

Finitely generated rings & algebras

Hilbert's Basis Theorem : If A is a Noetherian ring then so is $A[x]$.

Iterating this gives

Corollary : $F[x_1, \dots, x_n]$ is Noetherian when F is a field.

Proof of Hilbert's Basis Theorem

Suppose A is Noetherian.

We show all ideals of $A[x]$ are f.g.

Suppose $\{0\} \neq J \triangleleft A[x]$

Put

$I = \{ \text{leading coefficients of nonzero polys in } J \}$

$\lambda_0 + \lambda_1 x + \dots + \lambda_n x^n, \lambda_n \neq 0$

↑ leading coefficient

(B)

Clearly $I \triangleleft A$, so

$$I = \langle a_1, \dots, a_n \rangle \quad \exists a_1, \dots, a_n \in A$$

since A Noetherian,

then

$$(\forall i = 1, \dots, n) (\exists p_i(x) \in I)$$

$$p_i(x) = a_i x^{d_i} + (\text{lower terms})$$

Put

$$J' = \langle p_1(x), \dots, p_n(x) \rangle \subseteq I$$

and

$$d = \max \{d_1, \dots, d_n\},$$

Put

$$M = \{q(x) \in A[x] \mid \text{degree of } q(x) \leq d\}$$

$$= \langle 1, x, x^2, \dots, x^d \rangle,$$

$$\text{Claim: } I = \underbrace{(J \cap M)}_{\text{f.g.}} + J'$$

(f.g.) ✓✓✓

(f.g.) ✓✓

(f.g.) ✓

since $J \cap M \subseteq M$ f.g.

& A Noetherian

& part of Thm complete.

(c)

Part of Claim $\therefore J \cap M \subseteq J, J' \subseteq J$

$$\Rightarrow (J \cap M) + J' \subseteq J \quad \checkmark$$

$$\boxed{\text{WTS } J \subseteq (J \cap M) + J'}$$

Let $p(x) \in J, p(x) \neq 0$, say

$$\boxed{p(x) = ax^m + (\text{lower terms})}$$

↑
leading coefficient

$\boxed{\text{We show } p(x) \in (J \cap M) + J' \text{ by induction on } m.}$

If $m \leq d$ then $p(x) \in J \cap M \subseteq (J \cap M) + J'$

which starts the induction. $\checkmark$

Suppose $m \geq d$ and make an appropriate inductive hypothesis.

(D)

But $a \in I$, so $a = \sum_{i=1}^n u_i a_i$ ($\exists u_i$)

Since $I = \langle a_1, \dots, a_n \rangle$,

Put

$$q(x) = p(x) - \underbrace{\sum_{i=1}^n u_i x^{m-d_i} p_i(x)}_{\text{leading coefficient is } \sum u_i a_i = a}$$

leading coefficient

is $\sum u_i a_i = a$

so

$$\boxed{q(x) \text{ has degree } \leq m}$$

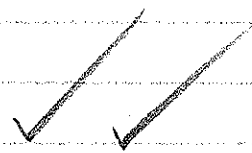
so, by the inductive hypothesis,

$$q(x) \in (J \cap M) + J'$$

so

$$p(x) = \underbrace{q(x)}_{\substack{m \\ (J \cap M) + J'}} + \underbrace{\sum_{i=1}^n u_i x^{m-d_i} p_i(x)}_{\substack{m \\ J'}}$$

$$\in (J \cap M) + J'$$



This proves the claim, and completes the proof of Hilbert's Basis Theorem. $\square$

(E)

Corollary : All finitely generated rings
and all finitely generated algebras
over a field are Noetherian.

Proof : If A is Noetherian and

$$B = \langle b_1, \dots, b_n \rangle_{A\text{-algebra}}$$

Then

$A[x_1, \dots, x_n]$ is Noetherian,

$$0 \rightarrow \ker \nu \hookrightarrow A[x_1, \dots, x_n] \xrightarrow{\nu} B \rightarrow 0$$

is exact where ν is evaluation :

$$p(x_1, \dots, x_n) \mapsto p(b_1, \dots, b_n),$$

so

B is Noetherian

In particular, if B is a ring and

$$A = \langle 1 \rangle_{\text{subring}} \cong \mathbb{Z} \text{ or } \mathbb{Z}_k \quad (\exists k)$$

then B is an algebra over A , so Noetherian

since A is. If A is a field, ditto.

