapsto 1, \quad 2, 5 \mapsto 2$$

✓ (1) $g(1) = 1, g(2) = 2$ units of $\mathbb{Z}_3$

✓ (2) $\ker g = \{0, 3\} = \{a \in \mathbb{Z}_6 \mid 2a = 0\}$

✓ (3) holds trivially since g onto

(iii) $A = F[x], \quad S = \{1, x, x^2, \dots\}$

$S^{-1}A \cong F[x, x^{-1}]$, ring of Laurent polys

$$\sum_{i \in \mathbb{Z}} a_i x^i \text{ of finite support}$$

under inclusion $F[x] \hookrightarrow F[x, x^{-1}]$

✓ (1) all powers of x are invertible

✓ (2) $\ker = \{0\} = \{p(x) \mid x^i p(x) = 0 \exists i\}$

✓ (3) all Laurent polys have form

$$x^{-N} p(x) = p(x) (x^N)^{-1}$$

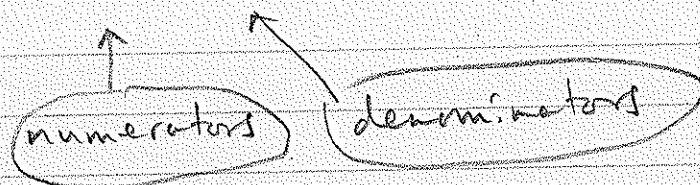
$$\exists N \geq 0, p(x) \in F[x]$$

(c)

Modules of fractions

Let M be an A -module and $S \subseteq A$ be multiplicatively closed. Form

$$M \times S = \{ (m, s) \mid m \in M, s \in S \}$$



and define a relation on $M \times S$:

$$(m, s) \equiv (m', s')$$

$$\Leftrightarrow \exists t \in S, t(sm' - s'm) = 0$$

easily checked to be an equivalence relation,

Write

$$m/s = \text{equivalence class of } (m, s)$$

and put

$$S^{-1}M = \{ m/s \mid m \in M, s \in S \}$$

with operations

$$m/s + m'/s' = \frac{sm' + s'm}{ss'}$$

$$a/x \cdot m/s = am/ts$$

element
of $S^{-1}A$

⑤

These operations are well-defined and $S^{-1}M$ forms an $S^{-1}A$ -module, called the module of fractions with respect to S . (quite a lot of checking)

Recall $A \xrightarrow{f} S^{-1}A, a \mapsto a/1$

is a homomorphism, so, by restriction of scalars (with respect to f),

$$\boxed{\begin{array}{l} S^{-1}M \text{ becomes an } A\text{-module} \\ \text{where } a \cdot \frac{m}{s} = \frac{a}{1} \frac{m}{s} = \frac{am}{s} \end{array}}$$

$\circlearrowleft A$
 $\uparrow$

$\circlearrowleft M$
 $\uparrow$

$\circlearrowleft f(a)$
 $\uparrow$

The mapping

$$M \xrightarrow{F} S^{-1}M, m \mapsto m/1$$

then becomes an A -module homomorphism, and

$$\boxed{\begin{array}{l} F \text{ is injective} \\ \Leftrightarrow (\forall x \in M, x \neq 0) \quad \text{Ann}(x) \cap S = \emptyset \end{array}}$$

(E)

Examples:

(1) Let $A = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$, so $S^{-1}A = \mathbb{Q}$,
and $M = \mathbb{Z} \oplus \mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{Z}\}$.

$$\text{In } S^{-1}M, \quad \frac{(a, b)}{s} = \frac{(c, d)}{t}$$

$$\Leftrightarrow u(t(a, b) - s(c, d)) = 0 \quad \exists u \neq 0$$

$$\Leftrightarrow (ta, tb) = (sc, sd)$$

$$\Leftrightarrow \left(\frac{a}{s}, \frac{b}{s}\right) = \left(\frac{c}{t}, \frac{d}{t}\right)$$

setting up an isomorphism: $S^{-1}M \cong \mathbb{Q} \oplus \mathbb{Q}$

need common denominators
to get onto:

$$\left(\frac{a}{s}, \frac{b}{t}\right) = \left(\frac{at}{st}, \frac{bs}{st}\right)$$

both as $\mathbb{Z}$ and
 $\mathbb{Q}$ -modules

More generally,

If A is an integral domain with field of fractions F
and M is a free A -module of rank n , then

$$S^{-1}M \cong F^n \quad (\text{n-dimensional vector space})$$

(F)

(2) Let $A = \mathbb{Z}$, $S = \{1, 3, 3^2, 3^3, \dots\}$,

so $S^{-1}A = \left\{ \frac{a}{3^i} \mid a \in \mathbb{Z}, i \in \mathbb{N} \right\}$

Put $M = \mathbb{Z}_6 \oplus \mathbb{Z}_6$.

Easy to check from the definition, in $S^{-1}M$

$$\frac{(a,b)}{3^i} = \begin{cases} \frac{(0,0)}{1} & \text{if } a, b \text{ even} \\ \frac{(0,1)}{1} & \text{if } a \text{ even, } b \text{ odd} \\ \frac{(1,0)}{1} & \text{if } a \text{ odd, } b \text{ even} \\ \frac{(1,1)}{1} & \text{if } a, b \text{ odd} \end{cases}$$

so that

$$S^{-1}M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

↑
as $\mathbb{Z}$, $S^{-1}\mathbb{Z}$ and $\mathbb{Z}_2$ -modules

noting $\text{Ann}(S^{-1}M) = \left\{ \frac{a}{3^i} \mid a \in 2\mathbb{Z}, i \in \mathbb{N} \right\} \triangleleft \max S^{-1}\mathbb{Z}$

$$\text{and } S^{-1}\mathbb{Z} / \text{Ann}(S^{-1}M) \cong \mathbb{Z}_2$$

Note also $\text{Ann}(\mathbb{Z}) = 3\mathbb{Z}$ and $3\mathbb{Z} \cap S \neq \emptyset$,

and $F: \mathbb{Z}_6 \oplus \mathbb{Z}_6 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the mod 2 map which is not injective.

9

Connection with tensor products:

Theorem: Always

$$S^{-1}M \cong S^{-1}A \otimes_A M = \left\{ \frac{1}{s} \otimes m \mid s \in S, m \in M \right\}$$

unique iso with property

$$am/s \mapsto \frac{a}{s} \otimes m$$

note ordinary brackets !!

because of common denominators:

$$\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^n \frac{a_i t_i}{s} \otimes m_i$$

$$\text{where } s = s_1 \cdots s_n$$

$$t_i = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n$$

$$= \sum \left(\frac{a_i t_i}{1} \cdot \frac{1}{s} \right) \otimes m_i$$

$$= \sum \frac{a_i t_i}{1} \left(\frac{1}{s} \otimes m_i \right)$$

$$= \sum (a_i t_i) \cdot \left(\frac{1}{s} \otimes m_i \right)$$

$$= \sum \frac{1}{s} \otimes (a_i t_i m_i)$$

$$= \frac{1}{s} \otimes \left(\sum a_i t_i m_i \right)$$

"buffer" scalar multiplication

restriction of scalars

