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Commutative Algebra Lecture 39 Wed 3/6/09

Finitely generated rings & algebras

Hilbert's Basis Theorem: If A is a Noetherian ring then so is $A[x]$.

Iterating this gives

Corollary: $F[x_1, \ldots, x_n]$ is Noetherian when F is a field.

Proof of Hilbert's Basis Theorem:

Suppose A is Noetherian.

We show all ideals of $A[x]$ are f.g.

Suppose
$$\{0\} \neq J \lhd A[x].$$

Put
$$I = \{\text{ leading coefficients of nonzero polys in } J \}$$

$$\lambda_0 + \lambda_1 x + \ldots + \underbrace{\lambda_n}_{\text{leading coefficient}} x^n, \quad \lambda_n \neq 0$$

(B)

Clearly $I \trianglelefteq A$, so

$$I = \langle a_1, \dots, a_n \rangle \quad \exists a_1, \dots, a_n \in A$$

since A Noetherian.

Then

$$(\forall i = 1, \dots, n)\ (\exists p_i(x) \in J)$$

$$\boxed{p_i(x) = a_i x^{d_i} + (\text{lower terms})}$$

Put

$$J' = \langle p_1(x), \dots, p_n(x) \rangle \subseteq J$$

and

$$d = \max\{d_1, \dots, d_n\}.$$

Put

$$M = \{ q(x) \in A[x] \mid \text{degree of } q(x) \leq d \}$$

$$= \langle 1, x, x^2, \dots, x^d \rangle.$$

$$\boxed{\underline{\text{Claim}}: \quad J = \underbrace{(J \cap M)} + J'}$$

f.g. ✓

f.g. ✓✓

since $J \cap M \leq M$ f.g. & A Noetherian

f.g. ✓✓✓

& proof of Thm complete.

Proof of Claim: $J \cap M \subseteq J$, $J' \subseteq J$

so $(J \cap M) + J' \subseteq J$ ✓.

$$\boxed{\text{WTS } \; J \subseteq (J \cap M) + J'}$$

Let $p(x) \in J$, $p(x) \neq 0$, say

$$\boxed{p(x) = ax^m + (\text{lower terms})}$$

leading coefficient

We show $p(x) \in (J \cap M) + J'$ by induction on m.

If $m \leq d$ then $p(x) \in J \cap M \subseteq (J \cap M) + J'$

which starts the induction. ✓

Suppose $m > d$ and make an appropriate inductive hypothesis.

But $a \in I$, so $a = \sum_{i=1}^{n} u_i a_i$ $(\exists u_i)$

since $I = \langle a_1, \dots, a_n \rangle$.

Put

$$q(x) = p(x) - \underbrace{\sum_{i=1}^{n} u_i x^{m-d_i} p_i(x)}_{\text{leading coefficient is } \sum u_i a_i = a}$$

so $\boxed{q(x) \text{ has degree} < m}$

so, by the inductive hypothesis,

$$q(x) \in (J \cap M) + J'$$

so

$$p(x) = \underbrace{q(x)}_{\substack{\cap \\ (J\cap M)+J'}} + \underbrace{\sum u_i x^{m-d_i} p_i(x)}_{\substack{\cap \\ J'}}$$

$$\in (J \cap M) + J'$$

This proves the Claim, and completes the proof of Hilbert's Basis Theorem. $\square$

Corollary: All finitely generated rings and all finitely generated algebras over a field are Noetherian.

Proof: If A is Noetherian and

$$B = \langle b_1, \dots, b_n \rangle_{A\text{-algebra}}$$

then

$$A[x_1, \dots, x_n] \text{ is Noetherian},$$

$$0 \longrightarrow \ker \nu \hookrightarrow A[x_1, \dots, x_n] \overset{\nu}{\twoheadrightarrow} B \longrightarrow 0$$

is exact where ν is evaluation:

$$p(x_1, \dots, x_n) \longmapsto p(b_1, \dots, b_n),$$

so $\boxed{B \text{ is Noetherian}}$.

In particular, if B is a ring and

$$A = \langle 1 \rangle_{\text{subring}} \cong \mathbb{Z} \text{ or } \mathbb{Z}_k \quad (\exists k)$$

then B is an algebra over A, so Noetherian since A is. If A is a field, ditto. $\square$